Math Circle for November 6, 2002

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Extinction & Population Dynamics (Part 2)

Challenge Problem (from Part 1)
Determine the long term dynamics of a population that behaves according to the equation:

$$N_{t+1} = \frac{rN_t}{1 + 4N_t^2}$$

A. Use the cobwebbing technique with the following values of r and N_0:
r = 2, 4, 5 and $N_0 = .1, .9$

B. Use your calculator to produce a short sequence of numbers using the initial population sizes given above.
When $r = 2$, $N^* = 0$ and $N^* = .5$

$$N_0 = .1 \rightarrow .19 \rightarrow .34 \rightarrow .46 \rightarrow .50$$
$$N_0 = .9 \rightarrow .42 \rightarrow .49 \rightarrow .50$$

When $r = 4$, $N^* = 0$ and $N^* = \frac{\sqrt{3}}{2} \approx .87$

$$N_0 = .1 \rightarrow .38 \rightarrow .97 \rightarrow .82 \rightarrow .89 \rightarrow .85 \rightarrow .87$$
$$N_0 = .9 \rightarrow .85 \rightarrow .87$$

When $r = 5$, $N^* = 0$ and $N^* = 1$

$$N_0 = .1 \rightarrow .48 \rightarrow 1.25 \rightarrow .86 \rightarrow 1.08 \rightarrow .95 \rightarrow 1.03 \rightarrow .98 \rightarrow 1.01 \rightarrow .99 \rightarrow 1.00$$
$$N_0 = .9 \rightarrow 1.06 \rightarrow .96 \rightarrow 1.02 \rightarrow .99 \rightarrow 1.01 \rightarrow 1.00$$

C. Algebraically solve for the equilibrium points for general values of r. How many equilibrium points are there?
Equilibrium points are $N^* = 0$ and $N^* = \frac{\sqrt{r-1}}{2}$. The second equilibrium only exists if $r > 1$.

D. Do you see new patterns appearing in the cobwebbing picture and numerical sequence that did not occur in Ponds 1 and 2? What are some possible biological explanations for the patterns you see?

In the previous examples (Ponds 1 and 2), the population approached the equilibrium points from one direction, i.e. if the initial population was less (greater) than the equilibrium level, all the intermediate steps were less (greater) than the equilibrium level. In this example, we see that the population levels sometimes cycle around the equilibrium level as it approaches equilibrium. When this phenomena occurs in biological systems it can be due to effects such as overcompensation or cannibalism.

LOGISTIC MAP

Cycling behavior is not unique to the model examined in the Challenge Problem. Lets consider the following model for population growth (the logistic map).

$$N_{t+1} = rN_t(1 - N_t) = rN_t - rN_t^2$$

This function is an upside down parabola with height $\frac{r}{4}$ when $N_t = \frac{1}{2}$. Again, we are assuming that N_t is the fraction of the pond (or other habitat) that is full, so $0 \leq N_t \leq 1$. To satisfy this, we use $0 \leq r \leq 4$.

Exercise 1a: Algebraically find the equilibrium points.

Solve

$$\begin{aligned} N &= rN(1 - N) \\ rN^2 + (1 - r)N &= 0 \\ N(rN + 1 - r) &= 0 \\ N^* = 0 \;\; \textit{and} \;\; N^* &= \frac{r - 1}{r} = 1 - \frac{1}{r} \end{aligned}$$

Exercise 1b: Using cobwebbing, predict how full the pond will be after ten years if:

$r = 2.8,\ 3.3,\ 3.83,\ 4$ and $N_0 = \frac{1}{2}$

For the case $r = 4$ also determine what happens when N_0 is close to $\frac{1}{2}$, but not equal to $\frac{1}{2}$.

Hint: If you are having trouble figuring out the dynamics from the cobwebbing, try calculating a sequence of numbers with your calculator. Start at $\frac{1}{2}$ and look for a pattern.

For $r = 2.8$ we see the population level cycling into the positive equilibrium point (damped oscillations).

Sample sequence: $.5 \rightarrow .7 \rightarrow .59 \rightarrow .68 \rightarrow .61 \rightarrow .67 \rightarrow .62 \rightarrow .66 \rightarrow .63$

For $r = 3.3$, the population oscillates between two different levels (period 2).

Sample sequence: $.5 \rightarrow .83 \rightarrow .48 \rightarrow .82 \rightarrow .48 \rightarrow .82 \rightarrow .48$

For $r = 3.83$, the population oscillates between three different levels (period 3).

Sample sequence: $.5 \rightarrow .96 \rightarrow .16 \rightarrow .5 \rightarrow .96 \rightarrow .16 \rightarrow .5$

For $r = 4$, if $N_0 = \frac{1}{2}$, the population goes extinct. If N_0 is just less than $\frac{1}{2}$ we find non-zero population levels after each step, but there doesn't seem to be a pattern (chaotic). One property to notice here is that even when we start with very similar population levels (.5 and .48 or .49), the dynamics over time are very different. This is an important property of chaotic systems.

Sample sequence: $.48 \rightarrow .998 \rightarrow .006 \rightarrow .025 \rightarrow .099 \rightarrow .36 \rightarrow .92 \rightarrow .30 \rightarrow .84 \rightarrow .53 \rightarrow .995 \rightarrow .017 \rightarrow .066 \rightarrow .25 \rightarrow .75 \rightarrow .76 \rightarrow .73 \rightarrow .79 \rightarrow .67 \rightarrow .88 \rightarrow .42 \rightarrow .97 \rightarrow .11$

Exercise 1c: What could be going on biologically in each of these cases?

****Exercise 1d:** Theoretically, how could you determine if 2, 3, 4, 8, 16, . . . cycles exist? If they do exist, how can you find the values they cycle between? Is there any overlap between solving for 2-cycles and 4-cycles, or other pairs of n-cycles?

For n-cycles, solve the equation $N_{t+n} = N_t$.

****Exercise 1e:** Using algebra, determine for which values of r the 2-cycle exists. Determine the values that the population cycles between for general r that also satisfy the 2-cycle criteria.

Solve the equation $N_{t+2} = N_t$.

$$N_{t+2} = rN_{t+1}(1 - N_{t+1})$$

$$\begin{aligned} N_{t+2} = r\ \{rN_t(1-N_t)\}\ (1-\ \{rN_t(1-N_t)\}) \\ N = r^2N(1-N)(1-rN+rN^2) \\ N = r^2N - r^2N^2 - r^3N^2 + r^3N^3 + r^3N^3 - r^3N^4 \\ r^3N^4 - 2r^3N^3 + (r^3+r^2)N^2 + (1-r^2)N = 0 \end{aligned}$$

We know that the fixed points must satisfy this equation as well since $N_{t+1} = N_t$ implies that $N_{t+2} = N_t$. Therefore, we can divide the polynomial by N and $(N - 1 + \frac{1}{r})$ to get a quadratic equation.

$$N\ (N-1+\frac{1}{r})\ r\ \{r^2N^2 - (r^2+r)N + (r+1)\}\ =\ 0$$

Solving the quadratic, we find that the additional two roots are:

$$N^* = \frac{r+1 \pm \sqrt{(r-3)(r+1)}}{2r}$$

These roots only exist when $r > 3$, so 2-cycles can only occur when $r > 3$. When they do occur, the population cycles between the two solutions of the quadratic.

TENT MAP

In mathematical biology, when we want to study the behavior of a complicated system, it often helps to examine a simpler approximation to that system. We can approximate the parabolic shape of the logistic using two straight lines (called the Tent Map).

$$\begin{aligned} N_{t+1} &= rN_t \qquad\quad if \quad N_t \leq \frac{1}{2} \\ &= r(1-N_t) \quad if \quad \frac{1}{2} \leq N_t \end{aligned}$$

The maximum value of this function is $\frac{r}{2}$ and occurs when $N_t = \frac{1}{2}$. To ensure that $0 \leq N_t \leq 1$ we require $0 \leq r \leq 2$.

We first notice that if $r < 1$, the slope of the left side of the tent is less than one, so the tent is always underneath the line $N_{t+1} = N_t$. Therefore, the only equilibrium point is at $N_t = 0$.

Consider the case when $r = 1$. Here we notice that the left side of the tent sits directly on the line $N_{t+1} = N_t$. Therefore, we have equilibria for all N_t satisfying $0 \leq N_t \leq \frac{1}{2}$. Thus, there are an infinite number of equilibrium points.

Now consider the case when $1 < r \leq 2$. Here the slope of the left side of the tent will always be greater than one, so the left side is above the line $N_{t+1} = N_t$. From a picture, we see that there is one non-zero equilibrium point. We can solve for this equilibrium point algebraically:

$$N = r(1 - N)$$
$$N^* = \frac{r}{1 + r}$$

Now consider the special case when $r = 2$. Notice that the peak of the tent for $r = 2$ has a height of 1, the same as the logistic map when $r = 4$. In fact, the behavior of the tent map with $r = 2$ is very similar to that of the logistic map with $r = 4$. We have already found the equilibrium points for general $r > 1$. Using this equation, we find $N^* = 0$ and $N^* = \frac{2}{3}$ for the case $r = 2$.

Let's look at a few iterations of the tent map with $r = 2$.

$$N_0 = 0.359375$$
$$N_1 = 2(0.359375) = 0.71875$$
$$N_2 = 2(1 - 0.71875) = 2(0.28125) = 0.5625$$
$$N_3 = 2(1 - 0.5625) = 2(0.4375) = 0.875$$

Our initial motivation for examining the tent map was to make things easier, but it looks like these calculations could become tedious. So, we'd like to figure out an easier way to iterate the tent map function.

BINARY DECIMALS

We first remember that:

$$\sum_{i=1}^{\infty} \frac{1}{2^i} = \frac{1}{2^1} + \frac{1}{2^2} + \frac{1}{2^3} + \frac{1}{2^4} + \frac{1}{2^5} + ... = \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \frac{1}{32} + ... = 1$$

Also, any number between zero and one can be expressed as a sum of the form:

$$\sum_{i=1}^{\infty} a_i \frac{1}{2^i} = a_1 \frac{1}{2^1} + a_2 \frac{1}{2^2} + a_3 \frac{1}{2^3} + a_4 \frac{1}{2^4} + ... = a_1 \frac{1}{2} + a_2 \frac{1}{4} + a_3 \frac{1}{8} + a_4 \frac{1}{16} + ...$$

where a_i equals 0 or 1. A decimal's binary representation is just a list of the coefficients a_i. We use the form:

$$0.a_1a_2a_3a_4... \Rightarrow a_1\frac{1}{2^1} + a_2\frac{1}{2^2} + a_3\frac{1}{2^3} + a_4\frac{1}{2^4} + ...$$

Let's go back to our initial example. We started with

$$N_0 = 0.359375 = \frac{23}{64} = \frac{1}{4} + \frac{1}{16} + \frac{1}{32} + \frac{1}{64} \Rightarrow 0.0101110$$
$$N_1 = 0.71875 = \frac{23}{32} = \frac{1}{2} + \frac{1}{8} + \frac{1}{16} + \frac{1}{32} \Rightarrow 0.101110$$
$$N_2 = 0.5625 = \frac{9}{16} = \frac{1}{2} + \frac{1}{16} \Rightarrow 0.10010 = 0.10001111111111...$$
$$N_3 = 0.875 = \frac{7}{8} = \frac{1}{2} + \frac{1}{4} + \frac{1}{8} \Rightarrow 0.1110$$

Notice that some numbers will have two possible binary decimal representations. For example $\frac{1}{2} \Rightarrow 0.1 = 0.0111111111....$

Exercise 2a: Use cobwebbing to determine the long term dynamics of the tent map with $r = 2$ for: $N_0 = \frac{3}{30}, \frac{4}{30}, \frac{5}{30}, \frac{6}{27}$, and $\frac{34}{75}$. Try your own values for N_0.

Starting at $N_0 = \frac{5}{30}$ we go to the equilibrium point. Starting at $N_0 = \frac{3}{30}$ we go to a 2-cycle. Starting at $N_0 = \frac{6}{27}$ we go to a 3-cycle. Starting at $N_0 = \frac{4}{30}$ we go to a 4-cycle. Starting at $N_0 = \frac{34}{75}$ we go to a 20-cycle.

Exercise 2b: Determine an algorithm for the tent map (with $r = 2$) using binary decimals. Look at a general number of the form:

$$0.a_1a_2a_3a_4... \Rightarrow a_1\frac{1}{2} + a_2\frac{1}{2^2} + a_3\frac{1}{2^3} + a_4\frac{1}{2^4} + ...$$

and remember that:

$$\sum_{i=1}^{\infty} \frac{1}{2^i} = 1$$

Try multiplying by 2 and subtracting from 1. Do you see a pattern?

Multiplication by 2: Take any binary decimal representation and multiply it by 2:

$$2\sum_{i=1}^{\infty} a_i \frac{1}{2^i} = \sum_{i=1}^{\infty} a_i \frac{1}{2^{i-1}}$$
$$2 * 0.a_1a_2a_3a_4... \Rightarrow a_1.a_2a_3a_4...$$

Multiplying a binary decimal by two involves shifting everything over to the left one decimal place, or equivalently, moving the decimal point one place to the right.

Subtraction from 1: Take any binary decimal representation and subtract it from 1:

$$1 - \sum_{i=1}^{\infty} a_i \frac{1}{2^i} = \sum_{i=1}^{\infty} a_i' \frac{1}{2^i}$$

where the prime indicates changing a 1 to a 0 and vice-versa.

General Rule:

$$0.0a_2a_3a_4a_5... \Rightarrow 0.a_2a_3a_4a_5...$$
$$0.1a_2a_3a_4a_5... \Rightarrow 0.0a_2'a_3'a_4'a_5'... \Rightarrow 0.a_2'a_3'a_4'a_5'...$$

In words, if the number is less than one half, shift everything one digit to the left. If the number is greater than or equal to one half, flip the digits ($1 \rightarrow 0$ and $0 \rightarrow 1$) and then shift everything one digit to the left.

Exercise 2c: Use the algorithm you developed in exercise 2b to find 2, 3, and 4 cycles when $r = 2$. Look for patterns in the binary representations of the numbers. For example, the equilibrium point:

$$\frac{2}{3} = \frac{1}{2^1} + \frac{1}{2^3} + \frac{1}{2^5} + \frac{1}{2^7} + ... = \frac{1}{2} + \frac{1}{8} + \frac{1}{32} + \frac{1}{128} + ... = 0.1010101010101010...$$

Exercise 2d: Use algebra to find 2, 3 and 4 cycles when $r = 2$. To set up the equations you need to specify the pattern that you are looking for. If small numbers are less than one half and big are greater than or equal to one half, look for cycles with small, small, big, big, or small, big, small, big, or small, small, small, big.

Exercise 2e: Use algebra to find 2, 3 and 4 cycles for general values of r.
Look for a 2-cycle with the pattern small, big:

$$\begin{aligned} N_1 &= rN_0 \\ N_2 &= r(1 - rN_0) \\ N &= r - r^2 N \\ N^* &= \frac{r}{1 + r^2} \end{aligned}$$

So, for $r = 2$ we should have a 2-cycle starting at $N_0 = 2/5 \Rightarrow 0.11001100110011001100......$

Now, look for a 3-cycle with the pattern small, small, big:

$$\begin{aligned} N_1 &= rN_0 \\ N_2 &= r^2 N_0 \\ N_3 &= r(1 - r^2 N_0) \\ N &= r - r^3 N \\ N^* &= \frac{r}{1 + r^3} \end{aligned}$$

So, for $r = 2$ we should have a 3-cycle starting at $N_0 = 2/9 \Rightarrow 0.001110001110001110001 11......$

Now, look for a 4-cycle with the pattern small, small, big, big:

$$\begin{aligned} N_1 &= rN_0 \\ N_2 &= r^2 N_0 \\ N_3 = r(1 - r^2 N_0) &= r - r^3 N_0 \\ N_4 &= r(1 - r + r^3 N_0) \\ N &= r - r^2 + r^4 N \\ N^* &= \frac{r - r^2}{1 - r^4} \\ N^* &= \frac{r}{(1 + r)(1 + r^2)} \end{aligned}$$

So, for $r = 2$ we should have a 4-cycle starting at $N_0 = 2/15 \Rightarrow 0.0010001000100010001.....$

CHALLENGE PROBLEM
Find all the 5-cycles in the tent map with $r = 2$, either algebraically, or using patterns of binary decimals.