1210-90 Final Exam

Fall 2013

Name KEY

Instructions. Show all work and include appropriate explanations when necessary. Answers unaccompanied by work may not receive credit. Please try to do all work in the space provided and circle your final answers.

1. (16pts) Find the following derivatives. Show your work below and circle your final answer.

   (a) (4pts) $D_x(5x^2 - 4x + 2)$

   4

   $10x - 4$

   (b) (4pts) $D_x(\cos(x^5))$

   4

   $-\sin(x^5)(5x^4) = -5x^4 \sin(x^5)$

   (c) (4pts) $D_x(\frac{x}{x+\sin x})$

   4

   $= \dfrac{(x+\sin x)(1) - (x)(1+\cos x)}{(x+\sin x)^2} = \dfrac{\sin x - x\cos x}{(x+\sin x)^2}$

   (d) (4pts) $D_x(x^2\sqrt{x+3}) = D_x\left(x^2(x+3)^{1/2}\right)$

   4

   $= 2x(x+3)^{1/2} + \frac{1}{2}x^2(x+3)^{-1/2} = 2x\sqrt{x+3} + \dfrac{x^2}{2\sqrt{x+3}}$

2. (8pts) Compute the following limits. Answers may be values, $\pm\infty$, or 'DNE'. Show your work below and circle your final answer.

   (a) (4pts) $\lim\limits_{x\to 1} \dfrac{x^2 - x}{x^2 - 1} = \lim\limits_{x\to 1} \dfrac{x(x-1)}{(x-1)(x+1)} = \lim\limits_{x\to 1} \dfrac{x}{x+1} = \dfrac{1}{2}$

   4

   (b) (4pts) $\lim\limits_{x\to 0^+} \dfrac{1}{x^2 - x} = \lim\limits_{x\to 0^+} \dfrac{1}{x(x-1)}$

   4

   Note that when $x \approx 0$ and $x > 0$, $x(x-1) < 0$, so expression is negative. So $\lim\limits_{x\to 0^+} \dfrac{1}{x^2-x} = -\infty$

   +2 DNE

24

3. (6pts) Find the equation of the tangent line to the graph of $f(x) = x^3 - 2x$ at the point $(2, 4)$.

$f(x) = x^3 - 2x \Rightarrow f(2) = 4.$

6

$f'(x) = 3x^2 - 2 \Rightarrow f'(2) = 10.$

2

Equ of tangent line at $x = a$:

$y = f(a) + f'(a)(x-a)$

So when $a = 2$

$y = 4 + 10(x-2) = 10x - 16.$

4. (6pts) A spherical balloon is being inflated. If air is being pumped in at a rate of 18π in^3/s, how fast is the radius of the balloon increasing when the radius is equal to 3 inches? Remember the volume of a sphere is $V = \frac{4}{3}\pi r^3$.

$V = \frac{4}{3}\pi r^3$

3

6

$\frac{dV}{dt} = 4\pi r^2 \frac{dr}{dt}$

So $18\pi = 4\pi (3)^2 \frac{dr}{dt}$

$\frac{dV}{dt} = 18\pi$ in^3/s

or

$r = 3$ in.

$\frac{dr}{dt} = \frac{1}{2}$ in/s.

5. (10pts) Use the **definition of the derivative** to compute the derivative of the function $f(x) = x^2 - 2x + 3$; that is, compute the limit

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

10

5

$$= \lim_{h \to 0} \frac{(x+h)^2 - 2(x+h) + 3 - (x^2 - 2x + 3)}{h}$$

$$= \lim_{h \to 0} \frac{\cancel{x^2} + 2xh + h^2 - \cancel{2x} - 2h + \cancel{3} - \cancel{x^2} + \cancel{2x} - \cancel{3}}{h}$$

missing limits

−2

$$= \lim_{h \to 0} \frac{2xh + h^2 - 2h}{h}$$

$$= \lim_{h \to 0} 2x + h - 2 = 2x - 2$$

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6. (18pts) Consider the function

$$f(x) = x^4 - \frac{8}{3}x^3 + 2x^2 + 1$$

(a) (3pts) Find $f'(x)$.

3

$$f'(x) = 4x^3 - 8x^2 + 4x$$

(b) (3pts) Find the critical point(s) of f.

3

$$0 = f'(x) = 4x^3 - 8x^2 + 4x = 4x(x^2 - 2x + 1) = 4x(x-1)^2$$

cps: $x = 0, 1$

(c) (2pts) Fill in the blanks: $f(x)$ is decreasing on the interval ($-\infty$, 0). **Note:** $\pm\infty$ are acceptable answers.

2

$-$ | $+$ | $+$ sign of f'
0   1

(d) (2pts) Which critical point is a local minimum?

2

$x = 0$ is a local min.

(e) (3pts) Find $f''(x)$.

3

$$f''(x) = 12x^2 - 16x + 4 = 4(3x^2 - 4x + 1) = 4(3x-1)(x-1)$$

(f) (3pts) Find the inflection point(s) of f.

3

$+$ | $-$ | $+$ sign of f''
$1/3$ 1

$x = 1/3, 1$ are inflection pts.
(or $(1/3, f(1/3))$ & $(1, f(1))$).

(g) (2pts) Fill in the blanks: $f(x)$ is concave down on the interval ($1/3$, 1). **Note:** $\pm\infty$ are acceptable answers.

2

7. (6pts) Evaluate the Riemann sum for $f(x) = \frac{2x}{1+x^2}$ on the interval $[-1, 5]$ using the partition of 3 subintervals of equal length with the sample points being the left-endpoints of each subinterval.

6

-1 1 3 5

$\Delta x = 2$ ✓

$$\Delta x\left(f(-1) + f(1) + f(3)\right) = 2\left(\frac{-2}{2} + \frac{2}{2} + \frac{6}{10}\right) = \frac{12}{10} = \frac{6}{5}$$

24

8. (9pts) Evaluate the following expressions related the the Fundamental Theorem of Calculus.

3 (a) (3pts) If $F(t)$ is an antiderivative of $f(t) = t^3 \sin t$, then $F'(t) =$ $t^3 \sin t$

(b) (3pts) $\frac{d}{dx}\int_1^x (t^3 \sin t)\, dt$

3

$$= \frac{d}{dx}\left(F(x) - F(1)\right) = F'(x) = x^3 \sin x$$

(c) (3pts) $\frac{d}{dx}\int_{x^2}^x (t^3 \sin t)\, dt$

3 $$= \frac{d}{dx}\left(F(x) - F(x^2)\right) = F'(x) - F'(x^2)(2x)$$

$$= x^3 \sin x - \left(x^6 \sin(x^2)\right)(2x) = x^3 \sin x - 2x^7 \sin(x^2)$$

9. (10pts) Find the following indefinite integrals. Remember $+C$!

(a) (5pts) $\int (4x^3 - x^2 + 7x + 5)\, dx$

$$= x^4 - \frac{1}{3}x^3 + \frac{7}{2}x^2 + 5x + C$$

-1 for missing +C

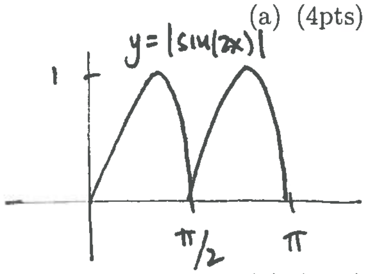
(b) (5pts) $\int x \sin(x^2)\, dx$

$u = x^2 \Rightarrow du = 2x\, dx$

$$= \frac{1}{2}\int \sin u\, du = -\frac{1}{2}\cos u + C = -\frac{1}{2}\cos(x^2) + C$$

10. (8pts) Use symmetry of the integrand to find the following definite integrals:

(a) (4pts) $\int_0^\pi |\sin(2x)|\, dx$ $= 2\int_0^{\pi/2} \sin(2x)\, dx$



$$= 2\left(-\frac{1}{2}\cos(2x)\right)\Big|_0^{\pi/2} = -\left[\cos(\pi) - \cos(0)\right] = 2$$

(b) (4pts) $\int_{-1}^1 \frac{x^3}{1+x^2}\, dx$

$f(x) = \frac{x^3}{1+x^2}$ is odd, since $f(-x) = \frac{(-x)^3}{1+(-x)^2} = -\frac{x^3}{1+x^2} = -f(x)$.

Since $[-1,1]$ is symmetric about $x=0$,

$$\int_{-1}^1 \frac{x^3}{1+x^2}\, dx = 0.$$

11. (26pts) Consider the region R in the first quadrant bounded by $y = x$ and $y = 3x - x^2$. Figure A below is a rough sketch of the region R.

(a) (5pts) Find the coordinates (x, y) of the point of intersection of the curves $y = x$ and $y = 3x - x^2$ labeled P in the sketch below.

5

$$x = 3x - x^2 \Longrightarrow 0 = 2x - x^2 = x(2-x)$$

$$x = 0, 2.$$

when $x = 2$, $y = x = 2$. So $P = (2,2)$

(b) (5pts) Find the area of the region R.

5

$$A = \int_0^2 (3x - x^2 - x)\,dx = \int_0^2 (2x - x^2)\,dx = \left(x^2 - \frac{1}{3}x^3\right)\Big|_0^2$$

$$= 4 - \frac{8}{3} = \frac{4}{3}$$

(c) (8pts) Find the volume of the solid obtained by rotating the region R around the x-axis.

8

$$V = \int_0^2 \pi\left[(3x - x^2)^2 - x^2\right]dx = \pi \int_0^2 (9x^2 - 6x^3 + x^4 - x^2)\,dx$$

$$= \pi \int_0^2 (8x^2 - 6x^3 + x^4)\,dx = \pi\left(\frac{8}{3}x^3 - \frac{3}{2}x^4 + \frac{x^5}{5}\right)\Big|_0^2$$

$$= \pi\left(\frac{64}{3} - \frac{48}{2} + \frac{32}{5}\right) = \frac{56\pi}{15}$$

(d) (8pts) Find the volume of the solid obtained by rotating the region R around the y-axis.

$$V = \int_0^2 2\pi x(2x - x^2)\,dx = 2\pi \int_0^2 (2x^2 - x^3)\,dx$$

8

$$= 2\pi\left(\frac{2}{3}x^3 - \frac{x^4}{4}\right)\Big|_0^2$$

$$= 2\pi\left(\frac{16}{3} - \frac{16}{4}\right) = \frac{8\pi}{3}$$

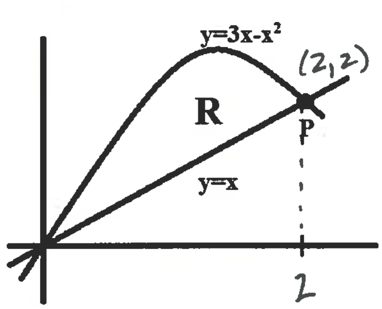


Figure A

26

12. (8pts) Find the arc length of the curve defined by the parametric equations

$$x = 3t^2 + 1 \qquad y = 2t^3 - 1$$

for $1 \le t \le 4$.

$\frac{dx}{dt} = 6t$ 2 $\qquad \frac{dy}{dt} = 6t^2$ 2

8

$$L = \int_1^4 \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\,dt = \int_1^4 \sqrt{36t^2 + 36t^4}\,dt = \int_1^4 6t\sqrt{1+t^2}\,dt.$$

2

$u = 1 + t^2$

$du = 2t\,dt$

$$= 3\int_2^{17} u^{1/2}\,du = 3\left(\frac{2}{3}u^{3/2}\right|_2^{17} = \boxed{2\left(17^{3/2} - 2^{3/2}\right)}$$

13. (9pts) A 2 meter-long rod is placed on the x-axis with one end at $x = 0$ and the other end at $x = 2$. Suppose in this position the density of the rod is given by $\rho(x) = 3x^2$ kg/m. Find the total mass of the rod, the moment of the rod with respect to the origin, and the location of the center of mass of the rod.

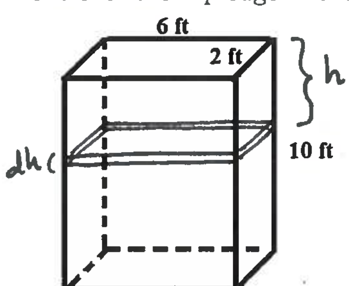
9

3 Total mass, $m=$ $\int_0^2 \rho(x)\,dx = \int_0^2 3x^2\,dx = \left(x^3\right|_0^2 = 8$ kg.

3 Moment, $M=$ $\int_0^2 x\rho(x)\,dx = \int_0^2 3x^3\,dx = \left(\frac{3}{4}x^4\right|_0^2 = 12$ kg·m

3 Center of mass, $\bar{x}=$ $\frac{M}{m} = \frac{12}{8} = 1.5$ m

14. (10pts) A rectangular tank is filled with an oil which has a density of 80 lbs/ft^3. The tank is 10 feet tall, 6 feet long, and 2 feet wide. Use an integral to determine much work is required to pump the oil out over the top edge of the tank? Give your answer in foot-pounds.



Volume of slice at location h: $(2)(6)\,dh = 12\,dh$ ft^3

Weight of slice at location h: $(80)(12)\,dh = 960\,dh$ lbs

Distance lifted: h ft.

10

$$W = \int_0^{10} 960h\,dh = 960\left(\frac{h^2}{2}\right|_0^{10} = 960(50)$$

$$= 48{,}000 \text{ ft}\cdot\text{lbs.}$$

another common setup:

$$W = \int_0^{10} 960(10-h)\,dh$$

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