1210-90 Final Exam

Spring 2013

Name KEY

Instructions. Show all work and include appropriate explanations when necessary. Answers unaccompanied by work may not receive credit. Please try to do all work in the space provided and circle your final answers.

1. (16pts) Find the following derivatives. Show your work below and circle your final answer.

   (a) (4pts) $D_x(3x^5 - x^3 + 2x)$

   4

   $$= 15x^4 - 3x^2 + 2$$

   (b) (4pts) $D_x(\frac{x}{x^2+1})$

   4

   $$= \frac{(x^2+1)(1) - (x)(2x)}{(x^2+1)^2} = \frac{1-x^2}{(x^2+1)^2}$$

   (c) (4pts) $D_x(x^2 \cos(3x))$

   4

   $$= 2x\cos(3x) - 3x^2\sin(3x)$$

   (d) (4pts) $D_x((3x + \frac{5}{x})^6)$

   4

   $$= 6\left(3x + \frac{5}{x}\right)^5 \left(3 - \frac{5}{x^2}\right)$$

2. (8pts) Compute the following limits. Answers may be ±∞ or 'DNE'. Show your work below and circle your final answer.

   (a) (4pts) $\lim\limits_{x \to 0} \frac{x^2 - x}{x} = \lim\limits_{x \to 0} \frac{x(x-1)}{x} = \lim\limits_{x \to 0} x - 1 = \boxed{-1}$

   4

   2 pts for correct ans w/o work

   (b) (4pts) $\lim\limits_{x \to 0} \frac{x^2 - 1}{x}$

   4

   $$\lim_{x \to 0^+} \frac{x^2-1}{x} = -\infty$$

   $$\lim_{x \to 0^-} \frac{x^2-1}{x} = +\infty$$

   > limit does not exist

24

3. (10pts) Consider the function $f(x) = \sqrt{1+x}$. $= (1+x)^{1/2}$

(a) (6pts) Find the equation of the tangent line to $f(x)$ at $x = 0$.

6

$f(0) = \sqrt{1+0} = \sqrt{1} = \boxed{1.}$

$f'(x) = \frac{1}{2}(1+x)^{-1/2} \Rightarrow f'(0) = \frac{1}{2}(1)^{-1/2} = \boxed{\frac{1}{2}.}$

$\boxed{y = f(0) + f'(0)(x-0)}$

$\boxed{y = 1 + \frac{1}{2}x}$

(b) (4pts) Use part (a) above to estimate the value of $\sqrt{1.1}$.

4

$\sqrt{1.1} = f(.1) \approx 1 + \frac{1}{2}(.1) = \boxed{1.05}$

4. (6pts) Use implicit differentiation to find an expression for $\frac{dy}{dx}$ in terms of x and y if

$$xy^2 - y = x^3 + 1$$

6

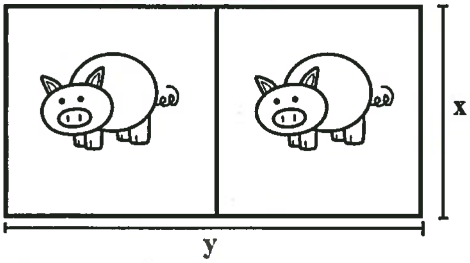
Diff both sides w.r.t x, treat y as a fcn of x

$y^2 + 2xy\frac{dy}{dx} - \frac{dy}{dx} = 3x^2$

$\frac{dy}{dx}(2xy-1) = 3x^2 - y^2 \Rightarrow \boxed{\frac{dy}{dx} = \frac{3x^2-y^2}{2xy-1}}$

5. (10pts) A farmer has 72 feet of fence and wants to build two pens for his pigs by building one large rectangular pen and splitting it down the middle with a length of fence. What dimensions (labeled x and y in the picture below) should the farmer use to maximize the area enclosed? **Note:** You must use calculus to get credit!

10



Want to maximize

$\boxed{A = xy}$

subject to the constraint

$\boxed{72 = 3x + 2y.}$

Since $72 = 3x + 2y \Rightarrow y = 36 - \frac{3}{2}x$

So

$A = xy = x(36 - \frac{3}{2}x) = 36x - \frac{3}{2}x^2$

Find cps

$0 = A'(x) = 36 - 3x \Rightarrow x = 12.$

$+$ | $-$ Sign of $A'(x)$

12

So $x = 12$ is a max.

missing -1

$\boxed{\begin{matrix} x = 12 \\ y = 18 \end{matrix}}$

26

6. (24pts) Consider the function

$$f(x) = x^3 - 6x^2 - 15x - 2$$

(a) (3pts) Find $f'(x)$.

3

$3x^2 - 12x - 15$

(b) (3pts) Find the critical point(s) of $f(x)$.

3

$0 = 3x^2 - 12x - 15 = 3(x^2 - 4x - 5) = 3(x-5)(x+1)$

cps: $x = 5, -1$

(c) (2pts) Fill in the blanks: $f(x)$ is decreasing on the interval (-1 , 5). **Note:** ±∞ are acceptable answers.

2

sign of f'

(d) (2pts) Which critical point is a local minimum?

2

$x=5$ is a local min.

(e) (3pts) Find $f''(x)$.

3

$6x - 12$

(f) (3pts) Find the inflection point(s) of $f(x)$.

3

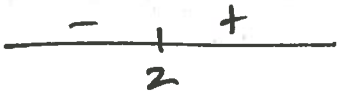
$0 = f''(x) = 6x - 12 \Rightarrow x = 2.$

So $x=2$ is an inflection pt.

(or $(2, -48)$)

(g) (2pts) Fill in the blanks: $f(x)$ is concave up on the interval (2 , ∞). **Note:** ±∞ are acceptable answers.

2



sign of f''

(h) (6pts) Find the extreme values of $f(x)$ on the interval $[-2, 1]$. Max= 6 Min= -22

6

$x=-1$ is only cp. in $[-2,1]$.

$f(-1) = 6$

$f(-2) = -4$

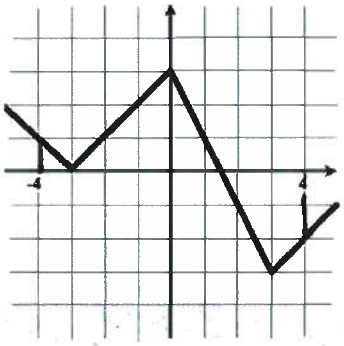
$f(1) = -22$

24

30

7. (6pts) Use the graph of $y = f(x)$ below to compute $\int_{-4}^{4} f(x)\,dx =$ $5/2$

6



$A = \frac{1}{2} + \frac{9}{2} - \frac{5}{2} = \frac{5}{2}$

8. (8pts) Find the following antiderivatives. **Remember:** $+C$!

missing +C
−2 total

(a) (4pts) $\int (4x^3 - 6x + 2)\,dx$

4

$\boxed{x^4 - 3x^2 + 2x + C}$

(b) (4pts) $\int (x+1)(x^2+2x)^5\,dx$ $= \frac{1}{2}\int u^5\,du = \frac{1}{12}u^6 + c$

4

$u = x^2 + 2x$

$du = 2x + 2\,dx$

$\boxed{= \frac{1}{12}(x^2+2x)^6 + c}$

9. (8pts) Evaluate the following definite integrals.

(a) (4pts) $\int_0^1 (x^2 - 3x + 1)\,dx$

4

$= \left(\frac{1}{3}x^3 - \frac{3}{2}x^2 + x\right|_0^1 = \frac{1}{3} - \frac{3}{2} + 1 = \frac{2}{6} - \frac{9}{6} + \frac{6}{6} = \boxed{\frac{-1}{6}}$

(b) (4pts) $\int_0^{\sqrt{\pi}} x\sin(x^2)\,dx$ $= \frac{1}{2}\int_0^{\pi} \sin u\,du = \frac{1}{2}\left(-\cos u\right|_0^{\pi}$

4

$u = x^2$

$du = 2x\,dx$

$= \frac{1}{2}(1+1) = \boxed{1}$

10. (6pts) Evaluate the Riemann sum for $f(x) = x^2 + 3$ on the interval $[-1, 3]$ using the partition of 4 subintervals of equal length with the sample points being the right-endpoints of each subinterval.

$\Delta x = \frac{4}{4} = 1.$

6

$f(0)(1) + f(1)(1) + f(2)(1) + f(3)(1)$

$= 3 + 4 + 7 + 12 = \boxed{26}$

28

11. (26pts) Consider the region R in the first quadrant bounded by $y = 3 - x^2$, $y = 2x$, and the y-axis. Figure A below is a rough sketch of the region R.

(a) (5pts) Find the coordinates (x, y) of the point of intersection of the curves $y = 3 - x^2$ and $y = 2x$ labeled P in the sketch below.

$$3-x^2=2x \Rightarrow x^2+2x-3=0$$

$$(x+3)(x-1)=0$$

$$\boxed{P=(1,2)}$$

5

(b) (5pts) Find the area of the region R.

5

$$A=\int_0^1((3-x^2)-(2x))\,dx = \int_0^1(3-x^2-2x)\,dx$$

3

$$=\left(3x-\tfrac{1}{3}x^3-x^2\right|_0^1 = 3-\tfrac{1}{3}-1 = \boxed{\tfrac{5}{3}}$$

(c) (8pts) Find the volume of the solid obtained by rotating the region R around the x-axis.

$$V=\int_0^1 \pi\left[(3-x^2)^2-(2x)^2\right]dx = \pi\int_0^1(9-6x^2+x^4-4x^2)\,dx$$

8

4

$$=\pi\int_0^1(9-10x^2+x^4)\,dx = \pi\left(9x-\tfrac{10}{3}x^3+\tfrac{1}{5}x^5\right|_0^1$$

$$=\pi\left(9-\tfrac{10}{3}+\tfrac{1}{5}\right) = \boxed{\frac{88\pi}{15}}$$

(d) (8pts) Find the volume of the solid obtained by rotating the region R around the y-axis.

4

$$V=\int_0^1 2\pi x(3-x^2-2x)\,dx$$

8

$$=2\pi\int_0^1(3x-x^3-2x^2)\,dx$$

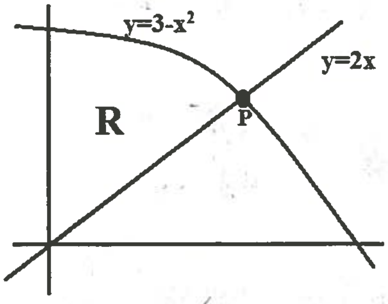


Figure A

$$=2\pi\left(\tfrac{3}{2}x^2-\tfrac{1}{4}x^4-\tfrac{2}{3}x^3\right|_0^1$$

$$=2\pi\left(\tfrac{3}{2}-\tfrac{1}{4}-\tfrac{2}{3}\right)$$

$$=\boxed{\frac{7\pi}{6}}$$

26

12. (10pts) Find the arc length of the curve defined by the parametric equations

$$x = \frac{2}{3}t^3 \qquad y = t^2$$

for $0 \le t \le 1$.

10

$$x'(t) = 2t^2 \qquad y'(t) = 2t.$$

$$L = \int_0^1 \sqrt{(x'(t))^2 + (y'(t))^2}\,dt = \int_0^1 \sqrt{4t^4 + 4t^2}\,dt$$

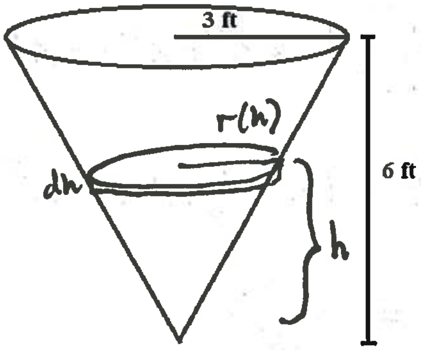
$$= 2\int_0^1 t\sqrt{t^2+1}\,dt = \int_1^2 u^{1/2}\,du = \left(\frac{2}{3}u^{3/2}\right|_1^2$$

$u = t^2+1$

$du = 2t\,dt$

$$= \frac{2}{3}\left(\sqrt{8} - 1\right)$$

13. (12pts) A tank in the shape of an inverted circular cone is filled with water. The tank is 6 feet tall and has a radius of 3 feet at its rim. How much work is required to pump the water over the top edge of the tank? To simplify your calculations, use the symbol ρ to denote the density of water in lbs/ft^3.



12

$$r(h) = \frac{1}{2}h.$$

$$A(h) = \pi\, r(h)^2 = \frac{\pi}{4}h^2.$$

Volume of "slice" of water at height h

$$V(h) = A(h)\,dh = \frac{\pi}{4}h^2\,dh \quad ft^3$$

Weight of "slice" of water at height h

$$W(h) = \rho V(h) = \rho\frac{\pi}{4}h^2\,dh. \quad lbs$$

Must lift it $(6-h)$ feet.

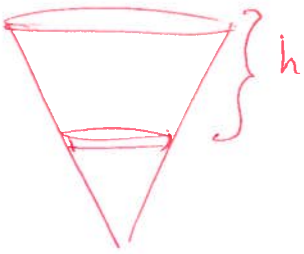
$$W = \int_0^6 \rho\frac{\pi}{4}h^2(6-h)\,dh = \frac{\pi\rho}{4}\int_0^6 (6h^2 - h^3)\,dh$$

$$= \frac{\pi\rho}{4}\left(2h^3 - \frac{h^4}{4}\right|_0^6$$

$$= \frac{\pi\rho}{4}\left(\overset{108}{\text{[illegible]}}\right) = \overset{27\pi\rho}{\text{[illegible]}} \quad ft\cdot lbs$$

or

$$r(h) = \frac{1}{2}(6-h)$$



$$\int_0^6 \rho\frac{\pi}{4}(6-h)^2 h\,dh$$