1210-90 Final Exam

Spring 2014

Name KEY

Instructions. Show all work and include appropriate explanations when necessary. Answers unaccompanied by work may not receive credit. Please try to do all work in the space provided and circle your final answers.

1. (16pts) Find the following derivatives. Show your work below and circle your final answer.

   (a) (4pts) $D_x(x^3 - 9x + 6)$

   4 $= 3x^2 - 9$

   (b) (4pts) $D_x(x^5 \sin x)$

   4 $= x^5 \cos x + 5x^4 \sin x$

   (c) (4pts) $D_x(\frac{\cos x}{x+2})$

   4 $= \dfrac{(x+2)(-\sin x) - (\cos x)(1)}{(x+2)^2} = \dfrac{-x\sin x - 2\sin x - \cos x}{(x+2)^2}$

   (d) (4pts) $D_x((x - \frac{3}{x})^4)$

   4 $= 4(x - 3x^{-1})^3 (1 + 3x^{-2})$

2. (8pts) Compute the following limits. Answers may be values, ±∞, or 'DNE'. Show your work below and circle your final answer.

   (a) (4pts) $\lim\limits_{x\to -1^+} \dfrac{x+1}{x^2 - 3x - 4} = \lim\limits_{x \to -1^+} \dfrac{x+1}{(x+1)(x-4)} = \lim\limits_{x\to -1^+} \dfrac{1}{x-4} = -\dfrac{1}{5}$

   4

   (b) (4pts) $\lim\limits_{x\to 4^-} \dfrac{x+1}{x^2 - 3x - 4} = \lim\limits_{x\to 4^-} \dfrac{x+1}{(x+1)(x-4)} = \lim\limits_{x\to 4^-} \dfrac{1}{x-4} = -\infty$

   +2 DNE

   4

   since when x is near 4, x-4 is near 0 and when x is less than 4, x-4 < 0.

24

3. (8pts) Consider the function $f(x) = x^{2/3}$.

(a) (5pts) Find the equation of the tangent line to the graph of $y = x^{2/3}$ at the point $(8, 4)$.

5

$f(x) = x^{2/3} \Rightarrow f(8) = 4$

$f'(x) = \frac{2}{3}x^{-1/3} \Rightarrow f'(8) = \frac{2}{3}\left(\frac{1}{2}\right) = \frac{1}{3}$

$y = f(a) + f'(a)(x-a) \Rightarrow y = 4 + \frac{1}{3}(x-8) = \frac{1}{3}x + \frac{4}{3}$

(b) (3pts) Use part (a) above to estimate $(8.1)^{2/3}$.

3

$(8.1)^{2/3} = f(8.1) \approx 4 + \frac{1}{3}(8.1-8) = 4 + \frac{1}{3}(.1) = 4 + \frac{1}{30}$.

since $8.1 \approx 8$

4. (6pts) Find the slope of the tangent line to the following curve at the point $(-1, 1)$

$$xy^5 + x^2 + 2 = 2y$$

Dift. both sides w.r.t x:

6

3 $\quad y^5 + 5xy^4\frac{dy}{dx} + 2x = 2\frac{dy}{dx}$

Plug in $(x,y) = (-1,1)$ and solve for $\frac{dy}{dx}$:

3 $\quad 1 - 5\frac{dy}{dx} - 2 = 2\frac{dy}{dx} \Rightarrow -1 = 7\frac{dy}{dx} \Rightarrow \frac{dy}{dx} = -\frac{1}{7}$

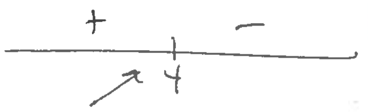
5. (8pts) A movie theater finds that the number of tickets it sells per show when it prices each ticket at x dollars is given by the formula $N(x) = 80 - 10x$. **Find the maximum amount of revenue** the theater can earn per show. **Note:** revenue=cost per ticket×number of tickets sold$= xN(x)$.

revenue $= R(x) = xN(x) = 80x - 10x^2$.

Find critical pts:

$0 = R'(x) = 80 - 20x \Rightarrow x = 4$ is a cp.

8



+1

sign of $R'(x)$

So $x=4$ is a maximum.

So max revenue $= R(4) = 80(4) - 10(16)$

$= 320 - 160$

$= 160$ +1

22

6. (27pts) Consider the function

$$f(x) = \frac{1}{3+x^2}$$

(a) (4pts) Find $f'(x)$.

4

$$f'(x) = \frac{(3+x^2)(0) - (1)(2x)}{(3+x^2)^2} = \frac{-2x}{(3+x^2)^2}$$

(b) (4pts) Find the critical point(s) of f.

4

$$0 = f'(x) = \frac{-2x}{(3+x^2)^2} \Longrightarrow x = 0 \text{ is cp.}$$

(c) (2pts) Fill in the blanks: $f(x)$ is decreasing on the interval (0 , $+\infty$). **Note:** $\pm\infty$ are acceptable answers.

2

$+ \quad | \quad -$ sign of f'
$\quad\ \ 0$

(d) (2pts) Classify each critical point you found in part (b) as a local minimum, a local maximum, or neither.

2

$x = 0$ is a local max (First Derivative Test)

(e) (4pts) Find $f''(x)$.

4

$$f''(x) = \frac{(3+x^2)^2(-2) - (-2x)(2)(3+x^2)(2x)}{(3+x^2)^4} = \frac{(3+x^2)(-2) + (2x)(2)(2x)}{(3+x^2)^3} = \frac{-6+6x^2}{(3+x^2)^3}$$

(f) (4pts) Find the inflection point(s) of f.

4

$$0 = f''(x) = \frac{-6+6x^2}{(3+x^2)^3} \Longrightarrow 0 = -6 + 6x^2 \Longrightarrow x^2 = 1 \Longrightarrow x = \pm 1.$$

−1

$+ \quad | \quad - \quad | \quad +$ sign of f''
$\quad\ -1 \qquad\quad 1$

Both $x = \pm 1$ are inflection pts

(g) (2pts) Fill in the blanks: $f(x)$ is concave down on the interval (−1 , 1). **Note:** $\pm\infty$ are acceptable answers.

2

(h) (5pts) **Mean Value Theorem for Derivatives**: If f is continuous on $[a, b]$ and differentiable on (a, b), then there is at least one number c in (a, b) such that

5

$$\frac{f(b) - f(a)}{b - a} = f'(c).$$

Find the value of c given by the Mean Value Theorem for Derivatives for $f(x) = \frac{1}{3+x^2}$ on the interval $[-1, 1]$.

$$f(-1) = f(1) = \frac{1}{4}.$$

$$\frac{f(1) - f(-1)}{1 - (-1)} = \frac{\frac{1}{4} - \frac{1}{4}}{2} = 0 = f'(c) = \frac{-2c}{(3+c^2)^2}$$

$$\Longrightarrow c = 0$$

27

7. (12pts) Use both versions of the Fundamental Theorem of Calculus to evaluate the following:

(a) (4pts) $\int_0^2 (x^3 - x)\, dx$ $= \left(\frac{x^4}{4} - \frac{x^2}{2}\right)\Big|_0^2 = \left[\frac{2^4}{4} - \frac{2^2}{2}\right] - \left[\frac{0^4}{4} - \frac{0^2}{2}\right]$

$= \left[\frac{16}{4} - \frac{4}{2}\right] - [0 - 0] = \boxed{2}$

4

(b) (4pts) $\frac{d}{dx} \int_0^x \frac{1}{t^2+1}\, dt$

Note: You do not need to know an antiderivative of $\frac{1}{t^2+1}$ to answer this question or (c) below.

$\boxed{= \frac{1}{x^2+1}}$ (Using 1st FTC)

4

(c) (4pts) $\frac{d}{dx} \int_1^{x^3} \frac{1}{t^2+1}\, dt$

Suppose $F(t)$ is an antiderivative of $\frac{1}{t^2+1}$ (So $F'(t) = \frac{1}{t^2+1}$)

4

$\frac{d}{dx}\left(\int_1^{x^3} \frac{1}{t^2+1}\, dt\right) = \frac{d}{dx}\left(F(x^3) - F(1)\right) = F'(x^3)(3x^2) \boxed{= \frac{3x^2}{x^6+1}}$

8. (12pts) Find the following antiderivatives. **Remember:** $+C$!

(a) (4pts) $\int (3x^2 + 6x + 1)\, dx$

4

$\boxed{= x^3 + 3x^2 + x + C}$

(b) (4pts) $\int \cos(3x)\, dx$

4

$\boxed{= \frac{1}{3}\sin(3x) + C}$

(c) (4pts) $\int 5\sin^4(x)\cos(x)\, dx$

Since $\frac{d}{dx}\sin x = \cos x$, using substitution or generalized power rule:

4

$= \boxed{\sin^5 x + C}$

24

9. (21pts) Consider the region R in the first quadrant bounded by $y = -x^2 + 3x$, the x-axis, and $x = 2$. Figure A below is a rough sketch of the region R.

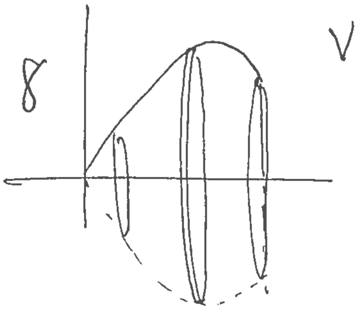
(a) (5pts) Find the area of the region R.

5

$$= \int_0^2 (-x^2+3x)\,dx = \left(-\frac{x^3}{3} + \frac{3x^2}{2}\right|_0^2 = \frac{-8}{3} + \frac{12}{2} = \frac{10}{3}$$

(b) (8pts) Find the volume of the solid obtained by rotating the region R around the x-axis.

8



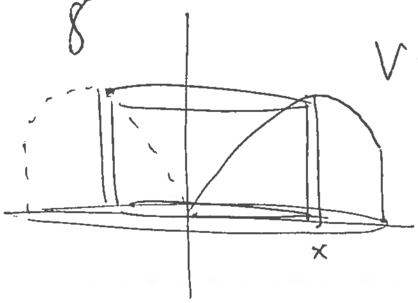
$$V = \int_0^2 \pi\left(-x^2+3x\right)^2 dx = \pi\int_0^2 (x^4 - 6x^3 + 9x^2)\,dx$$

$$= \pi\left(\frac{x^5}{5} - \frac{6x^4}{4} + 3x^3\right|_0^2 = \pi\left(\frac{32}{5} - \frac{96}{4} + 24\right)$$

$$= \pi\left(\frac{32}{5}\right)$$

(c) (8pts) Find the volume of the solid obtained by rotating the region R around the y-axis.

8



$$V = \int_0^2 2\pi x\left(-x^2+3x\right)dx = 2\pi\int_0^2 (-x^3+3x^2)\,dx$$

$$= 2\overset{\pi}{\cancel{\pi}}\left(-\frac{x^4}{4} + x^3\right|_0^2 = 2\pi\left(-\frac{16}{4} + 8\right) = 8\pi$$

10. (8pts) Consider the region S bounded by the curves $y = 2x^2$, the y-axis, and $y = 2$ sketched in Figure B below. Each integral below is the volume of a solid obtained by rotating S around a particular axis. Match the correct axis with the expression for volume by writing the appropriate letter in the blank provided. Each answer is used exactly once.

8

D $\int_0^1 \pi((3-2x^2)^2 - 1)\,dx$	A. x-axis
A $\int_0^1 \pi(4 - 4x^4)\,dx$	B. y-axis
C $\int_0^1 2\pi(x+1)(2-2x^2)\,dx$	C. $x = -1$
B $\int_0^2 \pi(\frac{y}{2})\,dy$	D. $y = 3$

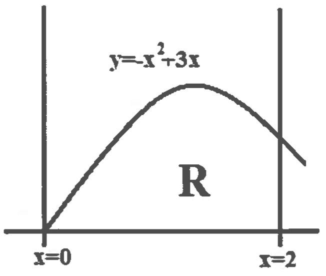


Figure A

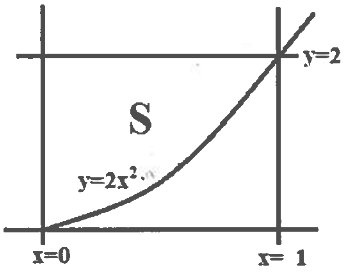


Figure B

29

11. (6pts) Find the arc length of the curve $y = \frac{4}{3}x^{3/2}$ between $x = 1$ and $x = 4$.

$$f(x) = \frac{4}{3}x^{3/2} \Longrightarrow f'(x) = \left(\frac{4}{3}\right)\left(\frac{3}{2}\right)x^{1/2} = 2x^{1/2}$$

6

$$L = \int_1^4 \sqrt{1+f'(x)^2}\,dx = \int_1^4 \sqrt{1+4x}\,dx = \frac{1}{4}\int_5^{17} u^{1/2}\,du = \frac{1}{4}\left(\frac{2}{3}u^{3/2}\right|_5^{17}$$

$u = 1+4x$

$du = 4\,dx$

$$= \frac{1}{6}\left(17^{3/2} - 5^{3/2}\right)$$

12. (6pts) A .2 meter-long spring requires a force of 12 Newtons to compress it .08 meters. How much work is required to compress the spring from it natural length to a length of .1 meters? **Remember:** springs satisfy Hooke's Law: $F = kx$, where k is the spring constant.

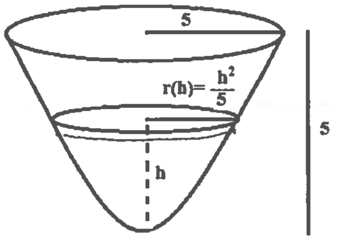
$$12 = k(.08) \Longrightarrow k = \frac{12}{.08} = 150 \text{ N/m.}$$

6

$$W = \int_0^{.1} 150x\,dx = 150\left(\frac{x^2}{2}\right|_0^{.1} = 150\left(\frac{1}{200}\right) = \frac{3}{4}\text{ J.}$$

Should read "has a radius of 5 ft at its rim".

13. (10pts) A tank in the shape of a paraboloid is filled with water which has a density of 60 lbs/ft^3. The tank is 5 feet tall and 5 feet wide at its rim. The radius of the tank is given by $r(h) = \frac{h^2}{5}$, where h denotes the distance up from the bottom of the tank (see the picture below). Use an integral to determine much work is required to pump the water out over the top edge of the tank. Give your answer in foot-pounds.



10

Volume of slice of H_2O at location h: $\pi r(h)^2\,dh$

$$= \pi\left(\frac{h^2}{5}\right)^2 dh$$

$$= \frac{\pi}{25}h^4\,dh.\ \text{ft}^3$$

Weight of slice at location h: $60\left(\frac{\pi}{25}\right)h^4\,dh$ lbs.

distance slice lifted: $5-h$ ft.

$$W = \int_0^5 60\left(\frac{\pi}{25}\right)h^4(5-h)\,dh = \frac{12\pi}{5}\int_0^5 (5h^4 - h^5)\,dh$$

$$= \frac{12\pi}{5}\left(h^5 - \frac{h^6}{6}\right|_0^5 = \frac{12\pi}{5}\left(5^5 - \frac{5^6}{6}\right)$$

$$= 5^4(12\pi)\left(1 - \frac{5}{6}\right)$$

$$= 5^4(12\pi)\left(\frac{1}{6}\right)$$

22

$$= 2\pi(5^4) = 2\pi(625) = 1250\pi \text{ ft}\cdot\text{lb}$$