Calculus I 1210-90 Final Exam

Summer 2014

Name KEY

Instructions. Show all work and include appropriate explanations when necessary. Answers unaccompanied by work may not receive credit. Please try to do all work in the space provided and circle your final answers.

1. (16pts) Find the following derivatives. Show your work below and circle your final answer.

   (a) (4pts) $D_x(x^3 - 9x + 6)$

4

$$= 3x^2 - 9$$

   (b) (4pts) $D_x(\sin x \cos x)$

4

$$= (\cos x)(\cos x) + (\sin x)(-\sin x) = \cos^2 x - \sin^2 x$$

   (c) (4pts) $D_x(\frac{2x+9}{x^2+3})$

4

$$= \frac{(x^2+3)(2) - (2x+9)(2x)}{(x^2+3)^2} = \frac{2x^2+6-4x^2-18x}{(x^2+3)^2} = \frac{-2x^2-12x}{(x^2+3)^2}$$

   (d) (4pts) $D_x(\sin(3x^4 - x))$

4

$$= \cos(3x^4 - x)(12x^3 - 1)$$

2. (8pts) Compute the following limits. Answers may be values, $\pm\infty$, or 'DNE'. Show your work below and circle your final answer.

   (a) (4pts) $\lim_{x \to 0} \frac{1}{|x|}$ As $x \to 0$, $|x| \to 0$ and is always positive. So

4

$$\lim_{x \to 0} \frac{1}{|x|} = +\infty$$

DNE +2

   (b) (4pts) $\lim_{x \to 0} \frac{x}{|x|}$

4

$$\lim_{x \to 0^+} \frac{x}{|x|} = \lim_{x \to 0^+} \frac{x}{x} = 1$$

$$\lim_{x \to 0^-} \frac{x}{|x|} = \lim_{x \to 0^-} \frac{x}{-x} = -1$$

So $\lim_{x \to 0} \frac{x}{|x|}$ DNE

3. (6pts) Find the equation of the tangent line to the graph of $y = (x-1)^9$ at the point $(2, 1)$.

$$f(x) = (x-1)^9 \Rightarrow f(2) = 1.$$

$$f'(x) = 9(x-1)^8 \Rightarrow f(2) = 9.$$

$$y = f(a) + f'(a)(x-a)$$

$$y = 1 + 9(x-2) = 9x - 17$$

6

4. (6pts) Use the **definition of the derivative** to compute the derivative of the function $f(x) = x^2 - 6$; that is, compute

$$\lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

$$\lim_{h \to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h \to 0} \frac{((x+h)^2 - 6) - (x^2 - 6)}{h}$$

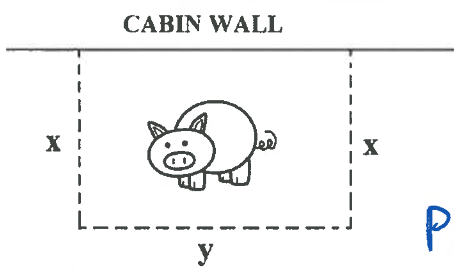
$$= \lim_{h \to 0} \frac{x^2 + 2xh + h^2 - \cancel{6} - \cancel{x^2} + \cancel{6}}{h}$$

$$= \lim_{h \to 0} \frac{2xh + h^2}{h} = \lim_{h \to 0} \frac{h(2x+h)}{h} = \lim_{h \to 0} 2x + h$$

$$= 2x$$

6

5. (10pts) Billy Joe wants to build a rectangular pig pen of area 8 square meters up against the side of his cabin; only three sides of the enclosure are fence since the side of his cabin will form a fourth wall (see picture below). What dimensions (labeled x and y in the picture below) should Billy Joe make the pen to use the least amount of fence? Minimize the perimeter $P = y + 2x$ subject to the constraint $xy = 8$. **Note:** You must use calculus to get credit!!



Minimize $P = y + 2x$

Subject to $A = xy = 8. \Rightarrow y = \frac{8}{x}.$

$$P(x) = \frac{8}{x} + 2x = 8x^{-1} + 2x.$$

10

Find cps.

$$0 = P'(x) = -8x^{-2} + 2 \Rightarrow 8x^{-2} = 2 \Rightarrow x^2 = 4 \Rightarrow x = 2$$

Check if a max or min:

$$P''(x) = 16x^{-3} \Rightarrow P''(2) = 2 > 0 \Rightarrow x = 2 \text{ is local min.}$$

So $x = 2$ meters

$y = 4$ meters

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6. (20pts) Consider the function

$$f(x) = 3x^5 - 20x^3 + 8$$

(a) (3pts) Find $f'(x)$.

3 $f'(x) = 15x^4 - 60x^2$

(b) (3pts) Find $f''(x)$.

3 $f''(x) = 60x^3 - 120x$

(c) (3pts) Find the critical point(s) of f.

3 $0 = f'(x) = 15x^4 - 60x^2 = 15x^2(x^2-4) = 15x^2(x+2)(x-2)$

So $x = 0, 2, -2$ are cps

(d) (2pts) Fill in the blank by circling the correct answer below: $f(x)$ is ______ at $x = 1$. $f'(1) = -45 < 0$

2 INCREASING DECREASING

(e) (2pts) Fill in the blank by circling the correct answer below: $f(x)$ is ______ at $x = 1$.

2 CONCAVE UP CONCAVE DOWN $f''(1) = -60 < 0$

(f) (4pts) Classify each critical point you found in part (c) as a local minimum, a local maximum, or neither.

$f'(x) = 15x^2(x+2)(x-2)$ sign of f': $+$ | -2 | $-$ | 0 | $-$ | 2 | $+$

4 $x = -2$ is a local max

$x = 0$ is neither

$x = 2$ is a local min

(g) (3pts) Find the inflection point(s) of f.

$f''(x) = 60x^3 - 120x = 60x(x^2-2) = 60x(x+\sqrt{2})(x-\sqrt{2})$

3 sign of f'': $-$ | $-\sqrt{2}$ | $+$ | 0 | $-$ | $\sqrt{2}$ | $+$

$x = -\sqrt{2}, \sqrt{2}, 0$ are inflection pts

7. (6pts) An object is thrown upwards off a 100 foot tall building. Its velocity after t seconds is given by $v(t) = -32t + 32$ feet per second. How far off the ground is the object after 3 seconds?

$s(0) = 100$

6 $s(t) = \int v(t)\,dt = \int(-32t+32)\,dt = -16t^2 + 32t + C$ 3

$100 = s(0) = C$

$s(t) = -16t^2 + 32t + 100$ 2

$s(3) = -16(9) + 32(3) + 100 = 52$ ft 1

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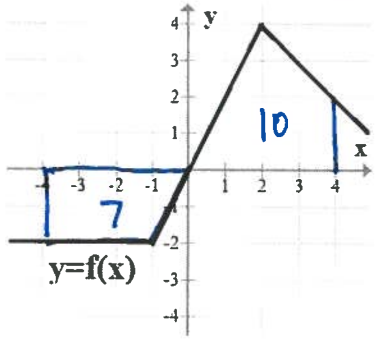
8. (6pts) Use the graph of $y = f(x)$ below to compute $\int_{-4}^{4} f(x)\,dx =$ <u>3</u>

6



$$\int_{-4}^{4} f(x)\,dx = 10 - 7 = 3$$

9. (8pts) Find the following antiderivatives. **Remember:** $+C$!

(a) (4pts) $\int (6x^2 - 5x + 2)\,dx$

missing +C = -1.

4

$$= 2x^3 - \frac{5}{2}x^2 + 2x + C$$

(b) (4pts) $\int \cos^2 x \sin x\,dx$ **Note:** $\cos^2 x$ is the same as $(\cos x)^2$.

4

$$= -\frac{1}{3}\cos^3 x + C$$

10. (8pts) Evaluate the following definite integrals using the Fundamental Theorem of Calculus.

(a) (4pts) $\int_1^2 (x^{-3})\,dx$

4

$$= \left(-\frac{1}{2}x^{-2}\right)\Big|_1^2 = -\frac{1}{2}\cdot\frac{1}{4} + \frac{1}{2}\cdot 1 = \frac{3}{8}$$

(b) (4pts) $\int_0^1 x(x^2-1)^7\,dx$

$u = x^2 - 1$, $du = 2x$ $\quad \int \frac{u^7}{2}\,du \quad \frac{u^8}{16} \quad \frac{1}{16}(x^2-1)^8\Big|_0^1$

4

$$= \left(\frac{1}{16}(x^2-1)^8\right)\Big|_0^1 = \frac{1}{16}(0) - \frac{1}{16} = -\frac{1}{16}.$$

11. (6pts) Evaluate the Riemann sum for $f(x) = x^3 - x$ on the interval $[-2, 2]$ using the partition of 4 subintervals of equal length with the sample points being the left-endpoints of each subinterval.

6

$-2 \quad -1 \quad 0 \quad 1 \quad 2$

$$\Delta x = \frac{2-(-2)}{4} = 1.$$

$$f(x) = x^3 - x$$

$$= 1\left(f(-2) + f(-1) + f(0) + f(1)\right) = -6 + 0 + 0 + 0 = -6$$

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12. (20pts) Consider the region R in the first quadrant bounded by $y=\sqrt{x^2+1}$, the x-axis, and $x=2$. Figure A below is a rough sketch of the region R.

(a) (10pts) Find the volume of the solid obtained by rotating the region R around the x-axis using the washer method.

10

$$V=\int_0^2 \pi\left(\sqrt{x^2+1}\right)^2 dx = \pi\int_0^2 (x^2+1)\,dx = \pi\left(\frac{x^3}{3}+x\right|_0^2$$

$$=\pi\left(\frac{8}{3}+2\right)=\frac{14\pi}{3}$$

$\pi r^2 dx$

(b) (10pts) Find the volume of the solid obtained by rotating the region R around the y-axis using the method of cylindrical shells.

10

$$V=\int_0^2 2\pi x\sqrt{x^2+1}\,dx = \pi\int_0^2 (2x)(x^2+1)^{1/2}dx$$

$u=x^2+1$

$du=2x\,dx$

$$=\pi\int_1^5 u^{1/2}du = \frac{2\pi}{3}\left(u^{3/2}\right|_1^5 = \frac{2\pi}{3}\left(5^{3/2}-1\right)$$

13. (8pts) Consider the region S bounded by the curves $y=2x^2$, the x-axis, and $x=1$ sketched in Figure B below. Each integral below is the volume of a solid obtained by rotating S around a particular axis. Match the correct axis with the expression for volume by writing the appropriate letter in the blank provided. Each answer is used exactly once.

6

B	$\int_0^1 2\pi x(2x^2)\,dx$	A. x-axis
D	$\int_0^1 \pi((2x^2+3)^2-3^2)\,dx$	B. y-axis
C	$\int_0^1 2\pi(x+3)(2x^2)\,dx$	C. $x=-3$
A	$\int_0^1 \pi(2x^2)^2\,dx$	D. $y=-3$

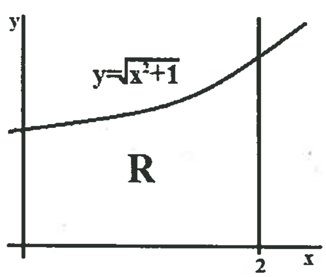

Figure A

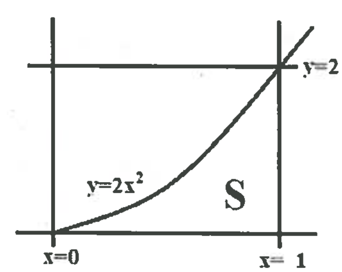

Figure B

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14. (6pts) Find the arc length of the parametric curve $x = 2t^2 + 4t + 9$, $y = \frac{3}{2}t^2 + 3t - 6$ between $t = 0$ and $t = 4$.

$$x'(t) = 4t + 4 = 4(t+1) \qquad y'(t) = 3t + 3 = 3(t+1)$$

$$L = \int_0^4 \sqrt{x'(t)^2 + y'(t)^2}\,dt = \int_0^4 \sqrt{16(t+1)^2 + 9(t+1)^2}\,dt = \int_0^4 \sqrt{25(t+1)^2}\,dt$$

$$= 5\int_0^4 (t+1)\,dt = 5\left(\frac{t^2}{2} + t\right|_0^4 = 5(8+4) = 60$$

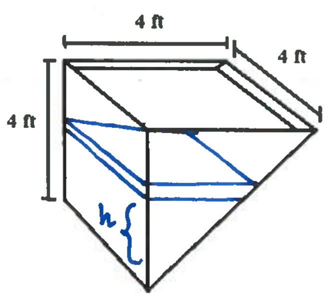
15. (6pts) A metal rod is located between $x = 0$ and $x = 3$ on the x-axis (units are centimeters). If the density of the rod at location x is given by $\rho(x) = 4 - x$ grams per centimeter, find the location of the center of mass of the rod.

$$m = \text{total mass} = \int_0^3 \rho(x)\,dx = \int_0^3 (4-x)\,dx = \left(4x - \frac{x^2}{2}\right|_0^3 = 12 - \frac{9}{2} = \frac{15}{2}\text{ g.}$$

$$M_y = \text{moment} = \int_0^3 x\rho(x)\,dx = \int_0^3 (4x - x^2)\,dx = \left(2x^2 - \frac{x^3}{3}\right|_0^3 = 18 - 9 = 9$$

$$\text{center of mass} = \frac{M_y}{m} = \frac{9}{(15/2)} = \frac{18}{15} = \frac{6}{5}.$$

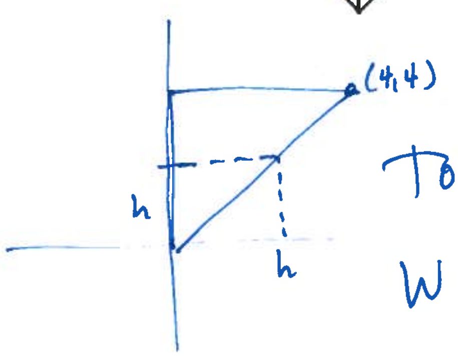
16. (10pts) A tank is 4 feet tall, 4 feet wide, and 4 feet long; when viewed from the side, the tank has the shape of a right triangle (see the picture below). This tank is filled with water which has a density of 60 lbs/ft^3. Use an integral to determine much work is required to pump the water out over the top edge of the tank. Give your answer in foot-pounds.



Volume of slice of water at height h : $4h\,dh$ ft^3

Weight of slice of water at height h : $(60)(4h)\,dh$
$= 240h\,dh$ lbs.

Distance lifted : $4 - h$



Work to lift slice at height h : $240h(4-h)\,dh$ ft·l

Total Work

$$W = \int_0^4 240h(4-h)\,dh = 240\int_0^4 (4h - h^2)\,dh =$$

$$= 240\left(2h^2 - \frac{h^3}{3}\right|_0^4 = 240\left(32 - \frac{64}{3}\right)$$

$$= 240\left(\frac{32}{3}\right) = (80)(32) = 2560 \text{ ft·lbs}$$