JW 15
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1210-90 Exam 1
Fall 2013

Name KEY

Instructions. Show all work and include appropriate explanations when space is provided. Correct answers unaccompanied by work may not receive full credit. Please circle your final answers.

1. (24pts) Compute the following limits. Be sure to show your work. **Note:** Answers can be values, $+\infty$, $-\infty$, or DNE (does not exist). An answer of DNE requires some explanation!

(a) $\lim_{x\to 0} \sqrt{x^2+4}$ $= \sqrt{0^2+4} = \sqrt{4} = 2$

4

(b) $\lim_{x\to 3} \frac{x^2-7x+12}{x-3}$ $= \lim_{x\to 3} \frac{(x-3)(x-4)}{(x-3)} = \lim_{x\to 3} x-4 = -1$

4

(c) $\lim_{x\to 0} \frac{\sin(2x)}{x}$ $= \lim_{x\to 0} \left(\frac{\sin(2x)}{2x}\right)\left(\frac{2x}{x}\right)$

$= \left(\lim_{x\to 0} \frac{\sin(2x)}{2x}\right)\left(\lim_{x\to 0} \frac{2x}{x}\right) = (1)(2) = 2$

4

(d) $\lim_{x\to 1} \frac{|x-1|}{x-1}$

If $x>1$, then $|x-1| = x-1$ and so $\lim_{x\to 1^+} \frac{|x-1|}{x-1} = \lim_{x\to 1^+} \frac{x-1}{x-1} = 1$.

If $x<1$, then $|x-1| = 1-x$ and so $\lim_{x\to 1^-} \frac{|x-1|}{x-1} = \lim_{x\to 1^-} \frac{1-x}{x-1} = -1$.

$\}$ limit DNE

4

(e) $\lim_{x\to +\infty} \frac{4x^2+9}{x^2-x}$

$= \lim_{x\to +\infty} \frac{x^2}{x^2}\frac{(4+9/x^2)}{(1-1/x)} = \lim_{x\to +\infty} \frac{4+9/x^2}{1-1/x} = 4$

4

(f) $\lim_{x\to 0} \frac{x^3-4}{x^2}$

When x is near 0, x^3-4 is near $-4<0$, while x^2 is near zero and positive. So $\lim_{x\to 0} \frac{x^3-4}{x^2} = -\infty$

4

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2. (12pts) Suppose c is a constant and consider the piecewise-defined function

$$f(x) = \begin{cases} x^2 - 4x + c^2, & x < 1 \\ -3cx + 1, & x \geq 1 \end{cases}$$

(a) (4pts) Compute $\lim_{x \to 1^-} f(x)$

4

$$\lim_{x\to 1^-} f(x) = \lim_{x \to 1^-} x^2 - 4x + c^2 = -3 + c^2$$

(b) (4pts) Compute $\lim_{x \to 1^+} f(x)$

4

$$\lim_{x \to 1^+} f(x) = \lim_{x\to 1^+} -3cx + 1 = -3c + 1.$$

(c) (4pts) Find the value(s) of c that make $f(x)$ continuous at $x = 1$.

4

$f(x)$ is continuous at $x=1$ if $\lim_{x\to 1^-} f(x) = \lim_{x \to 1^+} f(x)$

$$-3 + c^2 = -3c + 1 \Rightarrow c^2 + 3c - 4 = 0 \Rightarrow c = 1, -4$$

$$(c+4)(c-1) = 0$$

3. (10pts) Use **the definition of the derivative** to compute $f'(1)$ for $f(x) = x^3 + x$; that is, compute the limit

$$f'(1) = \lim_{h \to 0} \frac{f(1+h) - f(1)}{h}$$

$$f'(1) = \lim_{h\to 0} \frac{f(1+h) - f(1)}{h}$$

$$= \lim_{h \to 0} \frac{(1+h)^3 + (1+h) - 2}{h}$$

$$= \lim_{h\to 0} \frac{1 + 3h + 3h^2 + h^3 + 1 + h - 2}{h}$$

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$$= \lim_{h \to 0} \frac{4h + 3h^2 + h^3}{h}$$

$$= \lim_{h\to 0} 4 + 3h + h^2 = 4$$

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4. (20 pts) Compute the following derivatives. There is no need to simplify.

-1 small mistake
-2 right idea
-3 something right
-4 nothing right

(a) $D_x(x^7 - 5x^5 + 3x)$

4

$= 7x^6 - 25x^4 + 3$

(b) $D_x((x^3 + x)\sin x)$

4

$= (3x^2+1)\sin x + (x^3+x)\cos x$

(c) $D_x(\frac{x^3}{x^2+1})$

4

$$= \frac{(x^2+1)3x^2 - x^3(2x)}{(x^2+1)^2} = \frac{x^4+3x^2}{(x^2+1)^2}$$

(d) $D_x(\cos(7x^2 + 9))$

4

$= -\sin(7x^2+9)(14x)$

$= -14x\sin(7x^2+9)$

(e) $D_x((\frac{\sin x}{x})^5)$

4

$$= 5\left(\frac{\sin x}{x}\right)^4 \left(\frac{x\cos x - \sin x}{x^2}\right)$$

5. (6pts) Find $\frac{d^3y}{dx^3}$ if $y = 5x^3 + x - \sin x$.

6

$\frac{dy}{dx} = 15x^2 + 1 - \cos x$

$\frac{d^2y}{dx^2} = 30x + \sin x$

$\frac{d^3y}{dx^3} = 30 + \cos x$

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6. (8pts) Find the equation to the tangent line to the graph of the function $f(x) = \frac{1}{(x-2)^2}$ at the point $(1, 1)$.

$f(x) = \frac{1}{(x-2)^2} = (x-2)^{-2}$. $\quad f(1) = 1$.

2 ?

$f'(x) = -2(x-2)^{-3} \Rightarrow f'(1) = -2(-1)^{-3} = 2$.

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Eqn for tangent line at $x=a$

$y = f(a) + f'(a)(x-a)$ 2

So eqn for tangent line at $x=1$

$y = 1 + 2(x-1) = 2x - 1$. 2

7. (8pts) Let

$$f(x) = \left(\frac{x}{1+x^2}\right)^7$$

(a) (4pts) Find the derivative $f'(x)$.

4

$f'(x) = 7\left(\frac{x}{1+x^2}\right)^6 \left(\frac{(1+x^2)(1) - (x)(2x)}{(1+x^2)^2}\right) = 7\,\frac{x^6(1-x^2)}{(1+x^2)^8}$

(b) (4pts) At what three points x is the tangent line to the graph of $y = f(x)$ horizontal?

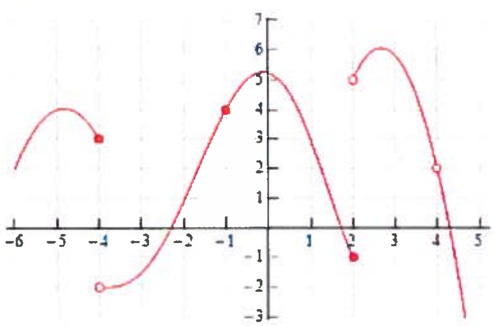
4

$f'(x) = 0 \Rightarrow$ tangent line horizontal (slope zero)

$0 = f'(x) = 7\,\frac{x^6(1-x^2)}{(1+x^2)^8} \Rightarrow 0 = x^6(1-x^2)$

$= x^6(1+x)(1-x) \Rightarrow x = 0, \pm 1$

8. (12pts) Examine the graph of the function $f(x)$ below and fill in the blanks.



2 pts each

(a) $\lim_{x\to -4^-} f(x) =$ <u>3</u>

(b) $\lim_{x\to -4^+} f(x) =$ <u>-2</u>

(c) List all values of x, $-6 < x < 5$, where $f(x)$ is **not** continuous. $x =$ <u>-4, 2, 4</u>

(d) <u>F</u> True (T) or False (F): $f'(1) > 0$.

(e) <u>T</u> True (T) or False (F): $f'(-2) > f'(-3)$.

(f) <u>F</u> True (T) or False (F): f is differentiable at $x = 4$.

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