Name ___*KEY*___

Instructions. Show all work and include appropriate explanations when space is provided. Correct answers unaccompanied by work may not receive full credit. Please circle your final answers.

1. (24pts) Compute the following limits. Be sure to show your work. **Note:** Answers can be values, $+\infty$, $-\infty$, or DNE (does not exist). An answer of DNE requires some explanation!

(a) $\displaystyle\lim_{x\to 0} \frac{\cos x}{1 + 3\sin x}$ $= \dfrac{\cos(0)}{1 + 3\sin(0)} = \dfrac{1}{1+0}$ $\boxed{= 1}$

(b) $\displaystyle\lim_{x\to 5} \frac{x - 5}{x^2 - 4x - 5}$ $= \displaystyle\lim_{x\to 5} \frac{x-5}{(x-5)(x+1)} = \lim_{x\to 5} \frac{1}{x+1}$ $\boxed{= \dfrac{1}{6}}$

(c) $\displaystyle\lim_{x\to 0} \frac{x - x\cos x}{x^2}$ $= \displaystyle\lim_{x\to 0} \frac{x(1 - \cos x)}{x^2} = \lim_{x\to 0} \frac{1 - \cos x}{x}$ $\boxed{= 0}$

(d) $\displaystyle\lim_{x\to 1^+} \frac{x^2 + 3}{(x-1)^3}$

when $x \to 1^+$, $x^2 + 3 \to 4$ and $(x-1)^3 \to 0$. Note that when $x > 1$, $(x-1)^3 > 0$. So

$\displaystyle\lim_{x\to 1^+} \frac{x^2 + 3}{(x-1)^3}$ $\boxed{= +\infty}$

Ans w/o explanation +2

DNE +2

(e) $\displaystyle\lim_{x\to 1} \frac{x^2 + 3}{(x-1)^3}$

when $x \to 1^-$, $x^2 + 3 \to 4$ and $(x-1)^3 \to 0$. Note that when $x < 1$, $(x-1)^3 < 0$. So

$\displaystyle\lim_{x\to 1^-} \frac{x^2+3}{(x-1)^3} = -\infty$. Along with 1d) this implies $\displaystyle\lim_{x\to 1} \frac{x^2+3}{(x-1)^3}$ $\boxed{\text{DNE}}$

(f) $\displaystyle\lim_{x\to -\infty} \frac{5x^2 + x}{x^3 - 9x}$ $= \displaystyle\lim_{x\to -\infty} \frac{x^2\left(5 + \frac{1}{x}\right)}{x^3\left(1 - \frac{9}{x^2}\right)} = \left(\lim_{x\to -\infty} \frac{1}{x}\right)\left(\lim_{x\to -\infty} \frac{5 + \frac{1}{x}}{1 - \frac{9}{x^2}}\right)$

$= 0 \cdot 5 = \boxed{0}$

2. (14pts) Consider the function

$$f(x) = \frac{x^2 + x}{x^2 - 3x} = \frac{x(x+1)}{x(x-3)}$$

(a) (2pts) $f(x)$ is continuous everywhere **except** $x = \underline{0}$ and $x = \underline{3}$.

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(b) (4pts) $f(x)$ has a horizontal asymptote at $y = \underline{1}$.

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$$\lim_{x \to \pm\infty} \frac{x^2 + x}{x^2 - 3x} = \lim_{x \to \pm\infty} \frac{x^2(1 + \frac{1}{x})}{x^2(1 - 3/x)} = \lim_{x \to \pm\infty} \frac{1 + \frac{1}{x}}{1 - 3/x} = 1.$$

(c) (4pts) Compute $\lim_{x \to 0} f(x)$.

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$$\lim_{x \to 0} f(x) = \lim_{x \to 0} \frac{x(x+1)}{x(x-3)} = \lim_{x \to 0} \frac{x+1}{x-3} = -\frac{1}{3}.$$

(d) (4pts) Find the value of c that make the following function continuous at $x = 0$:

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$$g(x) = \begin{cases} \frac{x^2+x}{x^2-3x} & \text{if } x \neq 0 \\ c & \text{if } x = 0. \end{cases}$$

$$C = -\frac{1}{3}$$

3. (10pts) Use **the definition of the derivative** to compute the derivative of $f(x) = 3x^2 + 7x$; that is, compute the limit

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

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$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h \to 0} \frac{3(x+h)^2 + 7(x+h) - 3x^2 - 7x}{h}$$

$$= \lim_{h \to 0} \frac{3(x^2 + 2xh + h^2) + 7x + 7h - 3x^2 - 7x}{h}$$

$$= \lim_{h \to 0} \frac{3x^2 + 6xh + 3h^2 + 7x + 7h - 3x^2 - 7x}{h}$$

$$= \lim_{h \to 0} \frac{6xh + 3h^2 + 7h}{h}$$

$$= \lim_{h \to 0} \cdot 6x + 3h + 7$$

$$= 6x + 7.$$

4. (20pts) Compute the following derivatives. There is no need to simplify.

(a) $D_x(x^3 - 2x^2 + x - 3)$

$= 3x^2 - 4x + 1$

(b) $D_x(\tan x) = \sec^2 x$

$D_x(\tan x) = D_x\left(\frac{\sin x}{\cos x}\right) = \frac{\cos x(\cos x) - \sin x(-\sin x)}{\cos^2 x} = \frac{\cos^2 x + \sin^2 x}{\cos^2 x} = \frac{1}{\cos^2 x}$

$= \sec^2 x.$

(c) $D_x(x^5 \sin x)$

$= D_x(x^5)\sin x + x^5 D_x(\sin x)$

$= 5x^4 \sin x + x^5 \cos x$

(d) $D_x\left(\frac{1-\cos x}{\sin x}\right) = \frac{\sin x \cdot D_x(1-\cos x) - (1-\cos x)\cdot D_x(\sin x)}{\sin^2 x}$

$= \frac{\sin^2 x - \cos x + \cos^2 x}{\sin^2 x} = \frac{1-\cos x}{\sin^2 x} = \csc^2 x - \csc x \cot x$

(e) $D_x(\cos(\sin(2x)))$

$= -\sin(\sin(2x)) \cdot D_x(\sin(2x))$

$= -\sin(\sin(2x)) \cos(2x) \cdot 2 = -2\sin(\sin(2x))\cos(2x)$

5. (6pts) Find $\frac{d^2 y}{dx^2}$ if $y = (1+\sin x)^3$.

$\frac{dy}{dx} = 3(1+\sin x)^2 \cos x$

$\frac{d^2 y}{dx^2} = 6(1+\sin x)\cos^2 x - 3(1+\sin x)^2 \sin x$

6. (8pts) Find the equation to the tangent line to the graph of the function $f(x) = (17 - x^2)^{-3}$ at the point $(4, 1)$.

$f(4) = 1.$

$f'(x) = -3(17-x^2)^{-4}(-2x) \implies f'(4) = -3(1)^{-4}(-8) = 24.$

$y = f(4) + f'(4)(x-4)$

$= 1 + 24(x-4)$

$= 24x - 95.$

7. (12pts) A ball is thrown upwards off the top of a 160-foot building. Its height (in feet) at time t (measured in seconds) is given by

$$h(t) = -16t^2 + 48t + 160.$$

(a) (3pts) Find the velocity of the ball at time t.

$v(t) = h'(t) = -32t + 48$

(b) (3pts) At what time does the ball reach its maximum height?

when $v = 0$

$0 = -32t + 48 \implies t = \frac{48}{32} = \frac{3}{2}$

(c) (3pts) At what time does the ball hit the ground?

$0 = -16t^2 + 48t + 160 = -16(t^2 - 3t - 10) = -16(t-5)(t+2)$

$t = 5$

(d) (3pts) What is the acceleration of the ball at time t?

$a(t) = v'(t) = h''(t) = -32$

8. (6pts) Explain, using the Intermediate Value Theorem, why the function

$$f(x) = x^4 - x^3 + 2x - 3$$

must have a root somewhere between $x = 1$ and $x = 2$.

$f(x)$ is continuous everywhere since it is a polynomial.

$f(1) = 1 - 1 + 2 - 3 = -1 < 0$ and $f(2) = 16 - 8 + 4 - 3 = 9 > 0$.

So by IVT, there is a number c with $1 < c < 2$ such that $f(c) = 0$.

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