1210-90 Exam 2

Spring 2014

Name KEY

Instructions. Show all work and include appropriate explanations when necessary. A correct answer unaccompanied by work may not receive full credit. Please try to do all work in the space provided. Please circle your final answers.

1. (16pts) For this problem, consider the function

$$f(x) = \frac{x^2}{1-x}.$$

   (a) (4pts) Find $f'(x)$.

4

$$f'(x) = \frac{(1-x)(2x) - (x^2)(-1)}{(1-x)^2} = \frac{2x-2x^2+x^2}{(1-x)^2} = \frac{2x-x^2}{(1-x)^2}$$

   (b) (4pts) Find the two critical points of $f(x)$.

4

$f'(x)$ is undefined at $x=1$ but so is $f(x)$, so not a cp.

$$0 = f'(x) = \frac{2x-x^2}{(1-x)^2} \Rightarrow 0 = 2x - x^2 = \cancel{2x}\, x(2-x) \Rightarrow x = 0, 2 \text{ are cps}$$

4

   (c) (4pts) Fill in the blanks: $f(x)$ is decreasing on the intervals ($-\infty$, 0) and (2 , $+\infty$). **Note:** $\pm\infty$ are acceptable answers.

4

   (d) (4pts) Use the **First Derivative Test** to classify each of the critical points you found above as a local maximum or a local minimum.

Sign of f': $-$ (0) $+$ (1) $+$ (2) $-$

$x=0$ is local min

$x=2$ is local max.

2. (10pts) Now consider the function

$$f(x) = x^5 + 5x^4 - 2$$

   (a) (4pts) Find $f''(x)$.

4

$$f'(x) = 5x^4 + 20x^3$$
$$f''(x) = 20x^3 + 60x^2$$

   (b) (4pts) Find the inflection point(s) of $f(x)$.

4

$$0 = f''(x) = 20x^3 + 60x^2 = 20x^2(x+3)$$

$x=-3$ is only inflection pt. or $(-3, f(-3))$.

Sign of f'': $-$ (-3) $+$ (0) $+$

   (c) (2pts) Fill in the blanks: $f(x)$ is concave down on the interval ($-\infty$, -3). **Note:** $\pm\infty$ are acceptable answers.

2

26

3. (6pts) Find the maximum and minimum values of the function $f(x) = 4x^3 - 3x^2 + 2$ on the interval $[0, 2]$.

6

Maximum value= 22

Minimum value= 7/4

Find critical pts:

$0 = f'(x) = 12x^2 - 6x = 6x(2x-1)$ 2

cps: $x = 0, \frac{1}{2}$. 2

Check values of $f(x)$ at cps and endpoints:

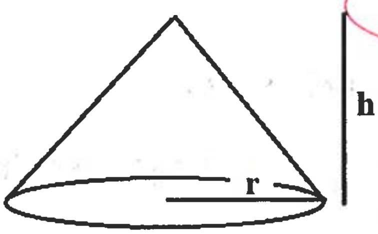
$f(0) = 2$

$f(2) = 32 - 12 + 2 = 22$

$f(\frac{1}{2}) = \frac{4}{8} - \frac{3}{4} + 2 = \frac{4-6+16}{8} = \frac{14}{8} = \frac{7}{4}$

4. (8pts) Sand is being dumped at a rate of 30π ft^3/s in a large conical pile. The size of the pile is growing, but the radius of the base is always equal to the height ($h = r$ in the picture below). Consequently, the volume of the sand in the pile is $V = \pi r^2 h = \pi h^3$. How fast is the height of the pile growing when the height is 10 feet?

8



$V = \pi h^3$

$\frac{dV}{dt} = 3\pi h^2 \frac{dh}{dt}$ 4

$\frac{dV}{dt} = 30\pi$, $h = 10$. 2

$30\pi = 3\pi(10)^2 \frac{dh}{dt} = 300\pi \frac{dh}{dt}$

So $\frac{dh}{dt} = \frac{1}{10}$ ft/s 2

5. (8pts) Find the equation of the tangent line to the following curve at the point $(1, 0)$.

$$xy + x^2 \cos(y) = 1.$$

8

Differentiate both sides w.r.t x:

$D_x(xy + x^2\cos(y)) = D_x(1)$

$y + x\frac{dy}{dx} + 2x\cos(y) - x^2\sin(y)\frac{dy}{dx} = 0$ 4

Plug in $x=1, y=0$:

$\frac{dy}{dx} + 2 = 0 \implies \frac{dy}{dx} = -2 = \text{slope}$ 2

Passes through the point $(1,0)$ $y = -2x + 2$ 2

6. (5pts) Compute x_2, the second approximation to the root of

$$x^3 - 2x^2 + 3x - 1 = 0$$

using Newton's Method with initial approximation $x_1 = 0$.

5

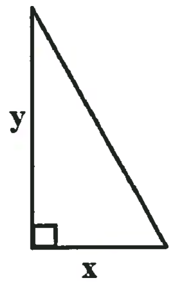
$x_2 = x_1 - \frac{f(x_1)}{f'(x_1)}$ 3

$f(x) = x^3 - 2x^2 + 3x - 1$

$f'(x) = 3x^2 - 4x + 3$

$= 0 - \frac{f(0)}{f'(0)} = 0 - \frac{-1}{3} = \frac{1}{3}$ 2

7. (10pts) Find the side lengths, labeled x and y in the picture below, of the right triangle of area 2 with the smallest hypotenuse. Instead of minimizing the length of the hypotenuse, it is easier to minimize the length of the hypotenuse squared, $H = x^2 + y^2$, subject to the area constraint $\frac{1}{2}xy = 2$. **Note:** You must use calculus to get credit!!



$\frac{1}{2}xy = 2 \Rightarrow xy = 4$ or $y = \frac{4}{x}$.

$H = x^2 + y^2 = x^2 + \left(\frac{4}{x}\right)^2 = x^2 + \frac{16}{x^2} = x^2 + 16x^{-2}$ 4

$0 = H'(x) = 2x - 32x^{-3} = 2x - \frac{32}{x^3} \Rightarrow 2x = \frac{32}{x^3}$

2

10

or $x^4 = 16$ 1

So $x = 2$ (-2 is not realistic) is a cp.

Check that $x=2$ is a local min: 2

$H''(x) = 2 + 96x^{-4} \Rightarrow H''(2) = 2 + \frac{96}{16} = 8 > 0$

So $x=2$ is a local min by 2^{nd} derivative test.

So $x = y = 2$ 1

8. (16pts) Find the indicated general antiderivatives: Remember $+C$!

(a) (4pts) $\int (x^2 - 4x + 9)\, dx$

4 $\frac{1}{3}x^3 - 2x^2 + 9x + C$

missing $+C$ -2

(b) (4pts) $\int \sqrt{x}\, dx = \int x^{1/2} dx$

4 $= \frac{2}{3}x^{3/2} + c$

(c) (4pts) $\int (\sin x - 1)\, dx$

4 $= -\cos x - x + C$

(d) (4pts) $\int (x^3 - 9)^7 (3x^2)\, dx$

4 Since $D_x(x^3 - 9) = 3x^2$,

$\int (x^3-9)^7 (3x^2) dx = \frac{1}{8}(x^3-9)^8 + c$

9. (6pts) Approximate $\int_{-1}^{7}(x^2-5)\,dx$ using a Riemann sum with 4 subintervals of equal length and the sample points being the left-endpoints of the subintervals.

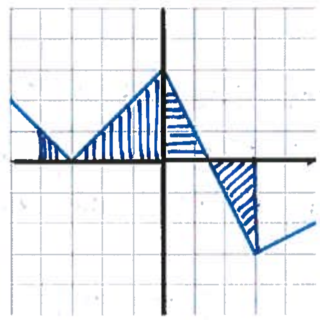
$\Delta x = \frac{7-(-1)}{4} = \frac{8}{4} = 2.$ $\quad f(x) = x^2-5$

6

$$\int_{-1}^{7}(x^2-5)\,dx \approx 2\left(f(-1)+f(1)+f(3)+f(5)\right)$$

$$= 2(-4-4+4+20) = 2(16) = 32$$

10. (8pts) Use the graph of $y = f(x)$ below and the area interpretation of the definite integral to evaluate the following:



8

4 (a) $\int_{-4}^{0} f(x)\,dx =$ 5

= area of

4 (b) $\int_{-4}^{3} f(x)\,dx =$ 5

= area of + area of

− area of

11. (7pts) Find the solution to the initial value problem

$$\frac{dy}{dx} = 3\cos x + \sin x + 5 \qquad y(0) = 3$$

Note: What this means is that you are looking for a function $y = f(x)$ whose derivative is given by the above and the whose value at $x = 0$ is 3.

Since $\frac{dy}{dx} = 3\cos x + \sin x + 5$, y is an antiderivative of $3\cos x + \sin x + 5$.

7

$$y = \int (3\cos x + \sin x + 5)\,dx = 3\sin x - \cos x + 5x + C$$

$y(0) = 3$ allows us to solve for C:

$$3 = y(0) = 3\sin(0) - \cos(0) + 5(0) + C$$

-3 for no i.c

$$= -1 + C \implies C = 4.$$

So

$$y(x) = 3\sin x - \cos x + 5x + 4$$