1210-90 Exam 2

Fall 2012

Name KEY

Instructions. Show all work and include appropriate explanations when necessary. Please try to do all work in the space provided. Please circle your final answer.

1. (14pts) For this problem, consider the function

$$f(x) = \frac{x}{1+x^2}.$$

(a) (4pts) Find $f'(x)$.

$$f'(x) = \frac{(1+x^2)(1) - x(2x)}{(1+x^2)^2} = \frac{1-x^2}{(1+x^2)^2}$$

(b) (4pts) Find the critical point(s) of $f(x)$.

$0 = f'(x) \Rightarrow 0 = 1 - x^2 = (1+x)(1-x)$ cps: $x = \pm 1$

$-$ $\quad$ $+$ $\quad$ $-$ sign of f'

-1 $\quad$ 1

(c) (2pts) Fill in the blanks: $f(x)$ is increasing on the interval (-1 , 1). **Note:** $\pm\infty$ are acceptable answers.

(d) (4pts) Use the First Derivative Test to classify each of the critical points you found above as a local maximum or a local minimum.

$-$ $\quad$ $+$ $\quad$ $-$ sign of f'

-1 $\quad$ 1

$x = -1$ is local min

$x = 1$ is local max

2. (10pts) Now consider the function

$$f(x) = 2x^2 + \cos^2 x.$$

(a) (4pts) Find $f''(x)$.

$f'(x) = 4x + 2\cos x(-\sin x)$

$f''(x) = 4 - 2\cos^2 x + 2\sin^2 x$

(b) (4pts) Find the inflection point(s) of $f(x)$ or show that there are no inflection points.

Since $0 \le \cos^2 x \le 1$ and $0 \le \sin^2 x \le 1$,

$2 \le 4 - 2\cos^2 x + 2\sin^2 x \le 6$

So $f''(x) > 0$ always, hence no inflection pts

(c) (2pts) Fill in the blanks: $f(x)$ is concave up on the interval ($-\infty$, $+\infty$). **Note:** $\pm\infty$ are acceptable answers.

3. (12pts) Consider the function

$$f(x) = \frac{1}{3}x^3 - x^2 + 1$$

(a) (6pts) Find the critical points of $f(x)$.

$0 = f'(x) = x^2 - 2x = x(x-2)$

$x = 0, 2$ are cps

(b) (6pts) Find the extreme values of $f(x)$ on the interval $[1,3]$. Max= 1 Min= $-1/3$

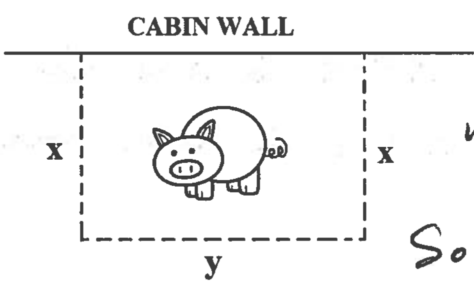
$x=2$ is only cp in $[1,3]$. So we test values of $f(x)$ at $1, 2,$ and 3.

$f(1) = \frac{1}{3}$

$f(2) = \frac{8}{3} - 3 = \frac{-1}{3}$

$f(3) = 9 - 9 + 1 = 1.$

4. (12pts) Billy Joe wants to build a rectangular pig pen up against the side of his cabin; only three sides of the enclosure are fence since the side of his cabin will form a fourth wall (see picture below). He has 20 feet of fence. What dimensions (labeled x and y in the picture below) should Billy Joe make his fence to maximize the area enclosed?



We want to maximize

$A = xy$

using the constraint

$20 = 2x + y \Rightarrow y = 20 - 2x.$

So

$A(x) = x(20-2x) = 20x - 2x^2.$

$A'(x) = 20 - 4x$

$0 = A'(x) = 20 - 4x \Rightarrow x = 5$ is a cp.

Note: x can only take on values between 0 and 10

$[0 \; \overset{+}{\text{———}} \; 5 \; \overset{-}{\text{———}} \; 10]$ sign of f'

So $x=5$ is local max and global max.

$x = 5$

$y = 10$

5. (10pts) Consider the function $f(x) = \frac{1}{x}$.

(a) (3pts) Find $f'(x)$.

$$f'(x) = \frac{-1}{x^2}$$

(b) (4pts) Find the value of c which satisfies the conclusion of the Mean Value Theorem on the interval $[1, 2]$.

$$\frac{-1}{c^2} = \frac{f(2)-f(1)}{2-1} = \frac{1}{2} - 1 = \frac{-1}{2} \Rightarrow c^2 = 2 \Rightarrow c = \sqrt{2}$$

(c) (3pts) Explain **in one sentence** why there is no value of c which satisfies the conclusion of the Mean Value Theorem on the interval $[-1, 1]$.

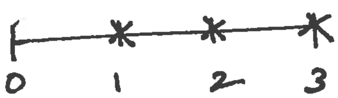
f is not continuous on $[-1,1]$ nor is it differentiable on $(-1,1)$, so hypotheses of MVT do not apply.

6. (15pts) For this problem, consider the definite integral

$$\int_0^3 (3x^2 - 2x)\, dx$$

(a) (6pts) Approximate the definite integral above using a Riemann sum. Use the partition of 3 subintervals of equal length with the sample points being the right end points of the subintervals.

$$\Delta x = \frac{3-0}{3} = 1$$



$$I \approx f(1)(1) + f(2)(1) + f(3)(1)$$

$$= 1 + 8 + 21 = 30$$

(b) (6pts) Evaluate the definite integral using the Fundamental Theorem of Calculus.

$$\int_0^3 (3x^2 - 2x)\,dx = \left(x^3 - x^2\right|_0^3 = (27-9) - (0-0)$$

$$= 18$$

(c) (3pts) What is the average value of the function $f(x) = 3x^2 - 2x$ on the interval $[0, 3]$?

$$f_{ave} = \frac{1}{3} \int_0^3 (3x^2 - 2x)\,dx = \frac{1}{3}(18) = 6$$

7. (10pts) A 6 foot-tall man tosses a ball up in the air with a velocity of 10 feet per second. **Note:** The acceleration of gravity is -32 feet per second per second (negative because it acts downward).

(a) (4pts) Find an expression for $v(t)$, the velocity of the ball as a function of time (in seconds).

$v'(t) = a(t) = -32$ and $v(0) = 10$, so

$v(t) = -32t + 10.$

(b) (4pts) Find an expression for $h(t)$, the height of the ball as a function of time (in seconds).

$h'(t) = v(t) = -32t + 10$ and $h(0) = 6$, so

$h(t) = -16t^2 + 10t + 6$

(c) (2pts) At what time does the ball hit the ground? (**Hint:** Use the quadratic formula)

$0 = -16t^2 + 10t + 6$ $\qquad \dfrac{-10 \pm \sqrt{100 + 4(16)(6)}}{-32}$

The positive solution is $t = \dfrac{10 + \sqrt{484}}{32}$

8. (12pts) Find the general antiderivatives of the following functions

(a) (4pts) $f(x) = x^2 + x^{-2}$

$F(x) = \frac{1}{3}x^3 - x^{-1} + C$

(b) (4pts) $f(x) = \sqrt[3]{x} = x^{1/3}$

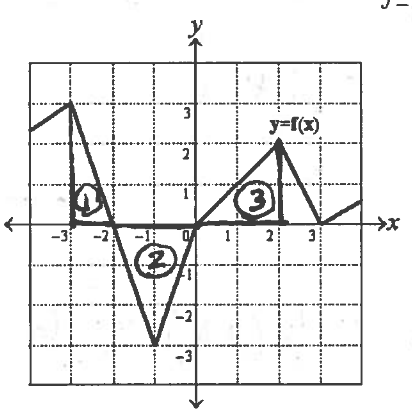
$F(x) = \frac{3}{4}x^{4/3} + C$

(c) (4pts) $f(x) = x(x^2 + 3)^7$

$u = x^2 + 3$
$du = 2x\,dx$

$\int x(x^2+3)^7 dx = \frac{1}{2}\int u^7 du = \frac{1}{16}u^8 + C = \frac{1}{16}(x^2+3)^8 + C$

9. (5pts) The graph of $f(x)$ is below. $\int_{-3}^{2} f(x)\,dx =$ $\underline{\frac{1}{2}}$.



$\int_{-3}^{2} f(x)dx$ = sum of areas of triangles, where area of triangle 2 is negative (below x-axis)

$= \frac{3}{2} - 3 + 2 = \frac{1}{2}.$