Calculus I
Exam 2, Fall 2002, Answers

1. A curve C in the plane is the graph of the relation $y^3 - xy^2 + x^3 = 5$ .Find the equation of the tangent line to the curve at the point (2,-1).

Answer. Differentiate implicitly:

$$3y^2\frac{dy}{dx} - y^2 - 2xy\frac{dy}{dx} + 3x^2 = 0 \ .$$

Put in the values $x = 2$ and $y = -1$ and solve for dy/dx:

$$3\frac{dy}{dx} - 1 + 4\frac{dy}{dx} + 12 = 0 \ ,$$

leading to $dy/dx = -11/7$.

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2. A cylindrical balloon is being inflated so that its volume is increasing at the rate of 3 in^3 per second. Assuming that the length of the balloon is held constantly at 9 in., at what rate is the radius increasing when it is 2 in? (The volume of a cylinder is $V = \pi r^2 h$.)

Answer. Since $h = 9$, we have $V = 9\pi r^2$. Differentiate with respect to t:

$$\frac{dV}{dt} = 18\pi r\frac{dr}{dt} \ .$$

Now, the given data are $dV/dt = 3$ in^3/sec and $r = 2$. This gives

$$3 = 18\pi(2)\frac{dr}{dt} \quad \text{so} \quad \frac{dr}{dt} = \frac{1}{12\pi} \text{ in/sec} \ .$$

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3. Let $y = (x^2 - 2)(x + 5)$. Find all local maxima, local minima and points of inflection of the graph.

Answer. Differentiate using the product rule

$$y' = 2x(x+5) + x^2 - 2 = 3x^2 + 5x - 2 \qquad y'' = 6x + 5 \ .$$

To find the local maxima and minima we set $y' = 0$ and solve. The roots are -2, 1/3. At the first root $y'' < 0$, so it is a local maximum, and at the second $y'' > 0$, so it is a local minimum. The point of inflection is where $6x + 5 = 0$, or $x = -5/6$.

4.Let $y = \sin x + \cos x$. Where between $-\pi/2$ and $\pi/2$ is there a critical point? Is this a maximum or a minimum?

Answer. We have $y' = \cos x - \sin x$. Thus $y' = 0$ where $\tan x = 1$, or $x = \pi/4$. Evaluate $y'' = -\sin x - \cos x$ at $x = \pi/4$, to get $y'' < 0$, thus this is a local maximum. Evaluate y at the endpoints and $x = \pi/4$: the values are -1, $\sqrt{2}$, 1 respectively, so $\pi/4$ is a maximum.

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5. $y = \dfrac{x^2}{(x-1)(x-2)}$ You must show enough work to explain how you found the various features of the graph.

Answer. The vertical asymptotes are the lines $x = 1,\ x = 2$. Since

$$y = \frac{x^2}{x^2 - 3x + 2}$$

the line $y = 1$ is a horziontal asymptote. Calculating the derivative, we find

$$y' = \frac{-3x^2 + 4x}{(x^2 - 3x + 2)^2}$$

so the critical points are at $x = 0,\ x = 1/3$. Since $y \geq 0$ for $x < 1$ and $x > 2$ we see that $x = 0$ is a local minimum, and $x = 1/3$ a local maximum and

$$\lim_{x\to 1^-} y = +\infty \quad \lim_{x\to 1^+} y = -\infty \quad \lim_{x\to 2^-} y = -\infty \quad \lim_{x\to 2^+} y = +\infty\ .$$