Calculus I
Exam 3, Fall 2002, Answers

1. Integrate:

a) $\int (x^4+4x+5)^3(x^3+1)dx =$

Answer. Let $u = x^4+4x+5, !du = 4(x^3+1)dx$. Then

$$\int (x^4+4x+5)^3(x^3+1)dx = \frac{1}{4}\int u^3 du = \frac{u^4}{16}+C = \frac{(x^4+4x+5)^4}{16}+C$$

b) $\int (\tan^2 x+1)\sec^2 x dx =$

Answer. Let $u = \tan x,\ du = \sec^2 x dx$. Then

$$\int (\tan^2 x+1)\sec^2 x dx = \int (u^2+1)du = \frac{u^3}{3}+u+C = \frac{\tan^3 x}{3}+\tan x+C$$

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2. Solve the differential equation: $\dfrac{dy}{dx} = xy^2, \quad y(2) = 0.$

Answer. We can separate the variables, getting

$$y^{-2}dy = xdx$$

which integrates to

$$-y^{-1} = \frac{x^2}{2}+C,$$

so that

$$y = \frac{-1}{x^2/2+C}\ .$$

Substituting the initial conditions:

$$0 = \frac{-1}{2+C}$$

which has no solution. Thus there is no function of this type satisfying this initial condition. Recall, however, that when we separated variables, we divided by y^2; this can only be done if $y \neq 0$. Thus the alternative $y = 0$ remains, and provides the solution: the function which is identically zero.

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3. Calculate the definite integrals:

a) $\int_0^4 (x^2-3x+1)dx$

Answer. $= (\frac{x^3}{3}-\frac{3}{2}x^2+x)\big|_0^4 = \frac{4^3}{3}-\frac{3}{2}4^2+4-0 = \frac{4}{3}$

b) $\int_0^{\pi/2} (\cos x \sin x)dx$

Answer. Let $u = \sin x,\ du = \cos x dx$. Then when $x = 0,\ u = 0$ and when $x = \pi/2,\ u = 1$, and we have

$$\int_0^{\pi/2} (\cos x \sin x)dx = \int_0^1 u du = \frac{1}{2}\ .$$

4. Find the area of the region in the first quadrant bounded by the curves $y = x(1-x)$ and $y = 4-4x^2$.

Answer. The second curve is always above the first, and both leave the first quadrant when $x = 1$. Thus the area is

$$Area = \int_0^1 ((4-4x^2)-(x-x^2))dx = \int_0^1 (4-x-3x^2)dx = (4x - \frac{x^2}{2} - x^3)\Big|_0^1 = \frac{5}{2}$$

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5. The region in the first quadrant bounded by the curves $y = x^2$ and $x = 1$ is rotated about the y-axis. What is the volume of the solid so produced?

Answer. We sweep out the volume along the x-axis, as x ranges from 0 to 1. The volume of the shell of width dx at the point x is $dV = 2\pi xy dx = 2\pi x^3 dx$. Thus the volume is

$$Volume = \int_0^1 2\pi x^3 dx = \frac{\pi}{2} .$$