Calculus I
Final Exam, Fall 2002, Answers

1. Find the derivatives of the following functions:

a) $f(x) = (x^2-3)^2(2x+4)$

Answer. $f'(x) = 2(x^2-3)(2x)(2x+4) + (x^2-3)^2(2) = 2(x^2-3)(5x^2+8x-3)$.

b) $g(x) = \tan x \sin x$

Answer. $g'(x) = \sec^2 x \sin x + \tan x \cos x = \sin x(\sec^2 + 1)$

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2. Integrate:

a) $\int (6x^2+1)^5 x dx$

Answer. Let $u = 6x^2+1,\ du = 12xdx$. Then

$$\int (6x^2+1)^5 x dx = \frac{1}{12}\int u^5 du = \frac{1}{72}(6x^2+1)^6 + C\ .$$

b) $\int \frac{dy}{y^{5/2}}$

Answer. $= \int y^{-5/2}dy = \frac{y^{-3/2}}{-3/2} + C = -\frac{2}{3y^{3/2}} + C$.

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3. Find the point in the first quadrant at which the tangent line to the curve $2y^2+6x-y=3$ has slope equal to 3.

Answer. We find the slope at any point by taking the differential: $4ydy+6dx-dy=0$, giving $dy/dx = -6/(4y-1)$. Set this equal to 3 and solve: $-6 = 3(4y-1)$, so $y = -1/4$. Put this value for y in the curve equation and solve for x:

$$2(-\frac{1}{4})^2 + 6x - (-\frac{1}{4}) = 3 \quad \text{so that} \quad x = \frac{7}{16}\ ,$$

so that the point in question is (7/16,-1/4).

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4. Consider the region in the first quadrant bounded by the curve $y = 12 - \frac{3}{4}x^2$. What are the dimensions of the largest rectangle with sides parallel to the coordinate axes which can be inscribed inside this region?

Answer. Let (x,y) be the dimensions of the rectangle,so that the area is $A = xy$. Since this point must be on the boundary of the region, its coordinates satisfy the equation: $y = 12-(3/4)x^2$. Thus, we have

$$A = x(12-\frac{3}{4}x^2) = 12x - \frac{3}{4}x^3\ .$$

To find the maximum, differentiate and set equal to zero:

$$\frac{dA}{dx} = 12 - \frac{9}{4}x^2 = 0\ ,$$

which has the solution $x = 4\sqrt{3}/3$. To find y put this in the original equation, getting $y = 8$. Thus the answer is $(4\sqrt{3}/3, 8)$.

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5. Sketch the graph of the function $y = 3x^4 - 4x^3 - 12x^2 + 2$. Find the x values of all local minima, maxima and points of inflection.

Answer. First, differentiate:

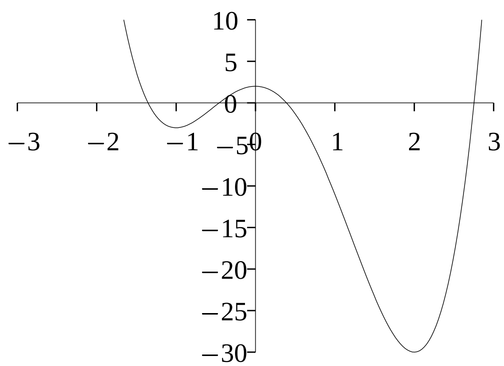
$$y' = 12x^3 - 12x^2 - 24x = 12x(x^2 - x - 2) = 12x(x-2)(x+1) \; .$$

Thus the critical points are at $x = -1, 0, 2$. Since the function is of fourth degree, it approaches $+\infty$ as x leaves the paper to the left or to the right. Thus, we expect $x = -1$ to be a local minimum, $x = 0$ a local maximum and $x = 2$ a local minimum. Taking the second derivative;

$$y'' = 36x^2 - 24x - 24 = 12(3x^2 - 2x - 2) \; .$$

Evaluating at the critical points, we verify that $y'' > 0$ at $x = -1, 2$, and $y'' < 0$ at $x = 0$, confirming our expectations. Using the quadratic formula, the roots are $x = (1 \pm \sqrt{7})/3$; these are the points of inflection. The curve is concave up until $x = (1 - \sqrt{7})/3$, then concave down, and finally concave up to the right of $x = (1 + \sqrt{7})/3$.

PSfrag replacements



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6. Solve the initial value problem: $\dfrac{dy}{dx} = y^2 x \qquad y(0) = 2$

Answer. We separate variables and then integrate, obtaining

$$\frac{dy}{y^2} = x\,dx \quad \text{which integrates to} \quad \frac{-1}{y} = \frac{x^2}{2} + C \; .$$

The initial condition gives us $C = -1/2$, so the solution is

$$\frac{-1}{y} = \frac{x^2}{2} - \frac{1}{2} \quad \text{or} \quad y = \frac{2}{1 - x^2} \; .$$

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7. Find the area between the curve

$$y = \frac{x+1}{x^3}$$

and the x-axis, as x ranges from 1 to 4.

Answer. The area is

PSfrag replacements

$$Area = \int_1^4 \frac{x+1}{x^3} dx = \int_1^4 (x^{-2} + x^{-3}) dx = (-x^{-1} - \frac{1}{2} x^{-2})\Big|_1^4 = \frac{39}{32}$$

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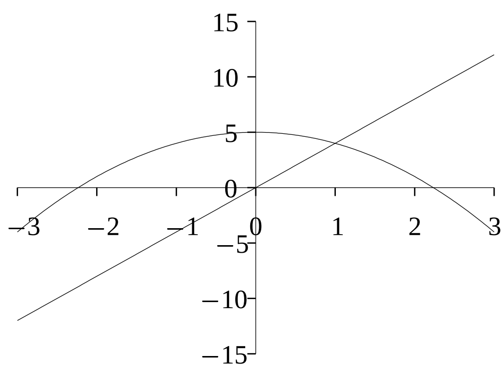
8. Find the volume of the solid obtained by rotating about the y-axis the region bounded by the curves $y = 4x$, $y = 5 - x^2$, $y = 0$.

Answer. Draw the graph. We see that the best way to calculate the volume is to accumulate the volume in the y direction, using the method of washers, The inner radius is at $x = y/4$, and the outer radius is at $x = \sqrt{5-y}$, and y ranges from 0 to the y value of the intersection of the two curves. We solve $4x = 5 - x^2$: the solutions are $x = -5, 1$, so $x = 1$ is the right value. There $y = 4x = 4$. Thus, the volume is

$$\int_0^4 (\pi R^2 - \pi r^2) dy = \pi \int_0^4 (5 - y - \frac{y^2}{16}) dy = (5y - \frac{y^2}{2} - \frac{y^3}{48})\Big|_0^1 = \frac{215}{48} .$$



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9. Consider a curve given parametrically by $x = 4\cos^2 t$, $y = 3\sin^2 t$. Find the length of the piece of this curve running from $t = 0$ to $t = \pi/2$.

Answer. To calculate the element of arc length, ds, we first take differentials:

$$dx = -8 \cos t \sin t dt , \quad dy = 6 \sin t \cos t dt ,$$

so that

$$ds^2 = (64 \cos^2 t \sin^2 t + 36 \cos^2 t \sin^2 t) dt^2 \quad \text{and} \quad ds = 10 \cos t \sin t dt .$$

Thus

$$Length = 10 \int_0^{\pi/2} \cos t \sin t dt = 10 \frac{\sin^2 t}{2}\Big|_0^{\pi/2} = 5 .$$

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10. The square with vertices at (0,0),(1,0),(1,1),(0,1) is filled with a material whose density at the point (x, y) is $\delta(x, y) = x(1-x)$ g/cm^2. What is the mass of this object? What is its moment about the y-axis?

Answer. The domain is that between the lines $y = 0$ and $y = 1$, for x running from 0 to 1. Since the mass is a function of x alone, we can accumulate the mass by sweeping along the x-axis. We get

$$Mass = \int_0^1 \delta(x)dx = \int_0^1 (x - x^2)dx = \frac{1}{6} \ .$$

and

$$Mom_{x=0} = \int_0^1 \delta(x)xdx = \int_0^1 (x^2 - x^3)dx = \frac{1}{12} \ .$$

In particular the x coordinate of the center of mass is $\bar{x} = 1/2$. We could have seen that in the beginning, since the density function is symmetric about the line $x = 1/2$.