**Mathematics 1210-90**
**Final Examination Answers**

**You must show your work. Just entering an answer will earn no points.**

1. Let $$f(x) = \frac{x^2}{1+x^2}$$

**Solution**. a) Note that $f(x) = 1 - (1+x^2)^{-1}$. Thus

$$f'(x) = -[-(1+x^2)^{-2}(2x)] = 2\frac{x}{(1+x^2)^2}$$

*b)* $$f''(x) = 2\frac{(1+x^2)^2 - x(2(1+x^2)(2x))}{(1+x^2)^4} = 2\frac{1-3x^2}{(1+x^2)^3}$$

2. **Solution**

*a)* $$\int (4x^2 + x - x^{-2})dx = \frac{4}{3}x^3 + \frac{x^2}{2} + \frac{1}{x} + C$$

b). Let $u = 1 + x^{3/2}$, $du = (3/2)x^{1/2}dx$, so that

$$\int \frac{\sqrt{x}}{(1+x^{3/2})^2}dx = \frac{2}{3}\int u^{-2}du = \frac{2}{3u} + C = \frac{2}{3(1+x^{3/2})} + C\ .$$

3. The volume of a cone of radius $r$ and height $h$ is $V = \frac{\pi}{3}r^2h$. Water is pouring into a conical cup of radius 8 cm and height 10 cm at the rate of 120 cm/min. At what rate is the height of water in the cup rising when it is at $h = 5$ cm (and $r = 4$)?

**Solution**. By similar triangles, when the height of the water is $h$, its radius is $r$, where

$$\frac{h}{10} = \frac{r}{8}\ , \quad \text{or} \quad r = \frac{4}{5}h\ .$$

Thus the volume of water is

$$V = \frac{\pi}{3}r^2h = \frac{\pi}{3}(\frac{4}{5})^2h^3\ .$$

$$\frac{dV}{dt} = \pi(\frac{4}{5})^2h^2\frac{dh}{dt}\ .$$

When $h = 5$, we get

$$120 = \pi(\frac{4}{5})^2 5^2\frac{dh}{dt} = 16\pi\frac{dh}{dt}\ ,$$

so

$$\frac{dh}{dt} = \frac{120}{16\pi} = 2.387 \quad \text{cm/min} \ .$$

4 $a$). Graph $\quad y = 2x + \frac{1}{x} \quad$ for $\quad x > 0$ .

b) What is the minimum value of $y$?

**Solution**. The function is always positive (for $x > 0$),. As $x \to 0$, $y \to \infty$ because of the second term, and as $x \to \infty$, $y \to \infty$ because of the first term. Since $dy/dx = 2 - x^{-2}$, the function is decreasing for $x < 1/\sqrt{2}$ and is increasing for $x > 1/\sqrt{2}$, and since $d^2y/dx^2 = 2x^{-3}$, the graph is always concave up.

Because of the above analysis, the minimum value of $y$ is taken when $x = 1/\sqrt{2}$, so is $2/\sqrt{2} + \sqrt{2} = 2\sqrt{2} = 2.828$.

5. Find the solution to the differential equation

$$\frac{dy}{dx} = \frac{3x}{y+1}$$

such that $y(0) = 4$.

**Solution**. Separating variables we have $(y+1)dy = 3xdx$, so

$$\frac{(y+1)^2}{2} = \frac{3x^2}{2} + C \ .$$

At $x = 0,\ y = 4$, so $C = 25/2$, giving us

$$(y+1)^2 = 3x^2 + 25 \ , \quad \text{or} \quad y = \sqrt{3x^2+25} - 1 \ .$$

6. A curve in the plane is given by the equation $x^3 - y^3 = 61$. What is the slope of the tangent line to the curve at the point (5,4)?

**Solution**. Take differentials: $3x^2dx - 3y^2dy = 0$, or $dy/dx = x^2/y^2 = 25/16 = 1.5625$.

7. Find the area of the region in the first quadrant bounded by the curve $y = 9x - x^2$.

**Solution**. Factoring, we have $y = x(9-x)$, so the region lies between the values $x = 0$ and $x = 9$. Thus

$$Area = \int_0^9 (9x - x^2)dx = (\frac{9}{2}x^2 - \frac{1}{3}x^3)_0^9 = 9^2(\frac{9}{2} - \frac{9}{3} = \frac{243}{2} = 121.5 \ .$$

8. The region in the first quadrant bounded by the curves $y = 9x$ and $y = x^3$ is rotated about the $y$-axis. Find the volume of the resulting solid.

**Solution**. This is the region between $x = 0$ and $x = 3$, bounded above by $y = 9x$ and below by $y = x^3$. Using the method of shells, a slab at a point $x$ of width $dx$ contributes the volume $dV = 2\pi x(9x - x^3)dx$. Thus the volume is

$$Volume = 2\pi \int_0^3 (9x^2 - x^4)dx = 2\pi(3x^3 - \frac{x^5}{5})_0^3 = 2\pi(3^4)(1 - \frac{3}{5}) = \frac{4(3^4)\pi}{5} \ .$$

If we sweep the volume out in the $y$ direction, and use the method of washers, we get

$$Volume = \pi \int_0^{27} [(y^{1/3})^2 - (\frac{y}{9})^2]dy = \pi \int_0^{27} (y^{2/3} - \frac{y^2}{81})dy = \pi[\frac{3}{5}y^{5/3} - \frac{y^3}{243}]_0^{27}$$

$$= \pi[\frac{3}{5}(243) - 81] = \frac{4(3^4)\pi}{5} = 64.8\pi \ .$$

9. What is the center of mass of the triangle bounded by the coordinate axes and the line $2x + y = 1$?

**Solution**. This is a right triangle whose legs are of length 1, 1/2, so has area 1/4. Now we compute the moments:

$$Mom_{x=0} = \int_0^{1/2} x(1 - 2x)dx = \frac{x^2}{2} - \frac{2}{3}x^3)_0^{1/2} = \frac{1}{8} - \frac{1}{12} = \frac{1}{24} \ .$$

$$Mom_{y=0} = \int_0^1 y\frac{1-y}{2}dy = \frac{1}{2}(\frac{y^2}{2} - \frac{y^3}{3})_0^1 = \frac{1}{12} \ .$$

Thus the center of mass is at (1/6,1/3).

10. Find $\int_0^{\pi/2} \sqrt{\sin x}\cos x dx$

**Solution**. Let $u = \sin x, \ du = \cos x dx$. The integral is

$$\int_0^{\pi/2} \sqrt{\sin x}\cos x dx = \int_0^1 u^{1/2}du = \frac{2}{3} \ .$$