Calculus I
Practice Problems 13: Answers

1. Find the work done in pumping all the oil (whose density is 50 lbs. per cubic foot) over the edge of a cylindrical tank which stands on end. Assume that the radius of the base is 4 feet, the height is 10 feet and the tank is full of oil.

Answer. The slab of oil of thickness dh at a depth h has to be lifted a height h. The work to do this is $dW = 50(\pi 4^2)dh \cdot h$. Thus

$$W = \int_0^{10} 800\pi h dh = 800\pi \frac{h^2}{2}|_0^{10} = 40,000\pi$$

foot-pounds.

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2. John Brown has a parabolic cistern in the ground with a depth of 12 feet and a diameter at the top of 4 feet. This can be viewed as formed by revolving the curve $y = 3x^2$ around the y-axis, where the line $y = 12$ represents ground level. How much work does it take to pump out the cistern when it is full of water (the density of water is 62.5 lb/ft^3)?

Answer. The slab at the level y feet above the ground has to be lifted a distance of $12 - y$ feet. The weight of this cylindrical slab of height dy is $62.5\pi x^2 dy$, since the radius is x. Since $x^2 = y/3$, the work it takes to raise this slab is

$$dW = (12 - y)(62.5)\pi(y/3)dy = 65.45(12y - y^2)dy\ .$$

Now, integrate from 0 to 12 to find the total work:

$$Work = 65.45\int_0^{12}(12y - y^2)dy = 65.45(12\frac{12^2}{2} - \frac{12^3}{3}) = 18,850 \text{ ft/lbs}\ .$$

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3. A cylindrical reservoir of base radius 50 feet and height 15 feet is built 300 feet above the surface level of a lake. How much work is required to fill the reservoir with lake water (assuming the lake is large enough that its surface level does not change during this process)? Recall that the density of water is 62.5 lb/ft^3.

Answer. The slab of width dh and at height h from the base of the reservoir weights $62.5\pi(50)^2 dh$ lbs, and has been lifted $300 + h$ feet. The work done for this slab is thus $dW = 62.5\pi(50)^2(300 + h)dh = (4.9 \times 10^5)(300 + h)dh$ ft/lbs. Integrating from 0 to 15:

$$Work = (4.9 \times 10^5)\int_0^{15}(300 + h)dh = (4.9 \times 10^5)(300(15) + \frac{15^2}{2}) = 2.26 \times 10^9$$

ft-lbs or 214 ton-miles.

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4. A 2 lb. weight will extend a certain spring 5 inches. How much work is done in extending the spring 14 inches?

Answer. By Hooke's law, the force F exerted at an extension of x inches is $F = kx$, where k is the spring constant. The given information gives $2 = k(5)$, so $k = 2/5$. Thus for this spring $F = .4x$. Thus the work to extend it 14 inches is

$$Work = \int F dx = .4 \int_0^{14} x dx = .4(\frac{14^2}{2}) = 39.2$$

in-lbs or 3.267 ft-lbs.

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5. A 10 kg mass extends a spring 45 cm, to a new equilibrium position. The spring is then extended another meter and released. With what velocity does it pass the equilibrium position?

Answer. We will use the energy equation for the motion: $mv^2 + kx^2 = C$. First, we find the spring constant: when $F = (10)(9.8)$ newtons, $x = .45$ meters. Thus $k = F/x = 98/.45 = 217.8$ newtons/meter. At the moment of release, $v = 0$ and $x = 1$, so the energy equation gives $C = 217.8(1)^2 = 217.8$. Now, the equilibrium positon is at $x = 0$, so the energy equation now gives $10v^2 = 217.8$, so $v = 4.67$ m/sec.

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6. Find the center of mass of the homogeneous region in the first quadrant bounded by the curve $x^4 + y = 1$.

Answer. The region is given by $0 \le y \le 1 - x^4,\ 0 \le x \le 1$. Its mass is

$$\int_0^1 (1 - x^4) dx = (x - \frac{x^5}{5})|_0^1 = 4/5.$$

The moment about the y-axis is

$$\int_0^1 x(1 - x^4) dx = (\frac{x^2}{2} - \frac{x^6}{6})|_0^1 = 1/3.$$

The moment about the x-axis is

$$\int_0^1 y(1-y)^{\frac{1}{4}} dy$$

which we integrate by the substitution $u = 1 - y$,

$$= \int_0^1 (1-u)u^{\frac{1}{4}} du = \int_0^1 (u^{\frac{1}{4}} - u^{\frac{5}{4}}) du = 16/45$$

Thus the center of mass has the coordinates $(\frac{1}{3}/\frac{4}{5}, \frac{16}{45}/\frac{4}{5}) = (\frac{5}{12}, \frac{4}{9})$.

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7. Find the center of mass of the region bounded by the curves $y = x - x^3$ and $y = x - x^2$.

Answer. We find mass and moments by sweeping out along the x-axis from 0 to 1. At a point x, the length of the line segment between the two curves is $L(x) = x - x^3 - (x - x^2) = x^2 - x^3$. Thus the mass of a strip of width dx is $dM = L(x)dx = (x^2 - x^3)dx$. Thus

$$Mass = \int_0^1 (x^2 - x^3) dx = \frac{1}{12}\ .$$

The moment about the y-axis is

$$Mom_{\{x=0\}} = \int_0^1 x(x^2 - x^3) = \frac{1}{20}\ .$$

Thus the x-coordinate of the centroid is $(1/20)/(1/12) = .6$. To find the moment about the x-axis by sweeping out in the x-direction, we have to locate the mass of the strip at a point x at the midpoint $m(x)$ of the segment. Since $m(x) = (1/2)(x - x^3 + (x - x^2)$ we have

$$dMom_{\{y=0\}} = m(x)L(x)dx = \frac{1}{2}(2x - x^2 - x^3)(x^2 - x^3)dx\ ,$$

$$Mom_{\{y=0\}} = \frac{1}{2}\int_0^1 (2x^3 - 3x^4 + x^6)dx = \frac{3}{140}\ .$$

Thus the y coordinate of the centroid is $(3/140)/(1/12) = .257$. The centroid is at (.6,.257).