Calculus I
Practice Problems 3: Answers

1. A point moves around the unit circle so that the angle it makes with the x-axis at time t is $\theta(t) = (t^2 + t)\pi$. Let $(x(t), y(t))$ be the cartesian coordinates of the point at time t. What is dy/dt when $t = 3$?

Answer. $y(t) = \sin((t^2 + t)\pi)$, so

$$\frac{dy}{dt} = \cos((t^2 + t)\pi)(2t + 1)\pi \ .$$

Evaluating at $t = 3$: $dy/dt = \cos(10\pi)(2(3) + 1) = (2(3) + 1)\pi = 7\pi$.

---

2. Find the derivative: $f(x) = \sin x \cos x$

Answer. $f'(x) = \sin x(-\sin x) + \cos x \cos x = \cos^2 x - \sin^2 x$.

---

3. Find the derivative: $g(x) = \dfrac{\sin x}{\cos x}$

Answer. This is $f(x) = \tan x$, so its derivative is $f'(x) = \sec^2 x$. If you use the quotient rule, you get

$$f'(x) = \frac{\cos x \cos x - \sin x(-\sin x)}{\cos^2 x} = \frac{1}{\cos^2 x} \ .$$

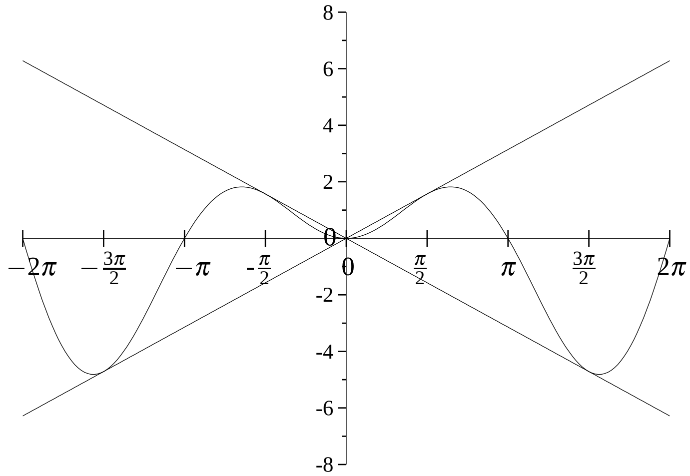
---

4. Let $f(x) = x \sin x$. Find the equation of the tangent line to the graph $y = f(x)$ at the points $x = (n + 1/2)\pi$ for any integer n.

Answer. The easy answer is to draw the graph and observe that the tangent line is $y = x$. See the graph. However, since the slope of the tangent line is given by the derivative, we calculate: $f'(x) = x \cos x + \sin x$, and evaluate at $x = (n + 1/2)\pi$, finding $f'((n + 1/2)\pi) = 1$. When $x = (n + 1/2)\pi$, we calculate that $y = (n + 1/2)\pi$ also, so the tangent line has the equation

$$\frac{y - (n + 1/2)\pi}{x - (n + 1/2)\pi} = 1 \ , \qquad \textit{or} \qquad y = (-1)^n x \ .$$

PSfrag replacements



---

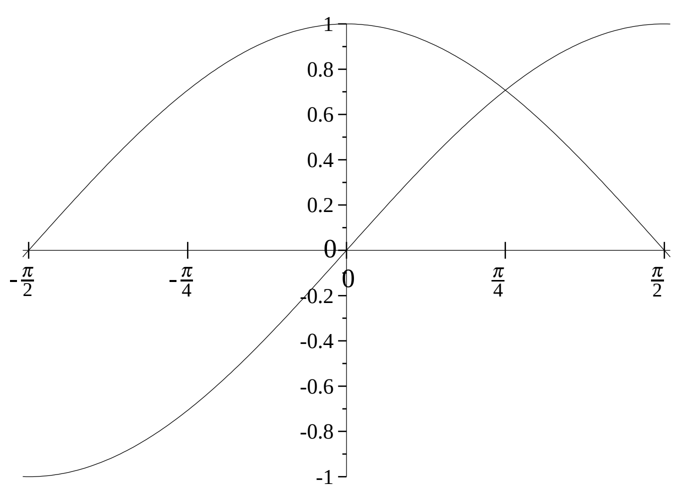
5. Consider the curves $C_1 : y = \sin x$ and $C_2 : y = \cos x$.
a) At which points x between $-\pi/2$ and $\pi/2$ do the curves have parallel tangent lines?
b) At which such points do they have perpendicular tangent lines?

Answer. At the point x, the tangents to the curves C_1 and C_2 have slope $\cos x$, $-\sin x$ respectively.

a) These lines are parallel if $\cos x = -\sin x$, or $\tan x = -1$, which has the solution $x = -\pi/4$.
b) These lines are perpendicular if $\cos x(-\sin x) = -1$, or $\sin x \cos x = 1$. But then

$$\sin(2x) = 2\sin x \cos x = 2$$

which has no solution: the curves never perpendicular tangent lines. Here are the graphs of the given curves.



PSfrag replacements

---

6. Differentiate: $f(x) = \dfrac{1+\tan x}{1-\tan x}$

Answer. Use the addition formula for the tangent: $f(x) = \tan(x+\pi/4)$. Then differentiate: $f'(x) = \sec^2(x+\pi/4)$. If you used the quotient rule, you probably ended up with

$$f'(x) = \frac{2\sec^2 x}{(1-\tan x)^2},$$

which is also the correct answer.

---

7. Let $y = x + 25x^{-1}$. Find an approximate value of y when $x = 3.2$.

Answer. If we start at $x = 3$, we find $y = 3 + 25/3 = 11.33$. Take the increment $dx = 0.2$, and now take differentials . Take the increment $dx = 0.2$ and now take differentials:

$$dy = dx - 25x^{-2}dx\ .$$

Substituting the values determined above: $dy = .2 - (25/9)(.2) = -.36$, so the approximate value of y is 11.33-.36 = 10.98. Note that at $x = 5$ we have dy = 0, so this technique will not work to approximate values of y for x near 5.

---

8. Find an approximate value of $\tan(0.26\pi)$.

Answer. Here we want to start at $x = \pi/4$, $y = 1$ and $dx = .01\pi$. We have $dy = \sec^2 x dx$, so at $x = \pi/4$, $dy = (\sqrt{2})^2(.01) = .02$. Thus the approximation to y is 1+.02=1.02.

---

9. Find the equation of the tangent line to $y = x^2(x^3-1)$ at (2,28).

Answer. Taking differentials,

$$dy = 2x(x^3-1)dx + x^2(3x^2dx)\ .$$

Since this gives the linear approximation to the graph, we get the equation of the tangent line by substituting $x = 2$, $dx = x-2$, $dy = y-28$:

$$y - 28 = (4)(7)(x-2) + 4(12)(x-2)$$

which simplifies to $y = 76x - 124$.

---

10. Find the equation of the tangent line to the curve $y = x\cos x$ at $(\pi/4, \pi\sqrt{2}/8)$.

Answer. The equation of differentials is $dy = -x\sin x dx + \cos x dx$. Substituting $x = \pi/4$, $dx = x - \pi/4$, $dy = y - \pi\sqrt{2}/8$:

$$y - \frac{\pi\sqrt{2}}{8} = \frac{\pi}{4}\frac{\sqrt{2}}{2}\left(x - \frac{\pi}{4}\right) + \frac{\sqrt{2}}{2}\left(x - \frac{\pi}{4}\right)$$

which simplifies to

$$y = \frac{\sqrt{2}}{2}\left(1 - \frac{\pi}{4}\right)x + \frac{\pi^2\sqrt{2}}{32}$$