Calculus I
Practice Problems 4: Answers

1. Find the the equation of the tangent line to the curve $y = x - x^{-2}$ at (2,7/4).

Answer. Here we get

$$dy = dx + 2x^{-3}dx$$

which leads to

$$y - \frac{7}{4} = x - 2 + \frac{2}{8}(x-2)$$

or $y = (5/4)x - 3/4$.

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2. Differentiate: $y = (x^2-1)\sin(x^2+1)$.

Answer.

$$\frac{dy}{dx} = 2x\sin(x^2+1) + (x^2-1)\cos(x^2+1)(2x) = 2x[\sin(x^2+1) + (x^2-1)\cos(x^2+1)]$$

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3. Find $f'(x)$: $f(x) = \frac{(x+1)^2}{(x-1)^2}$

Answer. Observe that

$$f(x) = \left[\frac{x+1}{x-1}\right]^2 = \left[1 + \frac{2}{x-1}\right]^2 .$$

Then

$$f'(x) = 2\left[1+\frac{2}{x-1}\right]\left[-\frac{2}{(x-1)^2}\right] = -4\frac{x+1}{(x-1)^3}$$

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4. Find $g'(x), g''(x)$: $g(x) = (x^3+1)^4$.

Answer. $g'(x) = 4(x^3+1)^3(3x^2) = 12x^2(x^3+1)^3$.

$$g''(x) = 24x(x^3+1)^3 + 12x^2(3)(x^3+1)^2(3x^2)$$

$$g''(x) = (x^3+1)^2[24x(x^3+1) + 108x^4]$$

$$g''(x) = (x^3+1)^2\left[132x^4+24x\right] = x\left(x^3+1\right)^2\left[132x^3+24\right]$$

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5. Find the derivative: $h(x) = (\cos(2x)+1)\sin(3x)$

Answer. $h'(x) = (\cos(2x)+1)\cos(3x)(3) + (-\sin(2x)(2))\sin(3x) = 3\cos(3x)(\cos(2x)+1) - 2\sin 3x \sin 2x$.

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6. Find the derivatives of the following functions:

a) $f(x) = \cos^2 x$

b) $g(x) = \dfrac{\sin^2 x}{\cos x}$

Answer. $f'(x) = 2\cos x(-\sin x) = -2\cos x \sin x$.

$$g'(x) = \frac{\cos x(2\sin x\cos x) - \sin^2 x(-\sin x)}{\cos^2 x} = \frac{2\cos^2 x\sin x + \sin^3 x}{\cos^2 x}$$
$$= \sin x(2+\tan^2 x)\ .$$

Alternatively, note that $g(x) = \sin x\tan x$, and find $g'(x) = \sin x(1+\sec^2 x)$.

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7. Find the first and second derivatives of $f(x) = x\sqrt{1-x^2}$

Answer. Rewrite using exponential notation: $f(x) = x(1-x^2)^{1/2}$. Use the product rule:

$$\frac{d}{dx}x(1-x^2)^{1/2} = x\frac{d}{dx}(1-x^2)^{1/2} + (1)(1-x^2)^{1/2}$$

Now use the chain rule to complete the differentiation. Here we take the intermediate variable to be $u = 1-x^2$:

$$\frac{d}{dx}x\left(1-x^2\right)^{1/2} = x\left(\frac{1}{2}\left(1-x^2\right)^{-1/2}(-2x)\right) + (1)\left(1-x^2\right)^{1/2} = \frac{-x^2}{\sqrt{1-x^2}} + \sqrt{1-x^2}$$

This simplifies when we put both terms over a common denominator:

$$f'(x) = \frac{1-2x^2}{\sqrt{1-x^2}}\ .$$

Now, we differentiate again for the second derivative:

$$f''(x) = \frac{(1-x^2)^{1/2}(-4x) - (1-2x^2)(1/2)(1-x^2)^{-1/2}(-2x)}{1-x^2}$$

Now, multiply numerator and denominator by $(1-x^2)^{3/2}$:

$$f''(x) = \frac{-4x+4x^3+x(1-2x^2)}{(1-x^2)^{3/2}} = \frac{x(2x^2-3)}{(1-x^2)^{3/2}}\ .$$

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8. Differentiate: $g(x) = (\sin(3x)+1)^3$.

Answer. $g'(x) = 3(\sin(3x)+1)^2(\cos(3x))(3)$, using the chain rule with $g(x) = u^3,\ u = -\sin v + 1,\ v = 3x$. Simplifying, we get $g'(x) = 9\cos(3x)(\sin(3x)+1)^2$.

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9. Differentiate: $h(t) = \dfrac{1-t^2}{1+t^3}$

Answer. Start with the quotient rule, and then simplify:

$$h'(t) = \frac{(1+t^3)(-2t) - (1-t^2)(3t^2)}{(1+t^3)^2} = \frac{t^4-3t^2-2t}{(1+t^3)^2}\ .$$

10. Differentiate: $f(x) = \sqrt{2x^2 - 3x + 1}$.

Answer. Here we first write $f(x) = (2x^2 - 3x + 1)^{1/2}$. Now with $u = 2x^2 - 3x + 1$, we use the chain rule:

$$f(x) = \frac{1}{2}(2x^2 - 3x + 1)^{-1/2}(4x - 3) \ .$$

11. Find the points on the curve $y = 3x^2 - 3x + 1$ whose tangent line is perpendicular to the line $x + 2y = 7$.

Answer. The slope of the given line is -1/2, so the sought for tangent lines must have slope 2. Differentiating: $y' = 6x - 3$; this is the slope of the tangent line. We set this equal to 2 and solve:

$$6x - 3 = 2 \ , \qquad 6x = 5 \ , \qquad x = 5/6 \ .$$

The corresponding value of y is $y = 3(5/6)^2 - 3(5/6) + 1 = 7/12$. The answer then is $(5/6, 7/12)$.

12. Consider the curves $C_1 : x^2 + y^2 = 1$, $C_2 : 2x^2 + y^2 = 2$ for y positive. For each x, the vertical line through $(x, 0)$ intersects the curves C_1, C_2 at the points (x, y_1), (x, y_2). Let $L(x)$ be the length of the line segment joining these two points. Find $L'(x)$.

Answer. From the given equations, we find $y_1 = \sqrt{1 - x^2}$, $y_2 = \sqrt{2 - 2x^2} = \sqrt{2}\sqrt{1 - x^2}$. Thus $L(x) = y_2 - y_1 = (\sqrt{2} - 1)(1 - x^2)^{1/2}$. Differentiating:

$$L'(x) = (\sqrt{2} - 1)\frac{1}{2}(1 - x^2)^{-1/2}(-2x) = -\frac{(\sqrt{2} - 1)x}{\sqrt{1 - x^2}} \ .$$

13. Let $\mathscr{P}$ be an upward-opening parabola whose axis is the y-axis and whose vertex is the origin. Suppose the line $y = C$ intersects the parabola in two points. Show that the tangent lines at these points intersect on the line on the axis of the parabola (the y-axis).

Answer. The equation of this parabola is $y = ax^2$ for some positive a. Also C must be positive for there to be points of intersection. These points are found by solving $C = ax^2$; they are $(\pm\sqrt{C/a}, C)$. The slope of the tangent line is found by differentiating: $y' = 2ax$, and setting $x = \pm\sqrt{C/a}$. We get $m = \pm 2a\sqrt{C/a} = \pm 2\sqrt{aC}$. The equations of the tangent lines at these two points are thus:

$$\frac{y - C}{x - \sqrt{C/a}} = 2\sqrt{aC} \ , \quad \frac{y - C}{x + \sqrt{C/a}} = -2\sqrt{aC} \ ,$$

which simplify to

$$y = 2\sqrt{aC}x - C \ , \qquad y = -2\sqrt{aC}x - C \ .$$

To find the point of intersection of these two lines, solve the equations simultaneously: we must have $2\sqrt{aC}x - C = -2\sqrt{aC}x - C$, which is possible only when $x = 0$; so the point of intersection is on the y-axis.

14. Suppose that a point moves along the x-axis according to the formula $x(t) = 1/(t^2 + 1)$. Let A(t) be the area of the circle with diameter joining the origin to the point $x(t)$. Find $A'(t)$ when $t = 3$.

Answer. The area of a circle of radius r is $A = \pi r^2$. Here $r = x(t)/2$. Thus

$$A(t) = \pi \left(\frac{1}{2} \frac{1}{t^2+1} \right)^2 = \frac{\pi}{4} \left(t^2+1\right)^{-2}$$

Then

$$A'(t) = \frac{\pi}{4} (-2) \left(t^2+1\right)^{-3} (2t) \ .$$

Evaluating at $t = 3$, we find

$$A'(t) = \frac{\pi}{4} (-2) \left(3^2+1\right)^{-3} (2(3)) = \frac{-3\pi}{1000} \ .$$