CHAPTER 1

Rate of Change, Tangent Line and Differentiation

§1.1. Newton's Calculus

Early in his career, Isaac Newton wrote, but did not publish, a paper referred to as the *tract of October 1666*. This was his sole work purely on mathematics, and contained the fundamental ideas and techniques of the calculus. While writing (1684-87) the *Principia Mathematica*, the fundamental exposition of his mathematical physics of "the system of the world", he reworked and expanded these ideas and included them as part of this treatise. Newton's central conception was that of objects in motion. To Newton motion is described by the position and velocity of the particle relative to a fixed coordinate system, as functions of time. These then are the fundamental variables: x, y, z, etc., and their velocities are denoted by $\dot{x}, \dot{y}, \dot{z}$, etc. Now, velocity is a measure of the rate of change of position (and acceleration, denoted $\ddot{x}$, etc., is a rate of change of velocity), and what was needed was a means to express this relationship, and a process of deriving relations among the various velocities and accelerations from given relations among the variables. This is the Calculus of Newton.

Calculus, as it is presented today starts in the context of two variables , or measurable quantities, x, y, which are related in the sense that values of one of the variables determine values of the other. A *function* $y = f(x)$ is a rule which specifies this relation between the *input* or *independent* variable x and the *output* or *dependent* variable y. This may be given by a formula, a table, or a computer algorithm; in fact, any set of rules which uniquely determine outputs for given inputs. The set of allowable values for the input variable is called the *domain* of the function, and the set of outputs, the *range*. In our context both the domain and the range are sets of real numbers.

An *interval* is the set of all real numbers between specified real numbers c and d. This is denoted by

$$(c,d) = \{x : c < x < d\} \tag{1.1}$$

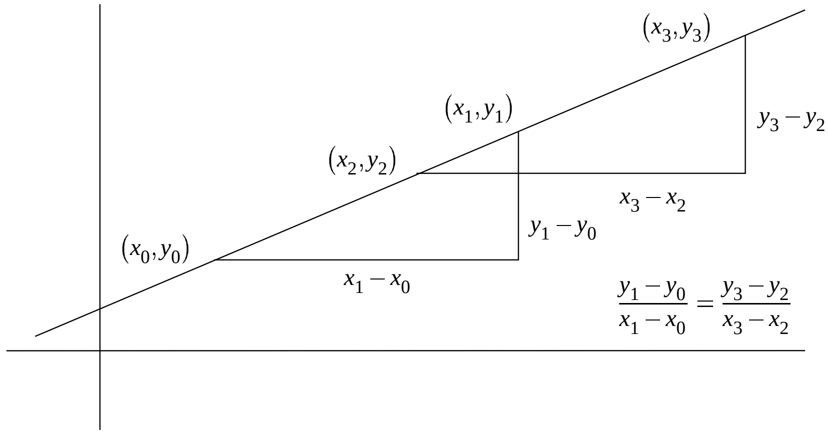
if neither endpoint is included, and by

$$[c,d] = \{x : c \le x \le d\} \tag{1.2}$$

if both are included. We shall say "I is an interval about a" to mean that a is between the endpoints of the interval I.

Now, suppose that $y = f(x)$ is a function defined for all x in an interval $I = (c,d)$. The **graph** of f is the set of all points (x,y), where x is in the interval (c,d) and $y = f(x)$. Calculus studies the behavior

Figure 1.1



of y as a function of x in terms of the way y changes as x changes. For x_0, x_1 points in the interval, and $y_0 = f(x_0)$, $y_1 = f(x_1)$, the ratio

$$\frac{\Delta y}{\Delta x} = \frac{y_1 - y_0}{x_1 - x_0} \tag{1.3}$$

is called the *average rate of change* of y with respect to x in the interval between x_0 and x_1. This is the ratio of the change in y (denoted Δy) with the change in x (denoted Δx).

Linear Functions

If the ratio (1.3) is the same for all points (x_0, y_0), (x_1, y_1) on the graph, we say that $y = f(x)$ is a **linear function** of x.This is because that condition is equivalent to saying that the graph of $y = f(x)$ is a straight line (which is easy to see using similar triangles; see figure 1.1).

For $y = f(x)$ a linear function, the ratio (1.3) is called the **slope** of the line, denoted m. Then another point (x, y) is on the line if and only if the calculation of (1.3) gives the slope. Thus, the condition for (x, y) to be on the line is

$$\frac{y - y_0}{x - x_0} = m \qquad \text{or} \qquad y - y_0 = m(x - x_0)\ . \tag{1.4}$$

This is called the **point-slope** equation of the line. If we wish to find the equation of the line through two points (x_0, y_0) and (x_1, y_1), we use those points to find the slope and then use the above equation. Thus, the condition for (x, y) to be on the line through these points is

$$\frac{y - y_0}{x - x_0} = \frac{y_1 - y_0}{x_1 - x_0}\ . \tag{1.5}$$

This is the **two-point equation** for the line.

Example 1.1 Find the equation of the line through the points (2,-1), (3,7). Then find the line through (6,-1) of the same slope.

Using (1.5) with $(x_0,y_0) = (2,-1)$ and $(x_1,y_1) = (3,7)$, we have that (x,y) is on the line when

$$\frac{y-(-1)}{x-2} = \frac{7-(-1)}{3-2} \quad \text{or} \quad \frac{y+1}{x-2} = \frac{8}{1}\,, \tag{1.6}$$

giving us the equation $y = 8x - 17$. This line has slope $m = 8$. Thus the line through (6,-1) of the same slope has the equation

$$\frac{y-(-1)}{x-6} = 8 \quad \text{or} \quad y = 8x - 49\,. \tag{1.7}$$

Example 1.2 Is P(5,12) on the line joining Q(2,7) and R(8,15)?

The slope of the line through Q and R is $(15-7)/(8-2) = 4/3$. The slope of the line through P and Q is $(12-7)/(5-2) = 5/3$. Since these two lines do not have the same slope, they cannot be the same line. Thus P is not on the line through Q and R.

Here are some facts about lines which will be useful when studying more general curves.

a. If L is a line of slope m, then

$$m = \tan\theta \tag{1.8}$$

where θ is the angle that L makes with a horizontal line. If the line is vertical, then L has *infinite* slope. Suppose we are given two lines: L_1 of slope m_1, and L_2 of slope m_2.Then

b. L_1 and L_2 are *parallel* if and only if $m_1 = m_2$.

c. L_1 and L_2 are *perpendicular* if and only if $m_1 m_2 = -1$.

d. The length of of the line segment between two points $P(x_1,y_1)$ and $Q(x_2,y_2)$ (the *distance* between the two points) is

$$|PQ| = \sqrt{(x_1-x_2)^2 + (y_1-y_2)^2}\,. \tag{1.9}$$

Example 1.3 Find the equation of the line through (2,3) which is perpendicular to the line $L:\ 2x+3y = 11$.

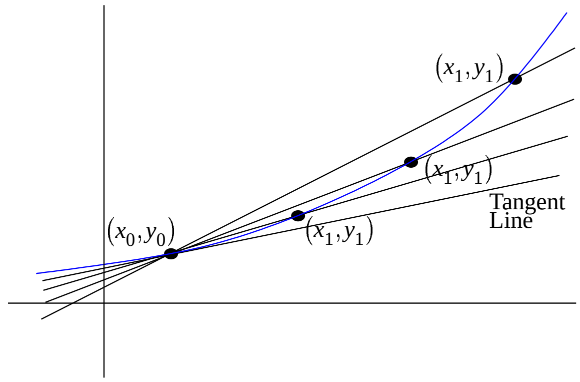
The line L has slope $m = -2/3$. Thus the line perpendicular to L has slope $-1/(-2/3) = 3/2$. Thus the equation of the line we seek is

$$\frac{y-3}{x-2} = \frac{3}{2} \quad \text{or} \quad y = \frac{3}{2}x\,. \tag{1.10}$$

Polynomial functions

For the general curve given by the equation $y = f(x)$, the ratio (1.3): $\Delta y/\Delta x$ is the slope of the line joining the two points (x_0,y_0) and (x_1,y_1) on the graph of f. But now, if the graph is not a line, this ratio changes as the point (x_1,y_1) moves. As x_1 approaches x_0, this ratio may approach a specific number. If it does, this number is called the **derivative** of y with respect to x, evaluated at the point x_0. It is the *instantaneous rate of change* of y with respect to x at x_0, and also the slope of the line which best approximates the curve at (x_0,y_0), called the *tangent line* to the curve (see Figure 1.2).

Figure 1.2



In this chapter we shall concentrate on finding the derivative of functions given by a formula; this process is called **differentiation**. It turns out to be quite simple for polynomial functions. But first, we make this discussion explicit.

Definition 1.1 *Let $y = f(x)$ be a function defined for all values of x in an interval about the point a. If the difference quotient*

$$\frac{f(x)-f(a)}{x-a} \tag{1.11}$$

approaches a specific number L, then we say that f is **differentiable** *at a, and the number L is called the* **derivative** *of f at a, denoted $f'(a)$. It is the slope of the* **tangent line** *of $y = f(x)$ at a.*

Example 1.4 Consider $f(x) = x^2$, Find the tangent line to this curve at the point $(a, f(a))$.

We take a point $(x, f(x))$ near $(a, f(a))$ and calculate the difference quotient

$$\frac{f(x)-f(a)}{x-a} = \frac{x^2-a^2}{x-a} = \frac{(x-a)(x+a)}{x-a} = x+a\,. \tag{1.12}$$

Clearly, as x approaches a, $x+a$ approaches $2a$, so we get $f'(a) = 2a$. Since this is true for any value a, we can conclude that if $f(x) = x^2$, its derivative is $f'(x) = 2x$.

Example 1.5 Find the equation of the line tangent to the curve $y = x^2$ at the point (3,9).

For $x = 3$, the derivative is $f'(3) = 2(3) = 6$. Thus the tangent line has the equation

$$\frac{y-9}{x-3} = 6 \quad \text{or} \quad y = 6x-9\,. \tag{1.13}$$

The ease with which we calculated the derivative for $y = x^2$ followed from simple algebraic facts. We shall see that this works in general for polynomials; but first, one more example:

Example 1.6 If $f(x) = x^3$, $f'(x) = 3x^2$.

Fix a point (a,a^3) on the graph, and let $\left(x,x^3\right)$ be a nearby point. We look at the slope of the line joining these points:

$$\frac{\Delta y}{\Delta x} = \frac{x^3-a^3}{x-a} \ . \tag{1.14}$$

Since the quotient of x^3-a^3 by $x-a$ is x^2+ax+a^2 this can be rewritten as

$$\frac{x^3-a^3}{x-a} = \frac{(x-a)(x^2+ax+a^2)}{x-a} = x^2+ax+a^2 \ , \tag{1.15}$$

and evaluating this at a, we get $f'(a)=3a^2$.

Now, for any polynomial $y=f(x)$, this process will work: divide $f(x)-f(a)$ by $x-a$, and evaluate the quotient at $x=a$ to calculate the derivative. Let's spell this out, starting with the division theorem of algebra:

Theorem 1.1 *Let f be a polynomial of degree d. Then, for any number a when we divide $f(x)$ by $x-a$, we get a quotient which is a polynomial of degree $d-1$ and a remainder of $f(a)$:*

$$\frac{f(x)}{x-a} = q(x) + \frac{f(a)}{x-a} \ . \tag{1.16}$$

Now, to apply this to the calculation of instantaneous rate of change, move the second term on the right to the left:

$$\frac{f(x)-f(a)}{x-a} = q(x) \ . \tag{1.17}$$

Now, as we let x approach a, the difference quotient $q(x)$ approaches $q(a)$, so the polynomial is differentiable at a, and its derivative is $q'(a)$.

Theorem 1.2 *A polynomial $y=f(x)$ is everywhere differentiable. Its derivative at $x=a$ is $q(a)$, where q is the quotient of $f(x)-f(a)$ by $x-a$.*

Now, Newton realized that using long division would be a tedious way to calculate derivatives, and with them instantaneous rates of change, so he had the genius to take a slightly more abstract approach to lead to an automatic way of calculating derivatives. First, we must make explicit what we mean by the phrase "the expression approaches a specific number" by introducing the notion of *limit*. Suppose that $y=g(x)$ defines a function in an interval about x_0. We say that g has the limit L as x approaches x_0 if we can make the difference $|g(x)-L|$ as small as we need it to be by taking x as close to x_0 as we have to. More precisely, but less intuitively,

Definition 1.2 $\lim_{x\to x_0} g(x) = L$ *if, for any $\varepsilon>0$ there is a $\delta>0$ such that $|x-x_0|<\delta$ implies $|g(x)-L|<\varepsilon$.*

We now observe that limits behave well under algebraic operations.

Proposition 1.2 *Suppose that f and g are functions defined in an interval about x_0 and that*

$$\lim_{x\to x_0} f(x) = L\ , \quad \lim_{x\to x_0} g(x) = M. \tag{1.18}$$

Then a)

$$\lim_{x\to x_0} (f(x) + g(x)) = L + M \tag{1.19}$$

b)

$$\lim_{x\to x_0} (f(x)\cdot g(x)) = L\cdot M \tag{1.20}$$

c) If $M \neq 0$, then

$$\lim_{x\to x_0} \frac{f(x)}{g(x)} = \frac{L}{M}\ . \tag{1.21}$$

Applying this proposition to the calculation of derivatives, we see how differentiation behaves under algebraic operations:

Proposition 1.3 *Suppose that f and g are functions defined and differentiable in an interval I. Then*

a) $f+g$ is differentiable in I, and $(f+g)' = f' + g'$.

b) fg is differentiable in I, and $(fg)' = f'g + fg'$.

We give a brief justification of these rules, which follow from the corresponding rules for limits (proposition 1.3). This is straightforward for part a), but for the product, the argument requires some preliminary algebraic manipulation. Suppose then, that f and g are differentiable at a, and let $h = fg$. Then to see that h is differentiable, we must take the limit, as x approaches a of

$$\frac{h(x)-h(a)}{x-a} = \frac{f(x)g(x)-f(a)g(a)}{x-a}\ . \tag{1.22}$$

The product rule for limits does not apply directly, for this is not a product. However, if we add and multiply $f(a)g(x)$, we get

$$f(x)g(x) - f(a)g(a) = [(f(x)-f(a))g(x)] + [f(a)(g(x)-g(a))]\ , \tag{1.23}$$

which leads to a sum of products

$$\frac{f(x)g(x)-f(a)g(a)}{x-a} = g(x)\frac{f(x)-f(a)}{x-a} + f(a)\frac{g(x)-g(a)}{x-a}\ . \tag{1.24}$$

Now, we can take the limits using proposition 1.3. We have to note that, since g is differentiable, we also have $\lim_{x\to a} g(x) = g(a)$.

Finally this brings us to the rule for differentiating polynomials.

Proposition 1.4

a) If f is constant, then $f' = 0$.

b) If $f(x) = ax^n$ for some positive integer n, then $f'(x) = anx^{n-1}$.
c) A polynomial is differentiated term by term, using b) for each term.

To verify a), we only have to note that a constant function is unchanging; its graph is a horizontal line, so has slope 0. c) follows from the fact that the limit of a sum is the sum of the limits. b) follows by a bootstrap method. We have already seen this for $n = 0, 1, 2, 3$. To proceed, we use the product rule. For example, take $n = 4$:

$$(x^4)' = (x^3x)' = (x^3)'x + x^3x' = (3x^2)x + x^3(1) = 4x^3 \ . \tag{1.25}$$

If we have the proposition for all integers up to $n-1$, then we have it for n by the same method:

$$(x^n)' = (x^{n-1}x)' = ((n-1)x^{n-2})x + x^{n-1}(1) = nx^{n-1} \ . \tag{1.26}$$

Example 1.7 Let $f(x) = 2x^2 - 3x + 3$. Find $f'(x)$. What is the equation of the line tangent to the curve given by $y = f(x)$, at the point (1,2)?

Using proposition 1.4, we have

$$f'(x) = 2(2x) - 3(1) + 0 = 4x - 3 \ . \tag{1.27}$$

This gives the slope of the tangent line at (1,2) by evaluating at $x = 1$: $f'(1) = 4(1) - 3 = 1$. Thus the equation of the tangent line is

$$\frac{y-2}{x-1} = 1 \quad \text{or} \quad y = x + 1 \ . \tag{1.28}$$

Example 1.8 If

$$f(x) = 2x^5 - x^4 + 8x^3 + 2x - 5 \ , \tag{1.29}$$

then

$$f'(x) = 2(5x^4) - 4x^3 + 8(3x^2) + 2x^0 - 0 = 10x^4 - 4x^3 + 24x^2 + 2 \ . \tag{1.30}$$

If a function f is differentiable at every point of an interval I, then the derivative is defined at every point in the interval I, and thus is a function on I. This function, denoted f', is defined by the rule: for all x in I,

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} \ . \tag{1.31}$$

In particular, since f' is now a function on I, it too may be differentiable. If so, its derivative is denoted $f''(x)$, and is called the **second derivative of** f with respect to x. Proceeding, we can define third and fourth derivatives and so forth.

So far, we have been interpreting the derivative as the instantaneous rate of change of y with respect to x, or the slope of the tangent line. Another fundamental interpretation is in terms of motion. Consider an object moving along a straight line. Let the variable t represent time, and x the displacement of the object from a fixed point, 0, on the line. Then the position of the object at time t is given by a function $x = x(t)$. The **velocity** (denoted at time t as $v(t)$) of the object is the instantaneous rate of change of x with respect to t. The **acceleration** of this object (denoted $a(t)$) is the instantaneous rate of change of v with respect to t. Thus, if $v(t)$ is the velocity of the object at time t, and $a(t)$ its acceleration, we have

$$v(t) = x'(t)\ , \qquad a(t) = v'(t) = x''(t)\ . \tag{1.32}$$

Example 1.9 Suppose an object is moving in a straight line so that its displacement at time t is given by $x(t) = 4t^2 + 12t$. Find the formulas for the velocity and acceleration of this object. What are the velocity and acceleration at time $t = 5$?

Differentiating, we find that $v(t) = x'(t) = 8t + 12$, $a(t) = x''(t) = 8$. Thus the velocity at time $t = 5$ is $v(5) = 8(5) + 12 = 52$, and the acceleration is $a(5) = 8$.

In many physical problems, an object is moving at constant acceleration. For example, an object falls near the surface of the earth at an acceleration of 32 ft/sec^2 downward (or 9.8 m/sec^2 downward, in the metric system). If the acceleration is constant, that tells us that ratio of the change in velocity over the change in time is constant: that is, the velocity is a linear function of time. Similarly, since the velocity is a linear function, the distance traveled must be given by a quadratic function; all we have to do is to use the given data to find the coefficients. We conclude:

Proposition 1.5 *Suppose an object moves at constant acceleration a. Then, if at time $t = 0$ the object is at position x_0 and has velocity v_0, then at any time we have*

$$x(t) = \frac{a}{2}t^2 + v_0 t + x_0\ , \qquad v(t) = at + v_0\ . \tag{1.33}$$

It is easy to check that these functions do have the desired properties, that is, that $x'(t) = v(t)$ and $v'(t) = a$, and that their values at $t = 0$ are as given. Furthermore, we can argue intuitively, as we have done above, that these are the precise formulas for distance and velocity. For, since the rate of change of velocity is constant, velocity must be linear, and since the rate of change of distance is linear, it must be quadratic. This was the way Newton argued; but there are loose ends as was pointed out very articulately by Newton's contemporary, George Berkeley. Why indeed, are these the only formulas with the desired properties? How do we know that there does not exist some as yet unknown mysterious functions which have the same values at $t = 0$, and the given acceleration? The third book of Newton's Principia gives formidable evidence that no such mysteries exist, and that work, together with much subsequent experimental evidence, carried the day. But Berkeley's objections were valid on logical grounds, and the issue was not satisfactorily resolved until the nineteenth century.

Example 1.10 An object is project upward at an initial velocity of 48 ft/sec. How high does it go?

We measure distance upward from the starting point, so that $x_0 = 0$ and $v_0 = 48$. The acceleration due to gravity is $a =$ -32 ft/sec, so (by proposition 1.5), the equations of motion (1.33) are

$$x(t) = -16t^2 + 48t\ , \qquad v(t) = -32t + 48\ . \tag{1.34}$$

If we complete the square for $x(t)$, we have

$$x(t) = -16(t - 3/2)^2 + 36\ . \tag{1.35}$$

Thus the greatest value of x is achieved at $t = 3/2$, and is $x(3/2) = 36$ feet. Note that at this highest point, $v(3/2) = 0$, confirming our intuition that at the moment the object turns around its velocity must be zero.

Example 1.11 An automobile is traveling at 60 mph. At what rate must it decelerate so as to stop in 100 yards?

Converting everything to feet and seconds, we have an initial velocity of 88 ft/sec, and we can take $s(0) = 0$. At some future time T, we have $s(T) = 300$ feet, $v(T) = 0$. Call the rate of deceleration a. The equations of motion 1.33 are

$$x(t) = -\frac{a}{2}t^2 + 88t\ , \qquad v(t) = -at + 88\ . \tag{1.36}$$

At time T we have $300 = -(a/2)T^2 + 88T$, $0 = -aT + 88$. From the second we get $T = 88/a$; putting that in the first we get

$$300 = -\frac{a}{2}\frac{88^2}{a} + 88\frac{88}{a} \quad \text{or} \quad 300 = 88^2\left(-\frac{1}{2a} + \frac{1}{a}\right) \quad \text{or} \quad 300 = \frac{88^2}{2a}\ , \tag{1.37}$$

so $a = 12.91$ ft/sec^2.

§1.1.1 More rules of differentiation

Eventually we will develop a full set of rules for finding the derivative of any function given by a formula. We turn now to the *quotient rule* to handle quotients of polynomials (called *rational functions*).

Proposition 1.6 *Suppose that f and g are differentiable at a point a, and $g(a) \neq 0$. Then $1/g$ and $h = f/g$ are differentiable at a, and*

$$\left(\frac{1}{g}\right)' = -\frac{g'}{g^2}, \qquad h' = \frac{gf' - fg'}{g^2}\ . \tag{1.38}$$

To show that $1/g$ is differentiable, we must calculate the limit as $x \to a$ of

$$\frac{\frac{1}{g(x)} - \frac{1}{g(a)}}{x - a}\ . \tag{1.39}$$

Once again, a little algebra helps us. Simplifying the compound fraction, we get

$$\frac{1}{x-a} \cdot \frac{g(a) - g(x)}{g(x)g(a)} = \frac{-1}{g(x)g(a)}\frac{g(x) - g(a)}{x - a}\ , \tag{1.40}$$

which has as its limit

$$\left(\frac{1}{g}\right)'(a) = -\frac{g'(a)}{(g(a))^2}\ . \tag{1.41}$$

Now the second equation of (1.38) follows from this and the product rule applied to f/g considered as $f\cdot(1/g)$.

(1.42)
$$\left(\frac{f}{g}\right)' = f\left(\frac{1}{g}\right)' + f'\left(\frac{1}{g}\right) = f\left(\frac{-g'}{g^2}\right) + f'\left(\frac{1}{g}\right) = \frac{gf' - fg'}{g^2}\ .$$

In particular, we have

(1.43)
$$\frac{d}{dx}\left(\frac{1}{x}\right) = -\frac{1}{x^2}\ .$$

Proposition 1.7 *Let n be any integer, positive, zero, or negative. Then*

(1.44)
$$\text{for}\quad f(x) = x^n \qquad \text{we have} \qquad f'(x) = nx^{n-1}\ .$$

By proposition 1.4b, this is true for n positive or zero. For negative exponents, we apply the quotient rule to $f(x) = 1/x^n$ with n positive:

(1.45)
$$f'(x) = -\frac{nx^{n-1}}{(x^n)^2} = (-n)x^{-n-1}\ ,$$

which is just (1.44) for the negative exponent $-n$.

Example 1.12 Find the derivative of $f(x) = x^2 - 2x + \frac{3}{x} - \frac{5}{x^2}$.

Rewrite the function in exponential notation: $f(x) = x^2 - 2x + 3x^{-1} - 5x^{-2}$. Now use (1.44): $f'(x) = 2x - 2 + 3(-x^{-2}) - 5(-2x^{-3})$, which can be rewritten as

(1.46)
$$f(x) = 2x - 2 - \frac{3}{x^2} + \frac{10}{x^3}\ .$$

Example 1.13. Let $f(x) = 30x + 2x^{-1}$. For what value of x is $f'(x) = 0$?

Differentiate: $f'(x) = 30 - 2x^{-2}$. Now solve $f'(x) = 0$:

(1.47)
$$0 = 30 - \frac{2}{x^2} \quad \text{so that} \quad x^2 = 15$$

and the answer is $x = \pm\sqrt{15}$.

Finally, we see how to differentiate the square root:

Proposition 1.8 *If* $f(x) = \sqrt{x}$ *for* $x > 0$, *then* $f'(x) = 1/(2\sqrt{x})$.

Here we use the fact that $x - a = (\sqrt{x} - \sqrt{a})(\sqrt{x} + \sqrt{a})$. Thus

(1.48)
$$\frac{\sqrt{x} - \sqrt{a}}{x - a} = \frac{\sqrt{x} - \sqrt{a}}{(\sqrt{x} - \sqrt{a})(\sqrt{x} + \sqrt{a})} = \frac{1}{\sqrt{x} + \sqrt{a}} \to \frac{1}{2\sqrt{a}}$$

as $x \to a$, for $a \neq 0$.

§1.2. Leibniz' Calculus of Differentials

Up to this point we have been following the development of the Calculus according to Newton. We have been considering variables y, z, u, v, etc. as functions of a particular variable (called the "independent variable") x, and discovering how to find rates of change of the dependent variables relative to the independent variable.

The ideas of Leibniz follow a different, but equivalent, set of ideas. Leibniz is concerned with a collection of variables $x, y, z, u, v, etc.$ and their "infinitesimal increments". This is a hard concept to get a hold on, but we can think of it this way. When we actually make measurements, we always have in mind, even if unspecified, an "error bar"; that is, a largest allowable error. Thus, our calculators display numbers to 8 decimal points, allowing for a "negligible" error of at most 10^{-8}. A more efficient computer has a smaller error bar, perhaps 10^{-32}, or 2^{-128}. Instruments of measurement, no matter how delicate, have to allow for such an error bar. So, if, say u is a measurable variable, it comes equipped with an error bar: an allowable increment in a measurement which does not change the accepted value of the measurement. It is this which we should call the "infinitesimal increment" in u, called by Leibniz the *differential* of u, denoted du. However, the important feature of this concept is that it is not tied down to the level of accuracy of today's instruments, but it represents the error bar for all time: du stands for the smallest measurable increment for all ways of measuring ever to come.

We get a more concrete interpretation of the differential by relating it to the linear approximation of the variables. More precisely, suppose the variables x, y are related by $y = f(x)$. The tangent line at a point (x_0, y_0) is the line which best approximates the curve. We have used the symbols $\Delta x, \Delta y$ to represent changes in the variables x and y along the curve; now we let dx, dy represent changes in the variables along the tangent line. Since the slope of the tangent line at (x_0, y_0) is $f'(x_0)$, we obtain this important relationship between the differentials: at the point $(x_0, f(x_0))$,

$$\frac{dy}{dx} = \lim_{\Delta x \to 0} \frac{f(x) - f(x_0)}{x - x_0} = f'(x_0), \tag{1.49}$$

or, without specifying the particular point,

$$dy = f'(x)\, dx. \tag{1.50}$$

This we can interpret as the equation of the tangent line, by replacing dx and dy by $x - x_0$ $y - y_0$.

Example 1.13 Find the equation of the tangent line to the curve $y = x^3 - 2x + 5$ at the point $(2, 9)$. First, we calculate the relation between the differentials:

$$dy = \left(3x^2 - 2\right) dx. \tag{1.51}$$

At $x = 2$, this gives $dy = 10dx$. Now we interpret this as the equation of the tangent line by replacing dy by $y - 9$ and dx by $x - 2$. The equation of the tangent line is thus

$$y - 9 = 10(x - 2) \qquad \textit{or} \qquad y = 10x - 11. \tag{1.52}$$

Finally, considering the equation $dy = f'(x)\, dx$ as the linear approximation to the equation $y = f(x)$ (at a particular point), we can make preliminary estimates of the change in y, given a change in x.

Example 1.14 The volume of a sphere of radius r is $V = (4/3)\pi r^3$. Suppose the surface of a sphere of radius 6 feet is covered by a 1 inch coat of paint. How much paint will be needed? From the defining equation we have $dV = 4\pi r^2 dr$; so letting $r = 6$ feet and $dr = 1/12$ feet, we can estimate the change in volume to be

$$dV = 4\pi(6)^2(\frac{1}{12}) = 37.7 \quad cu.ft. \tag{1.53}$$

Thinking of the derivative as the ratio of two quantities which eventually become zero has its philosophical problems, and was also subjected to the scathing criticism of Berkeley. He also objected to Newton's methods on the same ground: when we write

$$\frac{f(x)-f(a)}{x-a} = \frac{x^2-a^2}{x-a} = \frac{(x-a)(x+a)}{x-a} = x+a, \tag{1.54}$$

by what right are we now able to let x become a? If we did so one step sooner, we'd be dividing by zero, which is forbidden. So, in this set of equations x cannot be a. But in the next line we say, "let x be a"! These philosophical obstacles were eventually overcome; we shall proceed without resolution, as did Newton, Leibniz and their successors to enormous effect. Suffice it to say that this can all be put on a logical footing, while at the same time, the concept of differential as "smallest possible increment" is a powerful intuitive tool throughout mathematics and its applications. For example, we can easily give a heuristic derivation of the law of differentiation for composite functions.

§1.3. The Chain Rule

Suppose that y, u and x are variables such that u is a function of x: $u = f(x)$, and y is a function of u: $y = g(u)$. Then y can be viewed as a function of x by writing $y = h(x) = f(g(x))$. How do we find the rate of change of y with respect to x? Using differentials, we have: $dy = f'(u)du$, and $du = g'(x)dx$, so that $dy = f'(u)g'(x)dx$, it being understood that in this formula u is to be expressed in terms of x. A shorthand for this is

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}. \tag{1.55}$$

Example 1.15 Let $y = (4x+1)^3$. We introduce the intermediate variable $u = 4x+1$, so that $y = u^3$. Then $dy = 3u^2 du$, and $du = 4dx$, so that

$$\frac{dy}{dx} = 3u^2(4) = 12(4x+1)^2. \tag{1.56}$$

Example 1.16 If $y = x^{-n}$, we introduce $u = x^n$, so that $y = u^{-1}$. Then

$$\frac{d}{dx}(x^{-n}) = \frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx} = -\frac{1}{u^2}nx^{n-1} = -\frac{nx^{n-1}}{(x^n)^2} = -nx^{-n-1}, \tag{1.57}$$

giving another derivation of proposition 1.8.

Example 1.17 Let $y = ((2x+1)^3+5)^2$. Here we need to use the chain rule more than once. We think of $y = v^2$, where $v = u^3+5$, and $u = 2x+1$. Then

$$\frac{dy}{dx} = \frac{dy}{dv}\frac{dv}{du}\frac{du}{dx} = 2v(3u^2)(2) = 6((2x+1)^3+5)(2x+1)^2. \tag{1.58}$$

The statement of the chain rule is as follows.

Proposition 1.9 *Suppose that g is differentiable at the point a, and f is differentiable at $g(a)$. Then the composed function $h(x) = f(g(x))$ is differentiable at a and*

$$h'(a) = f'(g(a))g'(a). \tag{1.59}$$

In particular,

Proposition 1.10 *If f is differentiable at a, and n is any positive or negative integer, $h(x) = (f(x))^n$ is also differentiable at a and*

$$h'(x) = n(f(x))^{n-1}f'(x). \tag{1.60}$$

Of course, the Leibniz formulation 1.55 is easier to remember and apply than proposition 1.10. For that reason we shall begin to adopt the Leibniz notation for differentiation: if $y = f(x)$ is differentiable in an interval I, we write

$$f'(x) = \frac{dy}{dx} \tag{1.61}$$

and use $f'(x)$ and dy/dx interchangeably. The notation for higher derivatives is:

$$f''(x) = \frac{d^2y}{dx^2}, \qquad f'''(x) = \frac{d^3y}{dx^3}, \qquad etc. \tag{1.62}$$

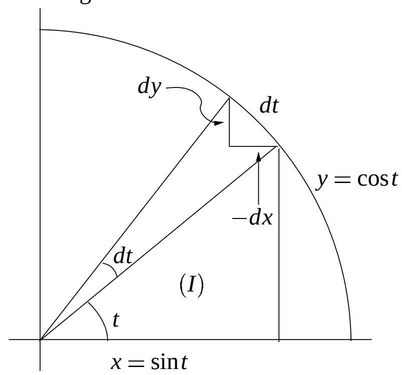
In this notation, Proposition 1.10 becomes simply

$$\frac{d}{dx}(y^n) = ny^{n-1}\frac{dy}{dx}. \tag{1.63}$$

§1.4. Trigonometric Functions

Consider a particle moving in the counterclockwise direction around the circle of radius 1 with constant angular velocity of 1 radian/second such that at time $t = 0$ it is at the point $(1,0)$. Then its position at time t is $(\cos t, \sin t)$. These functions are defined for all values of t, and are periodic of period 2π since in time 2π the particle will make one full circuit of the circle.

Figure 1.3: Pascal's derivation.



There are four other trigonometric functions defined by the equations

$$\tan t = \frac{\sin t}{\cos t}, \quad \cot t = \frac{\cos t}{\sin t}, \quad \sec t = \frac{1}{\cos t}, \quad \csc t = \frac{1}{\sin t}. \tag{1.64}$$

Assuming that these functions are differentiable, we can calculate the derivatives by an argument using differentials due to Blaise Pascal. In figure 1.3 we have located the moving point at $P = (\cos t, \sin t)$ at time t, and its position Q after an infinitesimal increment dt. Since we are on the unit circle t also measures arc length along the circle. The triangle with sides dx, dy, dt is called the "differential triangle". It may be of concern that dt represents an arc of the circle, but, remember, at the differential level an arc and a straight line are indistinguishable. Since the tangent line to the circle is perpendicular to the radius at the point P, the differential triangle is similar to the triangle (I) in figure 1.3.

Thus

$$\frac{-dx}{\sin t} = \frac{dt}{1}, \quad \frac{dy}{\cos t} = \frac{dt}{1}, \tag{1.65}$$

so

$$\frac{dx}{dt} = -\sin t, \quad \frac{dy}{dt} = \cos t. \tag{1.66}$$

Since $x = \cos t, y = \sin t$, we obtain the first part of the following.

Proposition 1.11

a) $\frac{d}{dt}(\sin t) = \cos t \qquad \frac{d}{dt}(\cos t) = -\sin t,$

b) $\frac{d}{dt}(\tan t) = \sec^2 t \qquad \frac{d}{dt}(\cot t) = -\csc^2 t,$

c) $\frac{d}{dt}(\sec t) = \sec t \tan t \qquad \frac{d}{dt}(\csc t) = -\csc t \cot t.$

b) and c) follow from the quotient rule. For example, b):

$$\frac{d}{dt}(\tan t) = \frac{d}{dt}\frac{\sin t}{\cos t} = \frac{\cos t \cos t - \sin t(-\sin t)}{\cos^2 t} = \frac{1}{\cos^2 t} = \sec^2 t. \tag{1.67}$$

Remember, that in the the above discussion we have assumed that the trigonometric functions were differentiable, and it was that assumption that allowed us to consider an arc of a (differential) circle as a straight line. These formulae imply the following limit results, just by the definition of the derivative.

Proposition 1.12

$$\lim_{x\to 0}\frac{\sin x}{x}=1, \qquad \lim_{x\to 0}\frac{\cos x-1}{x}=0. \tag{1.68}$$

These can be proven directly by geometric methods.

§1.5. Implicit Differentiation and Related Rates

Suppose that x and y are variables which are related by a functional equation: $F(x,y)=c$, a constant. We say that this relation defines y **implicitly** as a function of x. For, in principle, given a value of x, say $x=a$, we can solve the equation $F(a,y)=c$ for y, giving the "rule" defining y in terms of x. However, to find dy/dx we need not solve this equation. When y and x are so related, their differentials are related as well, and the chain rule can be used to find that relationship, as a function of both x and y. The idea is to think of $z=F(x,y)$ as another variable which, because of the relation $F(x,y)=0$ is constant, so $dz=0$. Now, apply the chain rule to the expression for z.

Example 1.18 Suppose that the variables x and y satisfy the relation

$$x^2-xy+2y^2=4. \tag{1.69}$$

Letting z represent this defining relation, we have $dz=0$. Now, using 1.31 and the rules for differentiation,

$$dz=2xdx-(xdy+ydx)+4ydy=0, \tag{1.70}$$

giving us

$$(-x+4y)dy=(-2x-y)dx, \tag{1.71}$$

leading to this expression for the derivative:

$$\frac{dy}{dx}=\frac{2x-y}{4y-x}. \tag{1.72}$$

Example 1.19 What is the equation of the tangent line to the curve given by 1.69 at the point (2,1)? We find the slope by substituting the values $x=2,y=1$ in equation 1.71:

$$m=\frac{dy}{dx}=\frac{2(2)+1}{4(1)-2}=\frac{5}{2}. \tag{1.73}$$

Then the equation of the line is

$$y-1=\frac{3}{2}(x-2) \qquad \textit{or} \qquad y=\frac{3}{2}x-4. \tag{1.74}$$

Notice that, if we substitute $x = 2, y = 1, dy = y - 1, dx = x - 2$ into equation 1.70 we get the same result. That is because equation 1.70 is the linear approximation to the relation between x and y, which is of course the same as the equation of the tangent line.

Example 1.20 Find the equation of the tangent line to the curve $y^3 + 2\cos^2 x = 0$ at the point $(\pi/4, -1)$. We differentiate implicitly:

$$3y^2 dy - 2\cos x \sin x dx = 0. \tag{1.75}$$

Now, at the point $(\pi/4, -1), y^2 = 1, \cos x = \sin x = \sqrt{2}/2$, so this becomes $3dy - dx = 0$. Replacing dx by $x - \pi/4$ and dy by $y - (-1)$ gives the equation of the tangent line:

$$3(y+1) - \left(x - \frac{\pi}{4}\right) = 0, \qquad \text{or} \quad 3y - x = \frac{\pi}{4} - 3. \tag{1.76}$$

Related Rates

Suppose we are in a situation where one or more variables are related, and the variables are functions of time. For example, if a spherical balloon is being inflated, then during this process the volume (V), area (A) and radius (r) are increasing with time. Since these are all related, we are able, by differentiation to relate the rates of growth. For example, suppose the balloon is being inflated by putting gas in at a steady rate of 3 cc/sec. We may ask "at what rate is the radius changing?" We start with the formula relating volume with radius: $V = (4/3)\pi r^3$. V and r are functions of time, so, differentiating with respect to time we obtain

$$\frac{dV}{dt} = \frac{4}{3}\pi\left(3r^2\frac{dr}{dt}\right) = 4\pi r^2\frac{dr}{dt}. \tag{1.77}$$

Putting in the datum $dV/dt = 3$, we find

$$\frac{dr}{dt} = \frac{3}{4\pi r^2} \qquad cm/sec, \tag{1.78}$$

so the change in radius depends on the radius at a particular time.

Here is a protocol for attacking such problems.

Step 1. Draw a picture (if appropriate), and identify the relevant variables: those things which can change. State the problem in terms of the variables.

Step 2. Find a relationship among the variables.

Step 3. Differentiate, to obtain a relationship among the variables and their rates of change.

Step 4. Put in the values of the variables at the time in question, and solve the resulting equation.

Example 1.21 Suppose as above a balloon is being inflated with gas at a rate of 3 cc/sec. At what rate is the area increasing when the radius is 14 cm? First, we identify the variables as volume: V, area: A, and radius: r. Now, these are related by the equations

$$V = \frac{4}{3}\pi r^3, \qquad A = 4\pi r^2. \tag{1.79}$$

Now, we differentiate these equations:

$$\frac{dV}{dt} = 4\pi r^2 \frac{dr}{dt}, \qquad \frac{dA}{dt} = 8\pi r \frac{dr}{dt}. \tag{1.80}$$

Now, at the specific time of interest, $r = 14$ cm, and

$$dV/dt = 3 \tag{1.81}$$

cc/sec. Substituting these values, we have:

$$3 = 4\pi(14)^2 \frac{dr}{dt}, \qquad \frac{dA}{dt} = 8\pi(14)\frac{dr}{dt}. \tag{1.82}$$

Then $dr/dt = 3/\left[4(14)^2\pi\right]$, so the second equation gives

$$\frac{dA}{dt} = 8\pi(14)\frac{3}{4(14)^2\pi} = \frac{6}{7} \qquad cm^2/sec. \tag{1.83}$$

Example 1.22 A ship leaves port at noon heading north at 25 knots (nautical miles per hour), and 2 hours later another ship leaves heading west at 30 knots. Assuming the ships travel in straight lines, at what rate is the distance between the ships increasing after an additional 3 hours?

First, the variables are: t, the time elapsed since noon, N, the distance traveled in that time by the ship heading north, W, the distance traveled by the ship heading west, and Z, the distance between them. The relations among these variables are:

$$Z^2 = N^2 + W^2, \tag{1.84}$$

from the Pythagorean theorem. In t hours after noon, the first ship has traveled $25t$ nautical miles: $N = 25t$, and, since the second ship started two hours later, it has traveled $30(t-2)$ nautical miles (notice, we are assuming that $t \geq 2$). Now since we have been given the rates of change of N and W, and want to find that of Z, we differentiate the first equation with respect to t to relate these rates:

$$2Z\frac{dZ}{dt} = 2N\frac{dN}{dt} + 2W\frac{dW}{dt}. \tag{1.85}$$

Now, at $t = 5$, we have $N = 125$, $W = 75$, $dN/dt = 25$, $dW/dt = 30$, giving

$$Z\frac{dZ}{dt} = 125(25) + 75(30) = 5375. \tag{1.86}$$

We find Z by the first relation $Z^2 = 125^2 + 75^2 = (25^2)(34)$, so $Z = 25\sqrt{34}$. Finally,

$$\frac{dZ}{dt} = \frac{5375}{25\sqrt{34}} = 36.87 \tag{1.87}$$

nautical miles/hour.

Example 1.23 Suppose that x and y are functions of t which satisfy the relation $x^3y^2 + 2y = 8$. Suppose that at the point $(1,2)$, the velocity of x is 3 in/sec. What is the velocity of y? Differentiating the relation implicitly, we get

$$3x^2\frac{dx}{dt}y^2 + x^3\left(2y\frac{dy}{dt}\right) + 2\frac{dy}{dt} = 0. \tag{1.88}$$

Now substituting $x = 1$, $y = 2$, $dx/dt = 3$, in (88)

$$3(1)^2(3)\left(2^2\right) + (1)^3\left(2(2)\frac{dy}{dt}\right) + 2\frac{dy}{dt} = 0. \tag{1.89}$$

Solving for dy/dt, we find $36 + 6dy/dt = 0$, or, the velocity of y is -6 in/sec.

Definition 1.3 *For integers p and q, the function*

$$y = x^{p/q} \tag{1.90}$$

is defined, for all positive x, as the positive solution of the equation $y^q = x^p$.

So, for example, $\sqrt{x}$ can be written as $x^{1/2}$, the cube root of x as $x^{1/3}$, etc. Using implicit differentiation we verify:

Proposition 1.13

$$\frac{d}{dx}x^n = nx^{n-1} \qquad \textit{for all rational numbers } n. \tag{1.91}$$

A rational number is a quotient p/q of integers. Differentiate the equation $y^q = x^p$ implicitly:

$$qy^{q-1}dy = px^{p-1}dx \qquad \textit{or} \qquad \frac{dy}{dx} = \frac{p}{q}\frac{x^{p-1}}{y^{q-1}}. \tag{1.92}$$

Replacing y by $x^{p/q}$, we get

$$\frac{dy}{dx} = \frac{p}{q}\frac{x^{p-1}}{x^{p-p/q}} = \frac{p}{q}x^{p-1-p+p/q} = \frac{p}{q}x^{(p/q)-1} \tag{1.93}$$

which is the desired result, since $n = p/q$.