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## CHAPTER 3

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# Extrema, Concavity, and Graphs

### §3.1. Extrema, Concavity, and Graphs

In this chapter we will be studying the behavior of differentiable functions in terms of their derivatives. Thus, whenever a function  $f$  is introduced, it is to be understood that it is defined and has first and second derivatives on an interval  $I$ .

**Definition 3.1** We say

- a)  $f$  is increasing on the interval  $I$  if, for  $a \leq b$ , we have  $f(a) \leq f(b)$ .
- b)  $f$  is strictly increasing on the interval  $I$  if, for  $a < b$ , we have  $f(a) < f(b)$ .
- c)  $f$  is decreasing on the interval  $I$  if, for  $a \leq b$ , we have  $f(a) \geq f(b)$ .
- d)  $f$  is strictly decreasing on the interval  $I$  if, for  $a < b$ , we have  $f(a) > f(b)$ .

**Proposition 3.1**

- a) If  $f'$  is always positive on  $I$ , then  $f$  is strictly increasing on the interval  $I$ .
- b) If  $f'$  is always negative on  $I$ , then  $f$  is strictly decreasing on the interval  $I$ .

For  $a < b$ , we have  $b - a > 0$ . By the mean value theorem, there is a  $c$  between  $a$  and  $b$  such that

$$(3.1) \quad \frac{f(b) - f(a)}{b - a} = f'(c).$$

Now in case the hypothesis of a) holds, this is positive. Since the denominator of the expression on the left is positive, that implies that  $f(b) - f(a) > 0$ . On the other hand, if b) holds, so  $f'(c) < 0$ , we conclude that the numerator is negative, so  $f(b) - f(a) < 0$ .

**Definition 3.2**

- a) Let  $a$  be a point in  $I$ . We say that  $f$  has a local maximum at  $a$  if, for all  $x$  sufficiently close to  $a$ ,  $f(x) \leq f(a)$ .
- b) We say that  $f$  has a local minimum at  $a$  if, for all  $x$  sufficiently close to  $a$ ,  $f(x) \geq f(a)$ .

If we want to graph the function  $y = f(x)$ , it is important to calculate  $f'$ , and determine the intervals in which it is positive or negative. Then we know that the graph must “go up” in an interval where  $f'$  is positive, and “go down” where  $f'$  is negative. Clearly, at points at which the sign of  $f'$  changes, there must be either a local maximum or a minimum. For example, if  $f'$  is negative to the left of  $a$ , and positive to the right of  $a$ , then  $f$  is decreasing to the left of  $a$  and increasing to the right of  $a$ , so it has a local minimum at  $a$ .

**Proposition 3.2 (First Derivative Test)**

- a) If  $f'(x) > 0$  for  $x < a$  and  $f'(x) < 0$  for  $x > a$ , then  $f$  has a local maximum at  $a$ .  
 b) If  $f'(x) < 0$  for  $x < a$  and  $f'(x) > 0$  for  $x > a$ , then  $f$  has a local minimum at  $a$ .

If the function has a continuous second derivative, the following test may be computationally easier:

**Proposition 3.3 (Second Derivative Test)**

- a) If  $f'(a) = 0$  and  $f''(a) < 0$ , then  $f$  has a local maximum at  $a$ .  
 b) If  $f'(a) = 0$  and  $f''(a) > 0$ , then  $f$  has a local minimum at  $a$ .

Take case (a). We must have  $f''$  negative in a neighborhood of  $a$ , so  $f'$  is decreasing in that neighborhood. Since  $f'(a) = 0$ , we must have  $f'$  positive to the left of  $a$  and negative to the right. So, by the first derivative test,  $f$  has a local maximum at  $a$ .

**Example 3.1** Let  $f(x) = 2x^3 - 15x^2 + 24x + 10$ . Then

$$(3.2) \quad f'(x) = 6x^2 - 30x + 24 = 6(x^2 - 5x + 4) = 6(x - 1)(x - 4),$$

$$(3.3) \quad f''(x) = 12x - 30 = 6(2x - 5).$$

We see that  $f'(1) = 0$ , and  $f'(4) = 0$ , and  $f' > 0$  for  $x < 1$ ,  $f' < 0$  for  $x$  between 1 and 4, and  $f' > 0$  for  $x > 4$ . Thus the graph of  $y = f(x)$  is increasing until  $x = 1$ , decreases from 1 to 4, increases again for  $x > 4$ , and  $x = 1$  is a local maximum,  $x = 4$  is a local minimum. We can confirm this by calculating the second derivative:  $f''(1) = -18 < 0$ , and  $f''(4) = 18 > 0$ . See figure 3.3 for the graph.

The knowledge about a function that we obtain from the first derivative suffices to solve practical optimization problems. Typically, the given situation leads to a function  $y = f(x)$  defined on a closed interval  $a \leq x \leq b$ , and the problem is to find the maximum or minimum value of the function on the interval. This can occur only at one of the following points:  $a$ ,  $b$ , any point in the interval at which  $f$  does not have a derivative, any point  $c$  on the interval at which  $f'(c) = 0$ . These are the *critical points* of the function. Evaluate the function at all of the critical points; the largest of these values is the maximum on the interval, and the smallest, the minimum.

**Example 3.2** A triangle in the first quadrant has vertices at the points  $(0,0)$ ,  $(t,0)$ ,  $(t,9-t^2)$ . Find the triangle of maximum area.

Here the variables are area,  $A$ , and value  $t$  determining the triangle. Area is the objective variable; in terms of  $t$  this area is

$$(3.4) \quad A(t) = \frac{1}{2}t(9-t^2) = \frac{1}{2}(9t-t^3).$$

Since the triangle is in the first quadrant, we must have  $t \geq 0$  and  $9 - t^2 \geq 0$ , so the interval on which the area is defined is  $0 \leq t \leq 3$ . Now

$$(3.5) \quad A'(t) = \frac{9 - 3t^2}{2}$$

so  $f'(t) = 0$  when  $t = \pm\sqrt{3}$ . The critical points on the interval in question are  $0, \sqrt{3}, 3$ . The values of  $f$  at these points are  $0, 3\sqrt{3}, 0$ , so the maximum value is  $3\sqrt{3}$ .

**Example 3.3** A box with a square bottom is to be made so as to contain  $150 \text{ in}^3$ . The cost of the material to make the sides is \$2 per  $\text{in}^2$  and the material for the top and bottom is \$3 per  $\text{in}^2$ . What are the dimensions of the box of minimal cost?

Here the variables are:  $x$ , the length of a side of the base of the box,  $h$ , the height, the area  $A_b$  of the base (which is the same as the area of the top), the area  $A_s$  of a side (of which there are 4), and the cost,  $C$ . Since we want to minimize  $C$ , this is the objective variable.

According to our data, the cost is given by  $C = 3(2A_b) + 2(4A_s)$ . Now  $A_b = x^2$ , and  $A_s = xh$ , so we obtain

$$(3.6) \quad C = 6x^2 + 8xh.$$

Now, we need a relationship between  $x$  and  $h$ : that is provided by the constraint that the box must contain  $150 \text{ in}^3$ . Thus  $x^2h = 150$ , or  $h = 150x^{-2}$ . The objective equation becomes

$$(3.7) \quad C = 6x^2 + 8x(150x^{-2}) = 6x^2 + 1200x^{-1}.$$

In principle, the variable  $x$  ranges over all positive numbers, but we note from equation (3.6) that  $C$  becomes arbitrarily large as  $x$  becomes very large or very small. Thus there is a minimum somewhere between: at a place where  $C'(x) = 0$ . Differentiating:

$$(3.8) \quad C'(x) = 12x - 1200x^{-2},$$

so  $C'(x) = 0$  when  $12x^3 = 1200$ , or  $x = 4.64$  in. Since this is the only point at which  $C' = 0$ , it is the value of  $x$  which minimizes the cost. Using  $x^2h = 150$ , we find  $h = 6.96$  in. Thus the box should be (approximately) 4.5 in square on the base, and of height 7 in.

### §3.2. Concavity and the Second Derivative

#### Definition 3.3

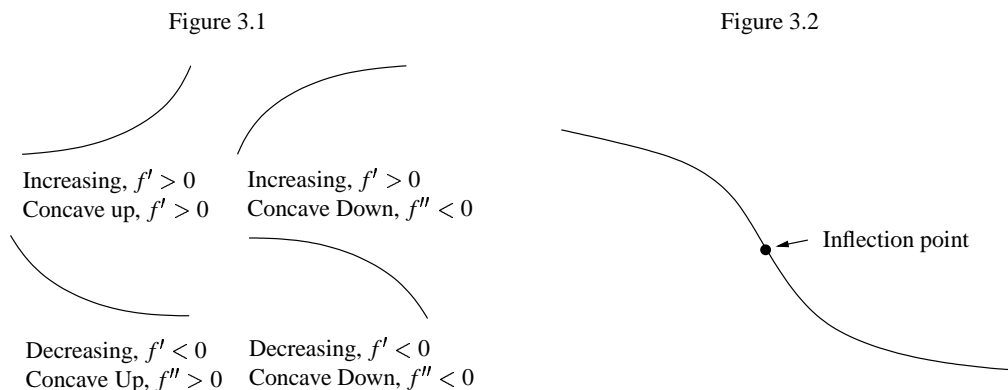
a) If, at every point  $a$  in  $I$ , the graph of  $y = f(x)$  always lies above the tangent line at  $a$ , we say that  $f$  is concave up.

b) If, at every point  $a$  in  $I$ , the graph of  $y = f(x)$  always lies below the tangent line at  $a$ , we say that  $f$  is concave down. (See figure 3.1).

#### Proposition 3.4

a) If  $f''$  is always positive in the interval  $I$ , then  $f$  is concave up in that interval.

b) If  $f''$  is always negative in the interval  $I$ , then  $f$  is concave down in that interval.



The reasoning is this: where  $f'' > 0$  the slope of the tangent line is increasing, so the graph of the function is curving upward. More precisely, let  $y = g(x)$  be the equation of the tangent line at  $a$ . We want to show that the graph of  $y = f(x)$  is always above the graph of  $y = g(x)$ , so we have to show that the difference  $h(x) = f(x) - g(x) \geq 0$  near  $a$ . Now  $h'(x) = f'(x) - f'(a)$ , since the slope of the tangent line is  $f'(a)$ . Thus  $h'(a) = 0$ , and  $h''(a) = f''(a)$  is positive by hypothesis, so  $h$  has a local minimum at  $a$ . Since  $h(a) = 0$ , it follows that  $h(x) \geq 0$  for all  $x$  near  $a$ .

By including this information given by the second derivative, we can accurately sketch the graph of the function  $y = f(x)$ , by determining the intervals in which  $f'$  and  $f''$  have given signs. The four possibilities are illustrated in figure 3.1.

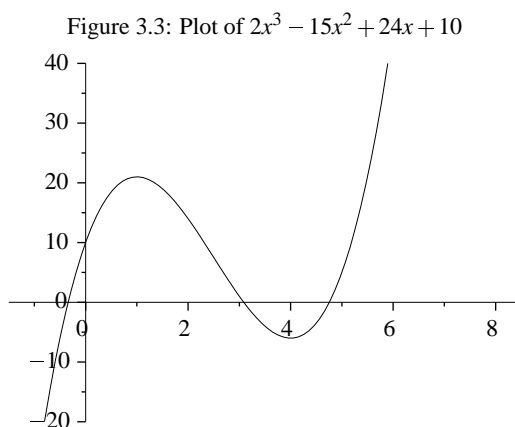
Points where the second derivative changes sign are points at which the concavity changes: these are called *points of inflection* (see figure 3.2). Clearly,

**Proposition 3.5** *If  $a$  is a point of inflection of  $f$ , and  $f''(a)$  exists, then it is zero.*

**Example 3.1 (continued).** For  $f(x) = 2x^3 - 15x^2 + 24x + 10$ , we calculated  $f''(x) = 6(2x - 5)$ . Thus  $f''$  is negative for  $x < 5/2$ , and positive for  $x > 5/2$ , so  $x = 5/2$  is a point of inflection. Putting together all the information we have obtained on this function we can draw its graph showing all important features. First we tabulate the data obtained:

$x$	$f$	$f'$	$f''$
$x < 1$	<i>incr</i>	$> 0$	<i>conc dwn</i>
1	21	0	-6
$1 < x < 2.5$	<i>decr</i>	$< 0$	<i>conc dwn</i>
2.5	7.5	-13.5	0
$2.5 < x < 4$	<i>decr</i>	$< 0$	<i>conc up</i>
4	-6	0	18
$x > 4$	<i>incr</i>	$> 0$	<i>conc up</i>

Based on these data, we draw the graph: figure 3.3.



### §3.3. Sketching Graphs of Rational Functions

To sketch a graph of a function  $y = f(x)$  follow these steps.

1. Determine where the function is not defined.
2. Determine the points  $x$  where  $f'(x) = 0$  and where  $f''(x) = 0$ .
3. Make a table of the values of  $f$ ,  $f'$ ,  $f''$  at the points found in steps 1 and 2.
4. Determine the signs of  $f'$  and  $f''$  in the intervals separated by the points found in steps 1 and 2.
5. Determine the vertical asymptotes of the function at the points found in step 1 (see section 2.2).
6. Determine the horizontal asymptotes of the function as  $x \rightarrow \pm\infty$  (see section 2.2).
7. Using all this information and the templates of figure 3.1, sketch the graph.

**Example 3.4** Let  $y = x^4 - 18x^2 + 2$ . We shall follow the steps above:

1. The function is defined everywhere.
2. Calculate the first and second derivatives:

$$(3.9) \quad f'(x) = 4x^3 - 36x^2 = 4x(x^2 - 9),$$

$$(3.10) \quad f''(x) = 12x^2 - 36 = 12(x^2 - 3).$$

$$(3.11) \quad f'(x) = 0$$

for  $x = 0, \pm 3$  and  $f''(x) = 0$  for  $x = \pm\sqrt{3}$ .

3. We create the table of relevant values at these points. In making this table, leave spaces for additional values of  $x$  which may be relevant.

$x$	$f$	$f'$	$f''$
-3	-79	0	
$-\sqrt{3}$	-43		0
0	2	0	
$\sqrt{3}$	-43		0
3	-79	0	

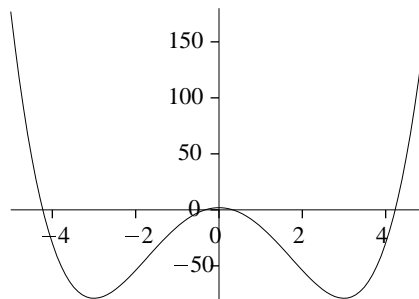
4. We still have to determine the signs of  $f''$  at  $x = 0, \pm 3$ , and the signs of  $f'$  at  $x = \pm\sqrt{3}$ , and finally at points before -3 and after 3 (for the latter, we can also determine the limits at infinity, as in Proposition 2.9). When that is done, we shall have completed the table:

$x$	$f$	$f'$	$f''$
$< -3$	$\rightarrow +\infty$	-	+
-3	-79	0	+
$-\sqrt{3}$	-43	+	0
0	2	0	-
$\sqrt{3}$	-43	-	0
3	-79	0	+
$> 3$	$\rightarrow +\infty$	+	+

5 and 6. The asymptotes as  $x \rightarrow \pm\infty$  are read off the first and last lines of the table.

7. To graph, first locate the points  $(x, f(x))$  for  $x$  on the table, then connect these points using the knowledge about  $f'$  and  $f''$ , and the templates of figure 3.1. The result is shown in figure 3.4.

Figure 3.4: Plot of  $f(x) = x^4 - 18x^2 + 2$



**Example 3.5** Graph  $y = \frac{x}{x-1}$ .

1. The function is not defined at  $x = 1$ .
2. We can rewrite the function (using long division) as

$$(3.12) \quad f(x) = 1 + (x-1)^{-1}.$$

Then

$$(3.13) \quad f'(x) = -(x-1)^{-2}, \quad f''(x) = 2(x-1)^{-3}.$$

Thus there are no points at which these are zero, so the only important data will be the signs of  $f$ ,  $f'$  and  $f''$  in the various intervals.

4. Calculation of  $f$  and its derivatives at sample points yields the table

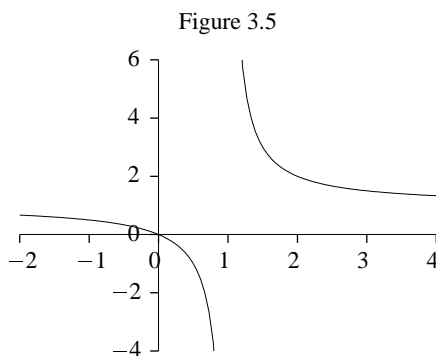
$x$	$f$	$f'$	$f''$
$< 0$	+	-	-
0	0	-	-
$0 < x < 1$	-	-	-
1	ND	ND	ND
$> 1$	+	-	+

5. The vertical asymptotes are at  $x = 1$ . Line 3 of the table tells us that as  $x \rightarrow 1^-$ ,  $f(x) \rightarrow -\infty$ , and line

5 of the table tells us that as  $x \rightarrow 1^+$ ,  $f(x)$  is positive so  $f(x) \rightarrow +\infty$

6. The form of the function (3.12) tells us that as  $x \rightarrow -\infty$ ,  $f(x) \rightarrow 1^-$  and as  $x \rightarrow +\infty$ ,  $f(x) \rightarrow 1^+$ .

7. The graph can now be drawn as in figure 3.5.



**Example 3.6** Graph  $y = \frac{x}{x^2 - 1}$ .

1. The function is not defined at the points  $1, -1$ .

2.  $f'(x) = -\frac{x^2 + 1}{(x^2 - 1)^2}$ ,  $f''(x) = \frac{2x}{(x^2 - 1)^3}$ .

3. We note that  $f'$  is never zero; in fact, it is always negative (except where undefined), and  $f''$  is zero just at  $x = 0$ . This gives the preliminary table

$x$	$f$	$f'$	$f''$
$< -1$	-	-	-
-1	ND	ND	ND
$-1 < x < 0$	-	-	-
0	0	-	0
$0 < x < 1$	-	-	-
1	ND	ND	ND
$x > 1$	+	-	+

4. A few calculations at intermediate points fills in the table:

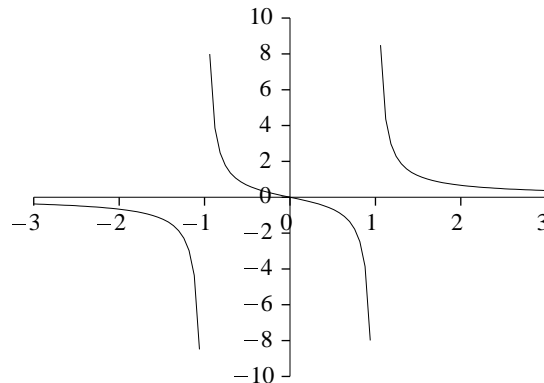
$x$	$f$	$f'$	$f''$
$< -1$	-	-	-
$-1$	<i>ND</i>	<i>ND</i>	<i>ND</i>
$-1 < x < 0$	+	-	+
$0$	0	-	0
$0 < x < 1$	-	-	-
$1$	<i>ND</i>	<i>ND</i>	<i>ND</i>
$x > 1$	+	-	+

5. We determine the nature of the asymptotes on each side of  $-1$  using lines 1 and 3 of the table, and on each side of  $1$  using lines 5 and 7.

6. Since the degree of the denominator is greater than the degree of the numerator, the line  $y = 0$  is the horizontal asymptote on each side. The approach as  $x \rightarrow -\infty$  is from below (by line 1), and from above (by line 7) as  $x \rightarrow +\infty$ .

7. Putting these data together, we obtain the graph in figure 3.6.

Figure 3.6: Plot of  $\frac{x}{x^2-1}$



### §3.4. Other Sketches

To graph more complicated functions, we follow the same steps, but may have to do some additional investigations.

Sometimes the asymptotic directions give us sufficient information, and it is not really necessary to calculate the first and second derivative (which for a rational function could be quite tedious). To illustrate this, we return to example 2.11.

**Example 3.7** Graph  $f(x) = \frac{x^3 + 2x^2}{x^2 - 3x + 2}$ .

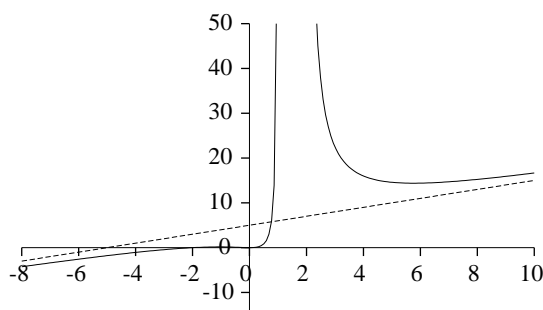
In section 2.2 we found the factorization  $f(x) = \frac{x^2(x+2)}{(x-1)(x-2)}$ , and the asymptotic behavior as  $x = 1, 2$ . Also, by long division, we saw that the horizontal asymptote is  $y = x + 5$ . Noting that the function is zero at the points  $x = 0, -2$ , we can tabulate the results just for the values of the function.



$x$	$< -2$	$-2$	$-2 < x < 0$	$0$	$0 < x < 1$	$1$	$1 < x < 2$	$2$	$x > 2$
$f(x)$	$-$	$0$	$+$	$0$	$-$	$ND$	$-$	$ND$	$+$

The simplest graph with these properties and the horizontal asymptote  $y = x + 5$  is shown in figure 3.7. We'd have to calculate the derivatives to be sure that there are no other ripples, but as a first sketch, this is satisfactory. (In fact, figure 3.7 does accurately show the local maxima and minima and the correct concavity).

Figure 3.7: Plot of  $\frac{x^3+2x^2}{x^2-3x+2}$  and the asymptote  $y = x + 5$



**Example 3.8** Graph  $y = \sqrt{x^2 - 1}$ .

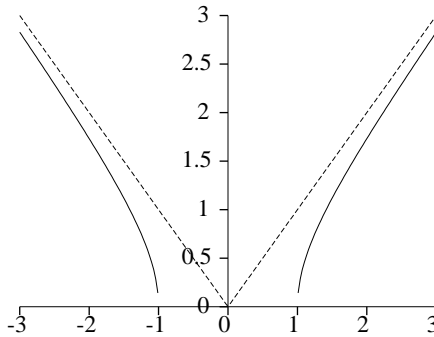
1. Unless otherwise specified, the symbol  $\sqrt{\quad}$  refers to the nonnegative square root. Since we cannot take the square root of a negative number, this function is undefined for  $x$  between  $-1$  and  $1$ .
2. We calculate:  $f'(x) = x(x^2 - 1)^{-1/2}$ ,  $f''(x) = -(x^2 - 1)^{-3/2}$ .
- 3 and 4. We obtain the following table of relevant values:

$x$	$f$	$f'$	$f''$
$< -1$	$+$	$-$	$-$
$-1$	$0$	$\infty$	$\infty$
$-1 < x < 1$	$ND$	$ND$	$ND$
$1$	$0$	$\infty$	$\infty$
$> 1$	$+$	$+$	$-$

Thus the graph is decreasing for  $x \leq 1$  and increasing for  $x \geq 1$  and is everywhere concave down. 5-6. There are no vertical asymptotes. To find the horizontal asymptotes, we rewrite the function as

$$(3.14) \quad y = |x| \sqrt{1 - \frac{1}{x^2}}.$$

Since the second factor tends to 1 as  $x \rightarrow \infty$  we conclude that the graph approaches that of  $y = |x|$  asymptotically from below. 7. See figure 3.8.

Figure 3.8: Plot of  $y = x^2 - 1$  and the asymptote  $y = |x|$ .**Example 3.9** Graph  $y = x \sin x$ .

The function is defined everywhere. Since  $\sin x$  oscillates, this function oscillates, but grows in height as  $|x|$  grows. In fact the graph will hit the lines  $y = \pm x$  at every point where  $|\sin x| = 1$ ; that is, at all points  $x = n\pi + \pi/2$ . This is enough information to graph the function, except around zero. But, since

$$(3.15) \quad f'(x) = \sin x + x \cos x$$

we see that  $f'(0) = 0$ , and the graph has a horizontal tangent at the origin. The graph now can be drawn as a sine curve “damped” by the lines  $y = \pm x$  (see figure 3.9).

Figure 3.9: Plot of  $y = \sqrt{x \sin x}$  and the asymptote  $y = \pm x$ .