CHAPTER 5

Applications of Integration

§5.1. Volume

In the preceding section we saw how to calculate areas of planar regions by integration. The relevant property of area is that it is *accumulative*: we can calculate the area of a region by dividing it into pieces, the area of each of which can be well approximated, and then adding up the areas of the pieces. To put it another way, we calculate area by adding piece by piece as we move through the region in a particular direction. Once we have obtained a formula for the differential increment in the area (such as $dA = L(x)dx$), we find the area by integration. This process can be used to calculate values of any accumulative concept, such as volume, arc length and work. This chapter is devoted to these calculations.

Example 5.1 To begin with, let us calculate the volume of a sphere of radius R. We first have to decide on a direction of accumulation; and for that we need a particular coordinate representation of the sphere. Start with the region in the plane bounded by the x-axis and the curve $C : x^2 + y^2 = R^2$. If we rotate this region in space about the x-axis, we obtain the sphere of radius (R) (see figure 5.1). We now accumulate the volume of the sphere by moving along the axis of rotation, (the x-axis), starting at $x = -R$, and ending at $x = R$. Let $V(x)$ be the volume accumulated when we reach the point x. Then the piece added when we move a distance dx further along the axis is a cylinder of width dx and of radius y, the length of the line from the x-axis to the curve C (see figure 5.1). The volume of this piece is thus

$$dV = \pi y^2 dx \ . \tag{5.1}$$

Now, along C, $y = \sqrt{R^2 - x^2}$, from which we obtain $dV = \pi(R^2 - x^2)dx$. The volume thus is the integral of this differential:

$$V = \int dV = \int_{-R}^{R} \pi(R^2 - x^2)dx = \pi \left[R^2 x - \frac{x^3}{3}\right] \Bigg|_{-R}^{R} \tag{5.2}$$

$$= \pi \left[R^2 (R) - \frac{R^3}{3} - \left(R^2 (-R) - \frac{(-R)^3}{3}\right)\right] = \frac{4}{3}\pi R^3 \ . \tag{5.3}$$

Figure 5.1

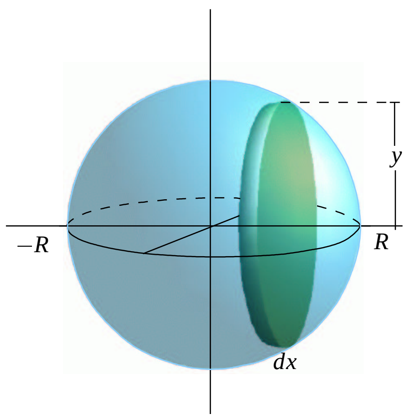
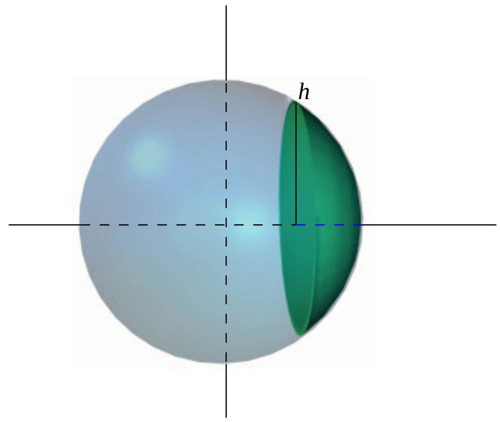


Figure 5.2



We can find the volume of a spherical segment of depth h the same way (see figure 5.2):

$$V = \int_{R-h}^{R} \pi(R^2 - x^2)dx = \frac{2\pi}{3}h(R^2 - h^2) \ , \tag{5.4}$$

after an even longer computation.

In general, we can calculate the volume of a solid by integration if we can see a way of sweeping out the solid by a family of surfaces, and we can calculate, or already know the area of those surfaces. Then we calculate the volume by integrating the area along the direction of sweep. In the above example we swept out the sphere by moving along the x-axis, and associating to each point x the area of the disc which is the perpendicular cross-section of the sphere at x.

§5.1.1 Volume: general method

The way to find the volume of a solid is this. Sketch the region under consideration. Choose a direction in which to accumulate the volume. Write down the expression for the differential increment in volume:

$$dV = A(x)dx \ , \tag{5.5}$$

where dx is an infinitesimal increment in the direction of accumulation, and $A(x)$ is the area of the section of the solid at the point x. Of course, if the solid is highly irregular, finding $A(x)$ may still be a problem. In this section we restrict attention to those cases where $A(x)$ is known or is easily found.

Example 5.2 Find the volume of a cone of base radius r and height h.

We sweep out the cone along its axis starting at the vertex (see figure 5.3), so x ranges from 0 to h. The cross-section of the cone at any x is a disc, let ρ be its radius. Then $dV = \pi\rho^2 dx$. Now, we can find

Figure 5.3

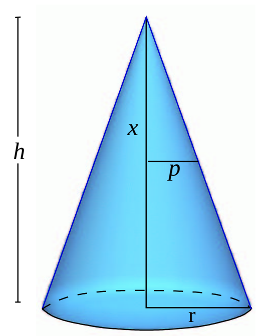
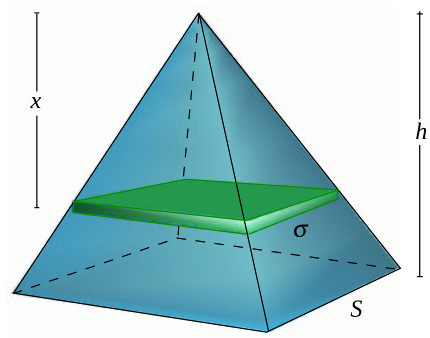


Figure 5.4



ρ as a function of x by similar triangles:

$$\frac{\rho}{r} = \frac{x}{h}\,, \tag{5.6}$$

so $\rho = rx/h$. Then

$$Volume = \int_0^h \pi \frac{r^2x^2}{h^2}dx = \pi \frac{r^2}{h^2}\frac{x^3}{3}\Big|_0^h = \frac{1}{3}\pi r^2 h\,. \tag{5.7}$$

Example 5.3 Find the volume of a pyramid of height h and square base of side length s.

Here again, we sweep the pyramid out along its axis, with x representing the distance from the vertex (see figure 5.4). Then x ranges from 0 to h, and at any x, the cross-section is a square of side length σ. The differential increment of volume at x is $dV = \sigma^2 dx$. Again by similar triangles $\sigma = sx/h$. Thus

$$Volume = \int_0^h \frac{s^2}{h^2}x^2 dx = \frac{s^2}{h^2}\frac{x^3}{3}\Big|_0^h = \frac{1}{3}s^2 h\,. \tag{5.8}$$

Example 5.4 Let R be the region in the plane bounded by the curves $y = x^2$, $y = -x^2$. Let K be a solid lying over this region whose cross-section is a semicircle of diameter the line segment between the curves (see figure 5.5). Find the volume of the piece of K lying between $x = 1$ and $x = 2$.

We sweep out along the x-axis. Then $dV = (1/2)\pi r^2 dx$, where r is the radius of the semicircle. Now $r = x^2$, so we obtain

$$V = \int_1^2 \frac{1}{2}\pi (x^2)^2 dx = \frac{1}{2}\pi \frac{x^5}{5}\Big|_1^2 = \frac{1}{10}\pi(32-1) = 3.1\pi\,. \tag{5.9}$$

Figure 5.5

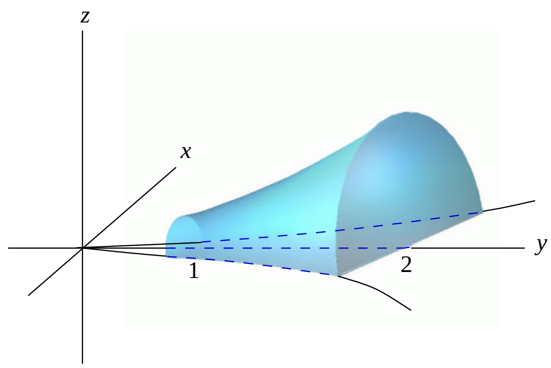
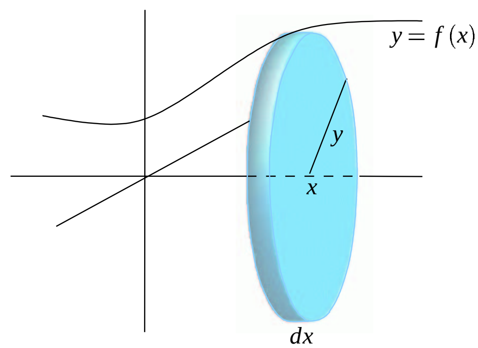


Figure 5.6



§5.1.2 Volumes of Revolution

A solid of revolution is obtained by revolving a region in the plane around an axis in the plane. There are three ways of calculating the volume of such a solid, depending upon how we sweep it out.
Disc method. Suppose, as in figure 5.6, the figure lies between a curve $C : y = f(x)$ and the x-axis, which is the axis of rotation. Then a cross section is a disc of radius y, so

$$dV = \pi y^2 dx = \pi(f(x))^2 dx \; . \tag{5.10}$$

Washer method. Suppose the figure lies between two curves $C_1 : y = f(x)$ lying above $C_2 : y = g(x)$ in the upper half-plane and the axis is the x-axis (see figure 5.7). Then a cross section is a *washer*: the region between two concentric circles. Its area is the difference of the areas of these circles. The differential increment of volume is then

$$dV = (\pi f(x)^2 - \pi g(x)^2) dx \; . \tag{5.11}$$

Figure 5.7

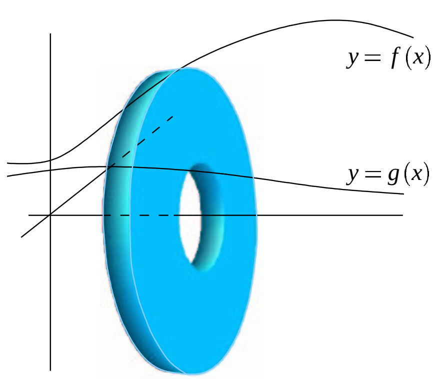
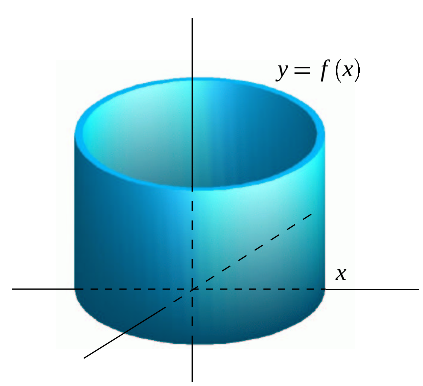


Figure 5.8



Shell method. This method is used when it is convenient to sweep out the volume along an axis perpendicular to the axis of rotation. To be more specific, suppose the region lies in the right half plane, and

Figure 5.9

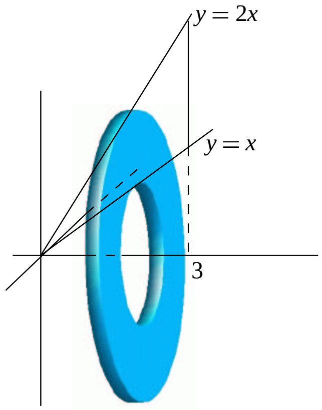
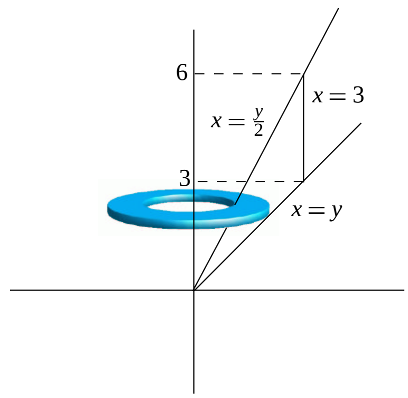


Figure 5.10



we rotate it about the y-axis. Then for each x, the incremental surface at x is the cylinder swept out by rotating the line segment perpendicular to the x-axis lying in the region (see figure 5.8) about the axis of rotation. The area of this surface is $2\pi xL$ where L is the length of the line segment. Thus

$$dV = 2\pi xLdx. \tag{5.12}$$

Example 5.5 Consider the region R between the lines $L_1 : y = x$, $L_2 : y = 2x$ and $x = 3$ (see figure 5.9). Suppose we generate a solid by rotating R about the $x - axis$. We sweep out the solid along the x-axis. At any x the cross-section is the washer generated by the line segment between L_1 and L_2. This is bounded by the circles of radius $2x$, x respectively. Thus $dV = \pi((2x)^2 - x^2))dx = \pi(3x^2)dx$, and so

$$V = \pi \int_0^3 (3x^2)dx = \pi x^3\Big|_0^3 = 27\pi\ . \tag{5.13}$$

Example 5.6 Suppose instead we rotate the same region R about the y-axis. Then (sweeping along the x-axis), the surface generated at any x is the cylinder of radius x and height the distance between the two curves: $2x - x$. Thus $dV = 2\pi x(x)dx$, and

$$V = 2\pi \int_0^3 x^2 dx = \frac{2}{3}\pi(3^3 - 0^3) = 18\pi\ . \tag{5.14}$$

Example 5.7 We could also do example 5.6 by accumulating volume along the axis of rotation (the y-axis), but this is more complicated. As we see from figure 5.10, y ranges from 0 to 6, and the curve describing the outer boundary changes at the point $y = 3$. So we have to split this up into two computations. For the first, y ranges from 0 to 3, and the region is bounded by the curves $x = y/2$, $x = y$, and for the second, y ranges from 3 to 6 and the region is bounded by the curves $x = y/2$, $x = 3$. In both cases, a slice at a fixed y is a washer, so

$$dV = \pi(R^2 - r^2)dy\,, \tag{5.15}$$

where R is the outer radius, and r is the inner radius. For the first computation then

$$V_1 = \int_0^3 \pi\left(y^2 - \left(\frac{y}{2}\right)^2\right)dy = \frac{\pi}{4}\int_0^3 3y^2dy = \frac{27}{4}\pi\,. \tag{5.16}$$

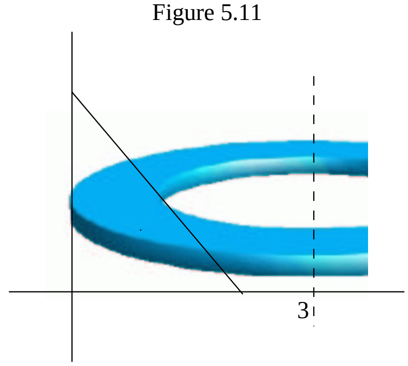
For the second:

$$V_2 = \int_3^6 \pi\left(3^2 - \left(\frac{y}{2}\right)^2\right)dy = \pi\int_3^6\left(9 - \frac{y^2}{4}\right)dy = \pi\left(9y - \frac{y^3}{12}\right)\Big|_3^6 = 9\pi + \frac{9}{4}\pi. \tag{5.17}$$

The total volume is $V_1 + V_2 = 18\pi$.

Example 5.8 Rotate the triangle bounded by the coordinate axes and the line $x + y = 1$ about the line $x = 3$, and find the volume (see figure 5.11).

Figure 5.11



We'll accumulate the volume in the direction of the axis of rotation, so the variable is y, ranging from 0 to 1. For a fixed y, the surface generated by the rotation is a washer of outer radius 3, and inner radius $3 - x = 3 - (1 - y) = 2 + y$. Thus $dV = \pi(3^2 - (2 + y)^2) = 5 - 4y - y^2$, and the volume is

$$\int_0^1 (5 - 4y - y^2)dy = \left(5y - 2y^2 - \frac{y^3}{3}\right)\Big|_0^1 = \frac{8}{3}\,. \tag{5.18}$$

§5.2. Arc Length

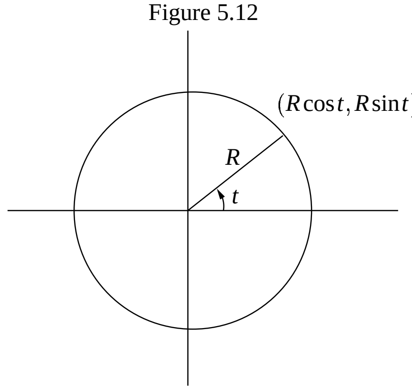
We have seen that a curve in the plane can be described *explicitly* as the graph of a function $y = f(x)$ or *implicitly*, as a relation $F(x,y) = C$ between the variable x and y. A third way a curve can be described is *parametrically* in the form

$$x = x(t) \qquad y = y(t) \tag{5.19}$$

as the *parameter* t ranges over an interval (a,b). For example, a particle may be moving on the plane, and its position at time t is $(x(t),y(t))$.

Example 5.9 The circle of radius R can be described parametrically by taking as the parameter the angle the ray through the point makes with the x-axis, so that $(R\cos t, R\sin t)$ are the coordinates of the point for a given angle t (see figure 5.12).

Figure 5.12



Since $\cos^2 t + \sin^2 t = 1$, these points all satisfy the implicit relation $x^2+y^2=R^2$.

Suppose an object is moving in a plane perpendicular to the earth's surface so that the only force acting on it is gravity. Let its position at time t be $(x(t),y(t))$. Then dx/dt is the horizontal velocity of the object, and dy/dt the vertical velocity, in the sense that these are the rates of change of the motion in those directions. We say that the pair $(dx/dt, dy/dt)$ is the *velocity* of the object. Similarly, its *acceleration* is the pair $(d^2x/dt^2, d^2y/dt^2)$, where d^2x/dt^2 is the *horizontal acceleration* and d^2y/dt^2 is the *vertical acceleration*. In our case, since the only force is vertical, that of gravity, we have

$$\frac{d^2x}{dt^2} = 0 \quad \text{and} \quad \frac{d^2y}{dt^2} = -32 \ \text{ft/sec}^2 \ . \tag{5.20}$$

We can integrate these equations to get the equations of motion of the object:

$$\frac{dx}{dt} = v_x \qquad \frac{dy}{dt} = -32t + v_y \tag{5.21}$$

where v_x is the initial (and constant) horizontal velocity, and v_y is the initial vertical velocity. The position is then given by integrating again:

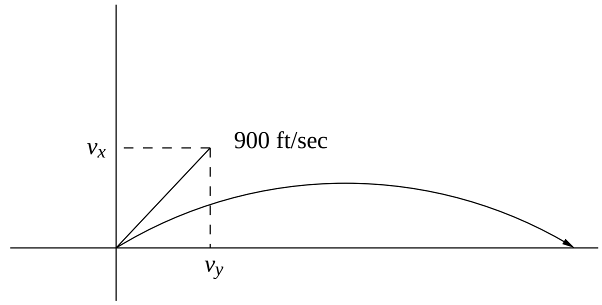
$$x(t) = v_x t + x(0)\ , \qquad y(t) = -16t^2 + v_y t + y(0) \tag{5.22}$$

where at time $t=0$ the particle is at $(x(0),y(0))$

Example 5.10 A rifle is fired from a prone position at an angle of 6 degrees from the horizontal. The bullet leaves the muzzle with an initial velocity of 900 ft/sec. Assuming the ground is level, how far does the bullet travel before it hits the ground again? How long does this take?

Let $(x(t),y(t))$ represent the position of the bullet at time t, assuming it starts at the origin, so $x(0)=0$ and $y(0)=0$. We are told that the tangent to the trajectory of the bullet at $t=0$ makes an angle of 6

Figure 5.13



degrees with the horizontal, and that it starts out in this direction at 900 ft/sec. In an increment of time dt it travels a distance $900dt$ feet along this line, so the corresponding horizontal and vertical displacements (see figure 5.13) are $dx = 900\cos 6dt = 895dt$ and $dy = 900\sin 6dt = 94dt$. Thus $v_x = 895$ ft/sec and $v_y = 94$ ft/sec. Thus, according to 5.5, the position of the bullet at time t is

$$x(t) = 895t \qquad y(t) = -16t^2 + 94t\ . \tag{5.23}$$

Now, the bullet is at ground level when $y(t) = 0$. Solving that equation, $t = 0$ or $t = 94/16 = 5.88$ sec, and $x = 895(5.88) = 5262$ feet.

Example 5.11 Suppose a ball is hit horizontally at a height of 5 ft. at 120 mph (176 ft/sec). How far from the hitter will it hit the ground?

Putting the origin of the coordinates at the batter's feet, we have $x(0) = 0,\ y(0) = 5$. Since the ball is struck horizontally, we have $v_x = 120,\ v_y = 0$. Thus, from (2), the equations of motion are

$$x(t) = 120t \qquad y(t) = -16t^2 + 5\ . \tag{5.24}$$

Ground level is at $y(t) = 0$, so we can solve the second equation for $t:\ 0 = -16t^2 + 5$, from which we find $t = \sqrt{5/16} = .559$. Thus the answer is $x(.559) = 120(.559) = 67.1$ ft.

When a curve C is given parametrically as $x = x(t),\ y = y(t)$, it may not be so easy to describe the curve. To do that, one tries to write the curve as a relation between x and y by eliminating the variable t.

Example 5.12 Suppose C is given parametrically by

$$x(t) = 15t + 10 \qquad y(t) = -16t^2 + 32t\ . \tag{5.25}$$

To eliminate t we solve the first equation for t in terms of $x:\ t = (x - 10)/15$, and then put that in the equation for y:

$$y = -16\left(\frac{x-10}{15}\right)^2 + 32\left(\frac{x-10}{15}\right), \tag{5.26}$$

from which we see that y is a quadratic function of x along C, so the curve is a parabola.

Example 5.13 Sometimes a little algebraic ingenuity is required. Given

$$x(t) = t^3 + t + 1,\ y(t) = t^2\ , \tag{5.27}$$

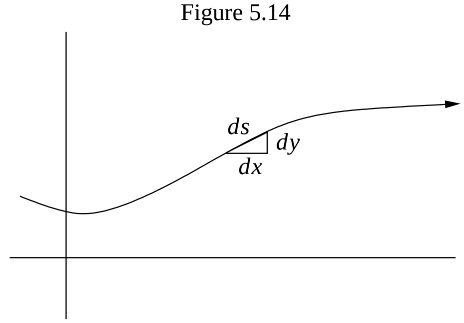
we eliminate t in this way:

$$x = t(t^2+1)+1 = \sqrt{y}(y+1)+1 \quad \text{or} \quad (x-1)^2 = y(y+1)^2\ , \tag{5.28}$$

so the relation between x and y is quadratic in x and cubic in y.

Finally, let us see how to calculate the length of a curve. Suppose that C is a curve in the plane running from point P to Q. First of all, observe that arc length is accumulative: if we break the curve up into many pieces, the length of the curve is the sum of the lengths of the pieces. We thus anticipate that arc length can be found by integration; in order to discover what to integrate, we seek a differential relationship between arc length and the coordinates. For R any point on the curve, we consider the length of the arc from P to R. It is customary to use the letter s as the function representing arc length along the curve: thus $s(R)$ is the length of the curve from P to R. If we now move along the curve C a small distance ds from R, then the variables are displaced by small amounts dx and dy (see figure 5.14).

Figure 5.14



Near R the curve is almost its tangent line, so we use the tangent line approximation to the length. By the pythagorean theorem we have

$$ds^2 = dx^2 + dy^2 \tag{5.29}$$

and thus the length of arc from P to Q is $L(C) = \int_P^Q ds = \int_P^Q \sqrt{dx^2+dy^2}$.

To do this integration, we use the equations describing a curve, explicit or parametric. We warn the reader that usually this computation leads to an expression we do not yet know how to integrate; in this case we shall leave it as a definite integral.

Example 5.14 What is the length of the line $y = 3x+2$ between the points $P(1,5)$ and $Q(3,11)$?

Here we can express the form 5.29 in terms of x. We have $dy = 3dx$, so

$$ds^2 = dx^2 + dy^2 = dx^2 + 3^2dx^2 = 10dx^2 \tag{5.30}$$

so $ds = \sqrt{10}dx$. Then

$$L = \int_P^Q ds = \int_1^3 \sqrt{10}dx = \sqrt{10}(3-1) = 2\sqrt{10} \tag{5.31}$$

which is, of course, the same answer we would have gotten from the distance formula of chapter I.

Example 5.15 Find the length of the arc of a circle of radius R subtended by an angle α.

Here we parametrize the circle as in example 5.9 (see figure 5.12): $x = R\cos t,\ y = R\sin t$. Then $dx = -R\sin t dt,\ dy = R\cos t dt$ and $2ds^2 = dx^2 + dy^2 = (-R\sin t)^2 t^2 + (R\cos t)^2 dt^2 = R^2(\sin^2 t + \cos^2 t)dt^2 = R^2 dt^2$. Our length is thus

$$L = \int ds = \int_0^\alpha R dt = R\alpha \ , \tag{5.32}$$

another standard formula.

Example 5.16 Find the length of the curve $x^2 = y^3$ from (0,0) to (27,9).

First we have to find a suitable parametrization of the curve. Since the variables are positive in this range, we can write $y = t^2$, and then $x^2 = y^3 = (t^2)^3 = (t^3)^2$, so we can take $x = t^3$, as t ranges from 0 to 3. Now $dy = 2t dt$ and $dx = 3t^2 dt$, so

$$ds^2 = dx^2 + dy^2 = 9t^4 dt^2 + 4t^2 dt^2 = t^2(9t^2 + 4)dt^2 \ ; \tag{5.33}$$

that is, $ds = t\sqrt{9t^2 + 4}dt$. Then the length is

$$L = \int ds = \int_0^3 t\sqrt{9t^2 + 4}dt == \frac{1}{18}\int_4^{85} u^{1/2} du \tag{5.34}$$

using the substitution $u = 9t^2 + 4, du = 18t dt$. The limits of integration in u are found by evaluating u at $t = 0$ and $t = 3$. The length is

$$L = \frac{1}{18}\frac{2}{3}u^{3/2}\Big|_4^{85} = \frac{1}{27}(85^{3/2} - 8) \ . \tag{5.35}$$

Example 5.17 Find the length of the parabolic segment $y = 5x - x^2$ from $x = 0$ to $x = 5$.

We have $dy = (5 - 2x)dx$, so $ds^2 = (1 + (5 - 2x)^2)dx^2$, and the length is

$$L = \int_0^5 \sqrt{1 + (5 - 2x)^2}dx = \frac{1}{2}\int_{-5}^5 \sqrt{1 + u^2}du \ , \tag{5.36}$$

making the substitution $u = 2x - 5,\ du = 2dx$. But that didn't help; we still end up with an integral we have yet to learn how to integrate.

§5.3. Work

Work is the product of force and distance. The unit of force in the British system is the foot-pound; in the metric system it is the joule, or newton-meter. For example, to move a box a distance of 12 feet against a force of friction of 40 lbs takes 40·12 = 480 foot-pounds. To lift a weight of 25 pounds a distance of 8 feet takes 25·8 = 200 foot-pounds. Remember that the weight (at the surface of the earth) of an object is a measure of its gravitational force, and is equal to mg, where m is the mass of the object and g is the acceleration of gravity. In the metric system, the gram and the kilogram are measures of mass, so to find the weight (in joules) we must multiply the mass (in kilograms) by g = 9.8 m/sec^2.

Example 5.18 How much work is done in lifting a mass of 1.2 kg a distance of 12 meters?
The weight of the mass is (1.2)(9.8) newtons, so the work is (1.2)(9.8)(12) = 141.12 joules.

Now, suppose that the force is not constant, but varies over the distance traversed. For example, when we put a rocket in orbit, we must calculate the energy (work) required. Now the force on the rocket (its effective weight) decreases as it leaves the surface of the earth. We shall take up this problem below.

So, suppose we have a force acting along a line; denote the value of the force at the point x as $F(x)$. If $F(x)$ is positive, it works in the direction of increasing x, and if negative, it works opposite to that direction. That is, if the force is positive, it does work as we progress, and if negative, it causes work as we progress. Work is accumulative: If we break the interval up into pieces, the total work done is the sum of the work done over each piece. We set up a work calculation by integration, once we know the differential relation. Let $W(x)$ be the work done to get from a to x. If we move a small distance dx further, we may assume the force to be constant, and equal to $F(x)$ over this interval. Then the increment in work is $dW = F(x)dx$. We now find the total work by integrating:

$$W = \int_a^b F(x)dx \,. \tag{5.37}$$

In particular, if this is positive, the force is doing work over the interval, and if negative, we must supply the work to overcome it.

Definition 5.1 *Let $F(x)$ be a force acting along the real axis, such that $F(x) > 0$ when F is acting in the direction of increasing x. Let $a < b$. The work done by the force over the interval $[a,b]$ is $W = \int_a^b F(x)dx$.*

Example 5.19 Let $F(x) = x(1-x^2)$ be a force acting on the x axis. How much work is done by the force a) between the points $0,1$? b) between the points $0,2$?

$$W = \int_0^1 (x-x^3)dx = \frac{x^2}{2} - \frac{x^4}{4}\Big|_0^1 = 1/2 \tag{5.38}$$

$$W = \int_0^2 (x-x^3)dx = \frac{x^2}{2} - \frac{x^4}{4}\Big|_0^2 = -2 \tag{5.39}$$

Thus the force does work for us between 0 and 1, but against us between 1 and 2.

§5.3.1 Springs

Consider a spring hanging vertically. If the spring is extended or compressed, a force will be exerted, called the *restoring force*, directed toward the initial rest position. If a weight is carefully attached to the spring, it will extend to a new rest position, at which the weight exactly balances the restoring force. This is the *equilibrium position* for that weight. According to Hooke's Law, the magnitude of the restoring force and the displacement from rest are proportional. That is, if we let x represent the displacement from the equilibrium position, and $F(x)$ the force in the spring, then we have

$$F(x) = -kx \tag{5.40}$$

for some constant k, called the *spring constant*. The negative sign indicates that the force acts in the direction opposite to the displacement.

Example 5.20 Suppose a 5 lb weight extends a spring by 1.5 inches. a) What is the spring constant? b) How far will a 12 lb. weight extend the spring?

The force in the spring at 1.5 inches, or .125 feet is -5 lbs, so by (5) $-5 = -k(.125)$, so $k = 40$ lb/ft, and Hooke's law for this spring is $F(x) = -40x$. When the 12 lb weight is attached, the restoring force is $F(x) = -12$, so we find x by solving $-12 = -40x$, or $x = .3$ ft or 3.6 inches.

Example 5.21 Suppose a 3 pound weight displaces a spring 2 inches. How much work is required to displace the spring two feet?

First we use Hooke's law with the given information to find the spring constant. In our situation, when $x = 1/6$ foot, $F = -3$ pounds. From Hooke's law, $-3 = -k(1/6)$, so the spring constant is $k = 18$, and Hooke's law for this spring is $F(x) = -18x$. Now, for work, we have $dW = F(x)dx$, so to extend the spring 2 feet, we calculate the work done by the spring over this distance:

$$W = \int_0^2 -18x dx = -9x^2 \Big|_0^2 = -36 \tag{5.41}$$

ft-lbs. Thus the work needed to extend the spring 2 feet must counterbalance this, so is 36 foot-pounds.

Now if a spring with an object at its end is extended a certain distance and then released, it will vibrate. To understand this motion, we recall the discussion of Example 4.11. There we saw that for a body in motion $vdv = adx$, where x is the displacement, v the velocity and a the acceleration. If we multiply by the mass m and use Newton's law, $F = ma$, we get

$$mvdv = Fdx. \tag{5.42}$$

The expression on the right is dW, the differential of work, and the expression on the left is the differential of $(1/2mv^2$, the kinetic energy: the change in kinetic energy is equal to the work done. This is then the differential expression of the law of conservation of energy. Now, for a spring, $F(x) = -kx$, so 5.42 becomes

$$mvdv + kxdx = 0 \tag{5.43}$$

for a spring, which integrates to

$$mv^2 + kx^2 = \text{constant} \tag{5.44}$$

during the motion of a vibrating spring of spring constant k and a mass m at its end.

Example 5.22 A spring of spring constant $k = 4$ ft/lb with an object of weight 3 lbs attached to its end is at rest. The object is extended a distance of 2 feet and then released. At what velocity does the object pass its equilibrium position?

The mass of the object is $m = w/g = 3/(32) = .0938$. Thus, in this situation 5.3.1 tells us that $.0938v^2 + 4x^2$ is constant during the motion. At the moment of release, $v = 0$ and $x = 2$, so this constant is $.0938(0)^2 + 4(2)^2 = 16$. At the moment the object passes the equilibrium point, $x = 0$, so we can solve:$.0938v^2 = 16$, for $v = 18.48$ ft/sec.

Example 5.23 A 1 kg mass extends a spring 8 cm. Suppose that a mass of 3 kg is attached to the spring, then extended 20 cm. beyond equilibrium and released. What will be the velocity of the mass as it passes the 10 cm. point?

First we must find the spring constant. The weight of a 1 kg mass is $w = mg = (1)(9.8) = 9.8$ newtons. From Hooke's law we have $-9.8 = -k(0.08)$, so $k = 9.8/0.08 = 122.5$ newtons/meter. For our 3 kg mass in motion, equation 5.3.1 is $3v^2 + 122.5x^2 = \text{constant}$. At the moment of release, $v = 0$ and $x = 0.2$, so the constant is $122.5(0.2)^2 = 4.9$. At the 10 cm point, we solve $3v^2 + 122.5(.1)^2 = 4.9$, giving $v = 1.107$ meters/second.

Example 5.24 What is the energy required to lift a payload of 1000 lbs a distance of 250 miles from the surface of the earth?

We have to calculate the work required to move the payload 250 miles vertically from the earth's surface. Let s represent distance from the center of the earth, and let R be the radius of the earth. According to Newton's law of gravitation, the force of gravity is

$$F(s) = -\frac{k}{s^2} \tag{5.45}$$

for some constant k. Now, when $s = R$, for our payload, $F(R) = -10^3$ lbs, so $k = 10^3R^2$, and $F(s) = -10^3R^2/s^2$. Thus the work "done by" gravity is

$$W = -\int_R^{R+250} \frac{10^3R^2}{s^2}ds = -10^3R^2\left(-\frac{1}{s}\right)\Bigg|_R^{R+250} = -10^3R^2\left(\frac{250}{R(R+250)}\right) . \tag{5.46}$$

Taking $R = 3900$ miles, we see that the energy required to lift this payload to 250 miles is

$$10^3(3900)\frac{250}{4150} = 2.34(10^5) \quad \text{mi}\cdot\text{lbs} = 12.4(10^9) \quad \text{ft}\cdot\text{lbs} \tag{5.47}$$

Example 5.25 A cistern is a hole in the ground (usually lined with steel or cement) used to collect water. Suppose that we have a cylindrical cistern of radius 8 feet and of depth 20 feet. If it is full of water, how much work is required to empty the cistern?

In this problem, the force is the weight of the water (one cubic foot of water weighs 62.5 lbs), but the distance the water has to be moved depends upon how deep into the cistern we have gotten. So, we again approach the problem dynamically, where the dynamic variable is the depth y of the water. At this depth, the work required to lift the next slab of water (of thickness dy) is $y\times$ (weight of that slab). The weight of that slab is $62.5\times$(volume). The slab is a cylinder of radius 8 feet and thickness dx. Thus

$$dW = (62.5)\pi(8)^2 y dy \ . \tag{5.48}$$

Since the cistern is 20 feet deep, the total work then is the integral

$$\int_0^{20} 62.5\pi(8)^2 y dy = 4000\pi \frac{y^2}{2}\Big|_0^{20} = 8\times 10^5\pi \tag{5.49}$$

ft $\cdot$lbs.

Example 5.26 Suppose instead that the cistern, of the same dimensions, has a parabolic profile, following the curve $y = (5/16)x^2$. Now, to lift the slab of thickness dy at a depth y takes the amount of work

$$dW = (62.5)\pi x^2 y dy \ , \tag{5.50}$$

because the radius at this level is now x. Since $x^2 = (16/5)y$, the total work is given by

$$\int_0^{20} 62.5\pi \frac{16}{5} y^2 dy = 200\pi \frac{y^3}{3}\Big|_0^{20} = 5.33\times 10^5\pi \tag{5.51}$$

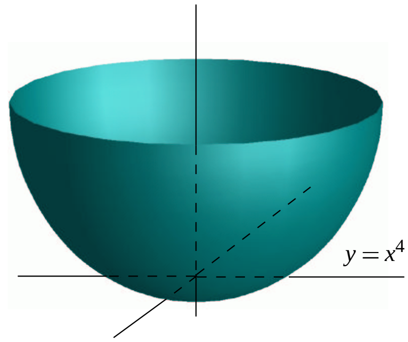
§5.4. Mass and Moments

Mass is another concept which is accumulative, and so can be calculated by integration. An object is said to be *homogeneous* if its composition is everywhere the same. In this case, mass is proportional to volume, where the constant of proportionality is denoted δ, the density. Thus, for example, the density of water is 1 g/cc, or 62.5 lbs/cu.ft. If the object is inhomogeneous, then its density will vary as we move around the object. For x a point on the object, we let $\delta(x)$ be its density at that point - in the sense that $dM = \delta(x)dV$ at the point: the ratio of the mass to volume of a small cube centered at x is approximately $\delta(x)$.

Example 5.27 A cistern is formed by rotating the curve $y = x^4$ from $x = 0$ to $x = 1$ yd. around the y-axis. The cistern is filled with muddy water which has settled, so that the density of the water at a height y from the bottom is $\delta(y) = (1.02 - .02y^2)62.5$ lbs/cu.ft (see figure 5.15). What is the total weight of the fluid contained in the cistern?

We calculate the weight by accumulating fluid along the y-axis, from $y = 0$ to 1. At a height y, the weight of the disc of height dy is $dW = \delta(y)dV$, and $dV = \pi r^2 dy$, where r is the radius of that disc. Since the profile is the curve $x = y^{1/4}$, we have $dW = \pi\delta(y)y^{1/2}dy$. Finally, since our spatial measurements

Figure 5.15



are given in yards, we convert the density to lbs/yd: $\delta(y) = (1.02 - .02y^2)(62.5)(27)$ lbs/cu.yd. Thus the weight is

$$W = \int_0^1 (62.5)(27)\pi\left(1.02 - .02y^2\right)y^{1/2}dy = 5301.3\int_0^1 \left(1.02y^{1/2} - .02y^{5/2}\right)dy \tag{5.52}$$

$$= 5301.3\left((1.02)\frac{2}{3}y^{3/2} - (.02)\frac{2}{7}y^{7/2}\right)\Bigg|_0^1 = 3574.6 \text{ lbs} \tag{5.53}$$

Example 5.28 A baseball bat can be considered as a cone of height 28 in, and of base radius 1.5 in. If the bat is made of hickory, of density 0.0347 lb/in^3, what is the weight of the bat?

We accumulate the weight of the bat from its vertex to its base, At a distance x in from the vertex, a slice of thickness dx has volume $dV = \pi r^2 dx$, where r is the radius at that point. By similar triangles, $r/1.5 = x/28$, so $r = 1.5x/28 = .0535x$. Then the weight of the bat is

$$W = \int \delta dV = \int_0^{28} (.0347)\pi(.0535x)^2 dx = (3.12\times 10^{-4})\int_0^2 8x^2 dx = 2.283 \text{ lbs} \tag{5.54}$$

§5.4.1 Moments

Suppose that we have two small objects situated on a plane, and L is a line which runs between the objects (see figure 5.16).

Archimedes observed that the objects are balanced around the line L if the product of the mass and the distance to the line is the same for the two objects. By "balance" we mean this: if the line L were a rod in space which is free to rotate, and the objects were attached to this rod by arms perpendicular to the rod (as in the figure), then the rod will not rotate if this condition is met. This product we call the *moment* of the object about the line L: $Mom_L = (mass)(distance)$. If we take distance to be directed:

Figure 5.16

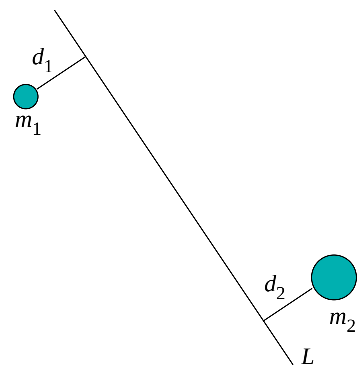
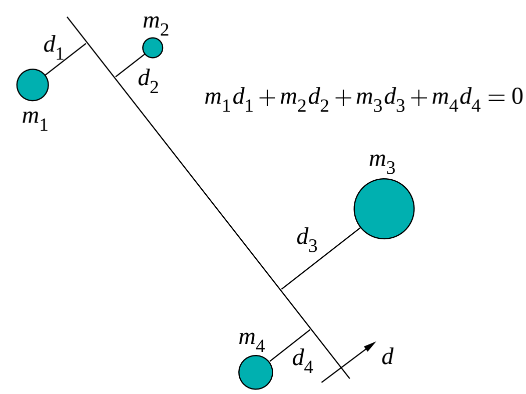


Figure 5.17



negative on one side of the line, and positive on the other, Archimedes' Law is that a system is in balance about a line L if the sum of the moments about L is zero. Phrased this way, the law applies to a system of many masses; for example the system in figure 5.17 is in balance.

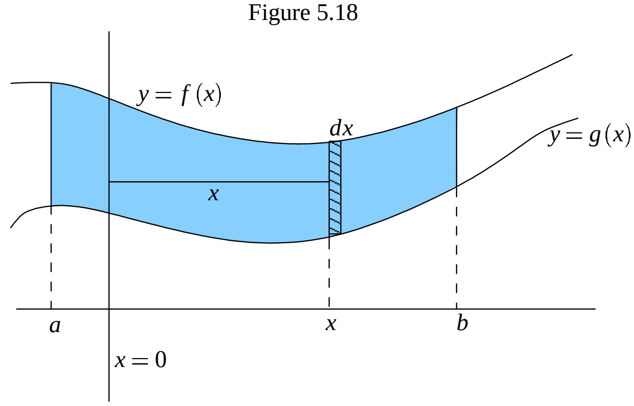
Thus, moment is an accumulative concept, and we can discover the moment for any object by integration. We shall consider only planar homogeneous objects of density 1, so that mass and area are the same. The first method for finding moment is to sweep out the region in the direction perpendicular to the line; at each stage adding the moment of a differential rectangle at a fixed distance from the line.

Suppose, to start with, we wish to find the moment about the y-axis $\{x = 0\}$ of a region is bounded by the curves $y = f(x)$, $y = g(x)$, from $x = a$ to $x = b$ (see figure 5.18). We calculate the moment by adding the moments of infinitesimal strips, starting at $x = a$, and going to $x = b$. At a point x, the next strip has height $f(x) - g(x)$ and width dx, so its mass is $(f(x) - g(x))dx$, and the moment about the y-axis is $dMom_{x=0} = \text{distance} \cdot \text{mass} = x(f(x) - g(x))dx$. The moment of the entire region is the integral of this differential from a to b:

(5.55)
$$Mom_{\{x=0\}} = \int_a^b x(f(x) - g(x))dx$$

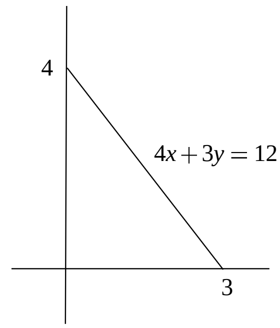
for the region of figure 5.18.

Figure 5.18



Example 5.29 Consider the region bounded by the coordinate axes and the line $4x+3y=12$ (see figure 5.19). Find the moments of this region about the coordinate axes.

Figure 5.19



To find $Mom_{\{x=0\}}$, we sweep out along the x axis from $x=0$ to $x=3$. The height of the strip at a point x is $y=(12-4x)/3$, and the distance from the line $x=0$ is x. Thus $dMom_{x=0}=(1/3)x(12-4x)dx$, and the moment is

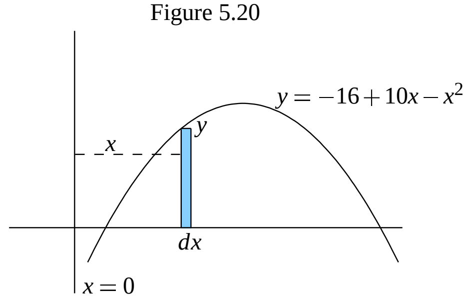
$$Mom_{\{x=0\}} = \int_0^3 \frac{1}{3}\left(12x-4x^2\right)dx = \frac{1}{3}\left(6x-\frac{4x^3}{3}\right)\Bigg|_0^3 = 6\ . \tag{5.56}$$

To find the moment about the x-axis, we sweep the region out in the vertical direction from $y=0$ to $y=4$. At a point y, the width of the strip is $x=(12-3y)/4$, so $dMom_{\{y=0\}}=(y/4)(12-3y)dy$, so

$$Mom_{\{y=0\}} = \int_0^4 \frac{1}{4}\left(12y-3y^2\right)dy = 8\ . \tag{5.57}$$

Example 5.30 Find the moment about the y-axis of the region bounded by the x axis and the curve $y=-16+10x-x^2$ (see figure 5.20).

Figure 5.20



The region will be swept out along the x-axis from the points where the curve intersects the x-axis. We solve $0=-16+10x-x^2=-(8-x)(2-x)$. The region thus lies between the lines $x=2$ and

$x = 8$. At an intermediate point x, the differential strip is at distance x from the y-axis, and the mass is $(-16+10x-x^2)dx$, so

$$dMom_{\{x=0\}} = x(-16+10x-x^2)dx \tag{5.58}$$

and

$$Mom_{\{x=0\}} = \int_2^8 x(-16+10x-x^2)dx = \int_2^8 (-16x+10x^2-x^3)dx \tag{5.59}$$

$$= \left(-8x^2+\frac{10}{3}x^3-\frac{x^4}{4}\right)\Big|_2^8 = 180\ . \tag{5.60}$$

Example 5.31 Find the moment about the line $x = 4$ of the same region.

Here we have the same analysis, but now the distance from the strip to the balance axis is $x-4$, so

$$dMom_{\{x=4\}} = (x-4)\left(-16+10x-x^2\right)dx\ . \tag{5.61}$$

Note that if $x < 4$ this will be negative, and if $x > 4$ it is positive - this is just what we want because we want the contributions of the pieces on either side of the axis to be opposite. To find the moment we integrate;

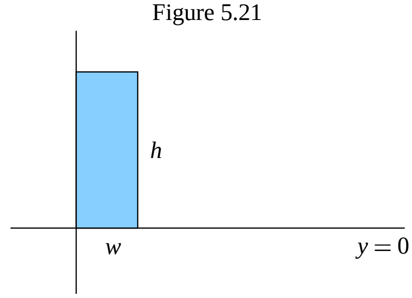
$$Mom_{\{x=4\}} = \int_2^8 (x-4)(-16+10x-x^2)dx = \int_2^8 (64-56x+14x^2-x^3)dx = 36\ . \tag{5.62}$$

Since the answer is positive, this region is overbalanced to the right. Note that the region is symmetric about the line $x = 5$, so that it is perfectly balanced about the line $x = 5$. If we calculate $Mom_{\{x=5\}}$, we'll get 0.

Example 5.32 Find the moment of a rectangular strip of width w and height h about the line through one end.

This is $Mom_{\{y=0\}}$ of the region sketched in figure 5.21. Sweeping out from $y = 0$ to $y = h$, we have $dMom_{\{y=0\}} = ywdy$, so $Mom_{\{y=0\}} = \int_0^h ywdy = \frac{1}{2}wh^2$.

Figure 5.21



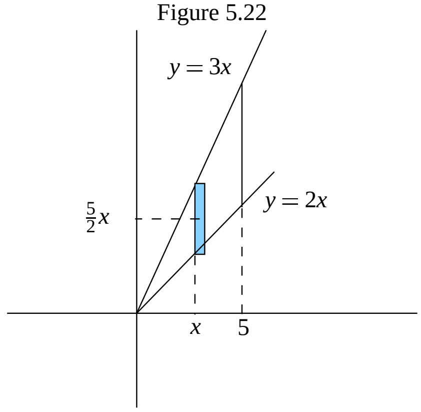
Writing the moment as $wh(h/2)$, we see that this is the moment of a point object of the same mass (wh situated at the midpoint of the strip. This observation leads to an alternative method for calculating the moment about $y = 0$ without having to change the variable of integration.

Consider the region bounded by the curves $y = f(x)$ and $y = g(x)$ and the lines $x = a$ and $x = b$, as in figure 5.18. We calculate the moment about the x-axis by sweeping out in the x direction. At any point x the differential piece to be added is a vertical rectangular strip of mass $(f(x) - g(x))dx$. The midpoint of this rectangle is $(1/2)(f(x) + g(x))$. Thus

$$dMom_{\{y=0\}} = \frac{1}{2}(f(x) + g(x))(f(x) - g(x))dx \; . \tag{5.63}$$

Example 5.33 Find the moment about the x-axis of the region in the first quadrant bounded by the lines $y = 3x,\ y = 2x,\ x = 5$ (figure 5.22).

Figure 5.22



We sweep out from $x = 0$ to $x = 5$. The differential strip at the point x has the mass $(3x - 2x)dx = xdx$, and its midpoint is $(1/2)(3x + 2x) = (5/2)x$. Thus

$$dMom_{\{y=0\}} = \frac{5}{2}x^2 dx \; , \tag{5.64}$$

so

$$Mom_{\{y=0\}} = \int_0^5 \frac{5}{2}x^2 dx = \frac{625}{6} \; . \tag{5.65}$$

To find the moment about the y-axis, we have

$$dMom_{\{x=0\}} = x(3x - 2x)dx = x^2 dx \tag{5.66}$$

so $Mom_{\{x=0\}} = \int_0^5 x^2 dx = 125/3$.

Example 5.34 Find the moment about the x-axis of the region of example 30, bounded by the x axis and the curve $y = -16 + 10x - x^2$.

At a point x between 2 and 8, the differential mass is ydx, and the midpoint of that strip is $y/2$, so $dMom_{\{y=0\}} = (y^2/2)dx$. Thus

$$Mom_{\{y=0\}} = \frac{1}{2}\int_2^8 (-16+10x-x^2)^2 dx = \frac{648}{5}\ . \tag{5.67}$$

§5.4.2 Centroids

Let R be a region in the plane and L a line in the plane. If R lies to the right of the line, then the moment of R about L, $Mom_L R$, is positive, and if R is to the left of L, then $Mom_L R$ is negative. So, if we look at all lines with a given slope, as we move from one side of R to the other, the moment about that line changes sign. Thus, there is a particular line with a given slope for which the moment of R is zero. The region R is *balanced* about this line: if R were to walk a tightrope in this direction, it would have the tightrope directly below this line. If we change the slope, we get another line of balance in the new direction. These two lines intersect in a point. It turns out that the line of balance of any slope goes through this point, called the *centroid*, or *center of mass* of the region R

Definition 5.2 *The centroid of a region R in the plane is that point in the plane such that for any line L through that point,* $Mom_L R = 0$.

To calculate the centroid, it is enough to look at lines of two different slope, in particular, horizontal and vertical lines. Suppose $(\bar{x}, \bar{y})$ is the centroid of R, so that the moment about the line $x = \bar{x}$ is zero. The distance from any point (x, y) to this line is $x - \bar{x}$, so we have

$$Mom_{\{x=\bar{x}\}}(R) = \int (x - \bar{x}) dA = 0 \tag{5.68}$$

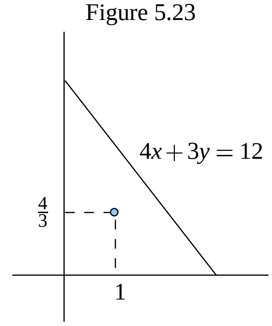
or $\int xdA = \bar{x}\int dA$, which says that $Mom_{\{x=0\}}(R) = \bar{x}Mass(R)$. Similarly we get $Mom_{\{y=0\}}(R) = \bar{y}Mass(R)$. Thus the coordinates for the centroid are

$$\bar{x} = \frac{Mom_{\{x=0\}}(R)}{Mass(R)}\ , \qquad \bar{y} = \frac{Mom_{\{y=0\}}(R)}{Mass(R)}\ . \tag{5.69}$$

Example 5.35 Find the centroid of the triangle in example 29.

This is a right triangle with sides of length 4, 3, so the mass is $(1/2(3)(4) = 6$. We found $Mom_{\{x=0\}} = 6$, $Mom\{y = 0\} = 8$, so the centroid is at $(\bar{x}, \bar{y}) = (6/6, 8/6) = (1, 1.33)$ (see figure 5.23).

Figure 5.23



Example 5.36 Find the centroid of the region in the first quadrant bounded by the lines $y = 3x,\ y = 2x,\ x = 5$ (figure 5.22).

In example 5.33, we found $Mom_{\{x=0\}} = 125/3,\ Mom_{\{y=0\}} = 625/6$. The mass is

$$Mass = \int_0^5 (3x - 2x)dx = \frac{x^2}{2}\Big|_0^5 = \frac{25}{2}\ . \tag{5.70}$$

Thus $\bar{x} = (125/3)/(25/2) = 10/3$ and $\bar{y} = (625/6)/(25/2) = 25/3$.

Example 5.37 Find the centroid of the region bounded by the y axis and the curve $y = -16 + 10x - x^2$ (see figure 5.20).

First of all, since the region is symmetric about the line $x = 5$, we have $\bar{x} = 5$. In example 5.34, we found $Mom_{\{y=0\}} = 648/5$. It remains to calculate the mass of the region which is

$$\int_2^8 \left(-16 + 10x - x^2\right) dx = \left(-16x + 5x^2 + \frac{x^3}{3}\right)\Big|_2^8 = 36. \tag{5.71}$$

Thus $\bar{y} = (648/5)/36 = 3.6$. The centroid is at $(5, 3.6)$. As a final application of moments, we derive

Theorem 5.1 (Pappus' Theorem) *Let R be a region in the right half plane, and consider the solid obtained by rotating R about the y-axis. The volume of this solid is the product of the area of R and the distance traveled by the centroid of R.*

This is easy to see using the shell method for finding the volume. By that method,

$$Volume = 2\pi \int xdA \tag{5.72}$$

where the integration is taken along the x-axis between the bounding lines. But $\int xdA = Mom_{\{x=0\}} = \bar{x}\,(Area)$, where the centroid has coordinates $(\bar{x}, \bar{y})$. So

$$Volume = 2\pi\bar{x}(Area)\ , \tag{5.73}$$

which is what is asserted by by Pappus' theorem. Although at Pappus' time the calculus didn't even exist (in fact, neither did algebra), he demonstrated this result in essentially the same way, using Archimedes' theory of moments.

Example 5.38 Find the volume of the solid ring obtained by rotating the disc $(x-5)^2 + y^2 = 16$ about the y-axis.

The region being rotated is the circle centered at $(5, 0)$ and of radius 4. Clearly the centroid of a disc is its center, so $\bar{x} = 5$. Thus the volume is $2\pi(5)\pi(4^2) = 160\pi^2$.