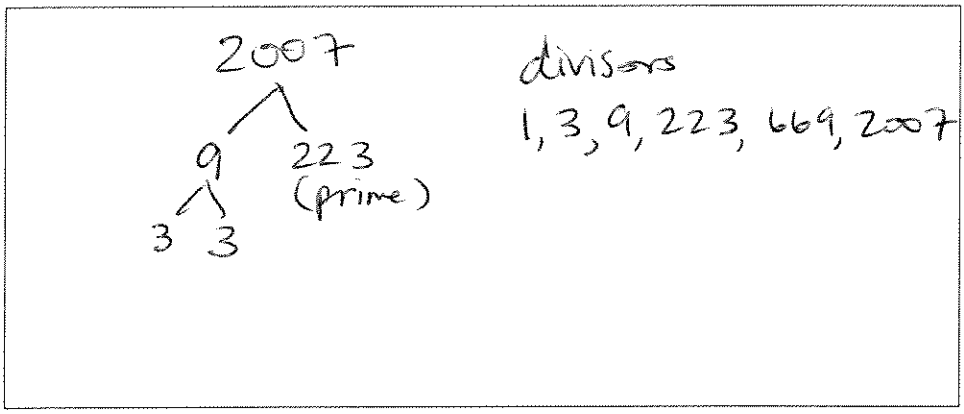
State Junior Mathematics Contest
Spring 2007

1. How many whole number divisors does the number **2007** have?

(*a*) 2 (*b*) 3 (*c*) 4 (*d*) 6 (*e*) 8



2. $66_{\text{eight}} + 132_{\text{four}} + 1011_{\text{two}} =$ _____$_{\text{eight}}$.

(*a*) 101 (*b*) 135 (*c*) 137 (*d*) 1131 (*e*) 1211

$66_8 = 6(8) + 6(1) = 48 + 6 = 54$
$132_4 = 1(16) + 3(4) + 2(1) = 30$
$1011_2 = 1(8) + 0(4) + 1(2) + 1(1) = 11$
$54 + 30 + 11 = 95$

$95 = 1(64) + 3(8) + 7(1) = 137_8$

3. The decimal $0.\overline{9} = 0.999\ldots$ is equal to

(a) 1 (b) $1-(\frac{9}{10})^{10}$ (c) $(\frac{9}{10})^{\frac{10}{9}}$ (d) 999/1000 (e) 9/10

let $n = 0.\overline{9} \Rightarrow 10n = 9.\overline{9}$

$$\Rightarrow \begin{array}{r} 10n = 9.\overline{9} \\ -\ n = 0.\overline{9} \\ \hline 9n = 9 \end{array}$$

$n = 1$

4. How many whole numbers between 99 and 999 are divisible by 4, 6 and 9?

(a) 13 (b) 24 (c) 25 (d) 27 (e) 31

LCM of 4, 6, 9 = 36

$$\begin{array}{r} 27 \\ 36\overline{)999} \\ -72 \\ \hline 279 \\ -252 \\ \hline 27 \end{array}$$

36 goes into 999 27 times but $36 + 72 < 99$

$\Rightarrow$ 25 divisors

5. If $M/5$ has a remainder of 2 and $N/5$ has a remainder of 4, then $(M+N)/5$ will have a remainder of what?

(a) 0 (b) 1 (c) 2 (d) 3 (e) 5

$\frac{M+N}{5} = \frac{M}{5} + \frac{N}{5}$

$\downarrow \quad \downarrow$

R2 + R4 $\Rightarrow$ There is a remainder of 6 $\Rightarrow$ 5 R1

6. How many children are there in a family in which each boy has as many sisters as brothers but each girl has twice as many brothers as sisters?

(a) 4 (b) 5 (c) 6 (d) 7 (e) 8

Try fitting condition 1:

GBB no

GGBBB no

GGGBBBB yes

7. Five players are on the Academic Team. Their names do not indicate their gender.

- Three are girls and two are boys.
- Two wear white shirts and three wear black shirts.
- MIKEN and CARRY wear different color shirts. ⇒ one is white, one is Black
- BARI and JAMIE wear the same color shirt. ⇒ must be Black
- PITA and CARRY are the same gender. ⇒ must be Girls
- JAMIE and MIKEN are different genders. ⇒ one is Girl, one is Boy
- The boy with the white shirt scored the most points.

Who is the leading scorer?

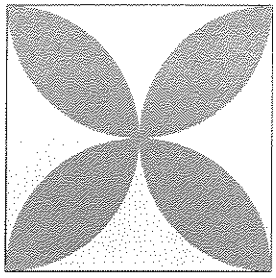
(a) *BARI* (b) *CARRY* (c) *JAMIE* (d) *MIKEN* (e) *PITA*

	Boy	Girl	White	Black
Miken				
no Bari				X
no Carry		X		
no Jamie				X
no Pita		X		

Bari can't be it because black shirt
Jamie " " " " " "
Carry " " " " girl.
Pita " " " " girl.

⇒ Miken is boy w white shirt.

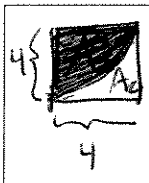
8. Find the area of the shaded region. The petals are formed by semicircles and the square is 8 cm on a side.



Consider ¼ of this figure.

we want $A = 4A_1$

(a) $32(\pi-2)$ (b) $8(\pi-1)$ (c) $16(\pi-1)$ (d) $64-32\pi$ (e) $16(\pi-2)$



$A_2 = A_{square} - \frac{1}{4}A_{circle} = 4^2 - \frac{1}{4}\left(\pi(4^2)\right)$

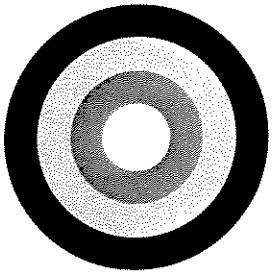
$= 16 - 4\pi$

$A_1 = 4^2 - 2A_2 = 16 - 2(16-4\pi)$

$= 16 - 32 + 8\pi = 8\pi - 16$

$\Rightarrow A = 4(8\pi-16) = 32\pi - 64 = 32(\pi-2)$

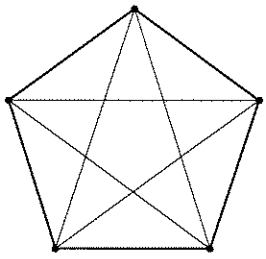
9. The radii of the circles on this target are 1, 2, 3, and 4 inches. What is the probability that a random shot that hits the target will hit the bull's eye (i.e., the inner circle)?



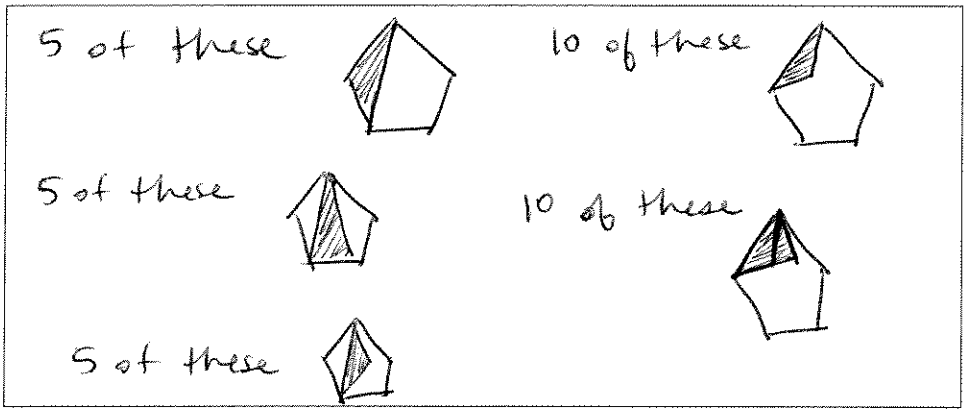
(a) 1/4 (b) 1/8 (c) 1/16 (d) $1/4\pi$ (e) $1/2\pi$

$$P(\text{bull's eye}) = \frac{\pi(1^2)}{\pi(4^2)} = \frac{\pi}{16\pi} = \frac{1}{16}$$

10. How many triangles are in this drawing?



(a) 10 (b) 20 (c) 25 (d) 30 (e) 35



11. Solve for x. $|3x-2| \leq 17$

(a) no solution
(b) $x \geq \frac{19}{3}$ or $x \leq -5$
(c) $x \geq -5$
(d) $x \leq \frac{19}{3}$
(e) $-5 \leq x \leq \frac{19}{3}$

$$-17 \leq 3x-2 \leq 17$$
$$+2 \quad +2 \quad +2$$
$$\frac{-15}{3} \leq \frac{3x}{3} \leq \frac{19}{3}$$
$$-5 \leq x \leq 19/3$$

12. Let r be a real number–positive, negative or zero. Which of the following numbers is always greater than r?

(a) r^2+1 (b) $2r$ (c) $\sqrt{|r|}+\frac{r}{2}$ (d) $(r+1)^3$ (e) r^{100}

(b) if $r<-1$, $2r<r$

(c) if $r=100$, $\sqrt{100}+\frac{100}{2}=10+50=60<r$

(d) if $r<-3$, $(r+1)^3<-8<r$

(e) if $|r|<1$, $r^{100}<r$

13. Ms Trong gets a ten percent raise every year. Her salary after four such raises has gone up by about what percent?

(a) 40% (b) 42% (c) 44% (d) 46% (e) 48%

Let x = salary.

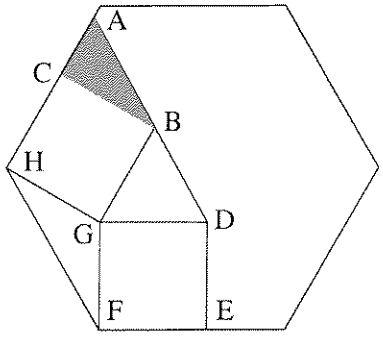
After 1 yr, salary is $1.1x$

After 4 yrs, salary is $1.1^4 x$

$1.1^4 = 1.21(1.21) = 1.4641$

$\Rightarrow$ ~46% increase

14. Given this regular hexagon of side $\sqrt{3}$, squares DEFG and CBGH, line ABD, find the area of the triangle ABC.



(a) $\frac{1}{4}$ (b) $\frac{1}{2}$ (c) $\frac{\sqrt{3}}{3}$ (d) $\frac{\sqrt{3}}{6}$ (e) *none*

measure of interior angles of regular hexagon $= 120°$

$m\angle GFE = 90°$ (square) $\Rightarrow m\angle GFH = 30°$

also $m\angle GHF = 30°$

$\Rightarrow \triangle GFH$ isosceles $\Rightarrow \overline{HG} \cong \overline{FG} \Rightarrow \square DEFG \cong \square CBGH$

$\Rightarrow m\angle DGB = 60°$ and it's an isosceles $\triangle$ since $\overline{BG} \cong \overline{DG} \Rightarrow \triangle BGD$ equilateral $\triangle$

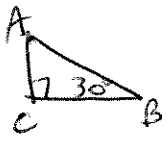
$\Rightarrow m\angle DBG = 60°$

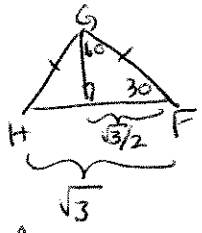
but $m\angle DBG + m\angle GBC + m\angle CBA = 180°$ (straight line)

$60° + 90° + m\angle CBA = 180°$

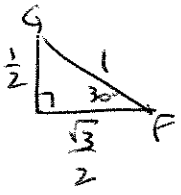
$m\angle CBA = 30°$

$\Rightarrow \triangle ABC$ is 30/60/90 $\triangle$





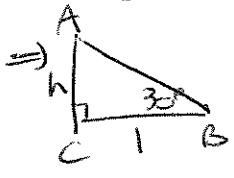
$\Rightarrow \frac{1}{2}$



i.e. $GF = 1$

but $\overline{GF} \cong \overline{GB} \cong \overline{BC}$

$\Rightarrow$



$\frac{1}{\frac{\sqrt{3}}{2}} = \frac{h}{\frac{1}{2}} \Leftrightarrow h = 1/\sqrt{3}$

$\Rightarrow$ area $= \frac{1}{2}(1)\left(\frac{1}{\sqrt{3}}\right) = \frac{1}{2\sqrt{3}} = \frac{\sqrt{3}}{6}$

15. For a function defined for all natural numbers by

$$f(n+1) = f(n) + f(n-1),$$

and beginning with $f(1) = 1, f(2) = 1$, for which value of n is $f(n)$ a multiple of 4?

(a) If $n = 3k$, then $f(n)$ is a multiple of 4
(b) If $n = 4k$, then $f(n)$ is a multiple of 4
(c) If $n = 5k$, then $f(n)$ is a multiple of 4
(d) If $n = 6k$, then $f(n)$ is a multiple of 4
(e) It is not possible to predict which terms will be a multiple of 4.

$f(1)=1$
$f(2)=1$
$f(n+1)=f(n)+f(n-1)$

$\Rightarrow f(3)=f(2)+f(1)=1+1=2$
$f(4)=f(3)+f(2)=2+1=3$
$f(5)=f(4)+f(3)=3+2=5$

Fibonacci sequence

1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, 144, ...
$n=1$ $n=2$ $n=3$ $n=4$ $n=5$ $n=6$

16. If you lose 20% on an investment during the first year and gain 25% the following year, what is your net gain over the two years?

(a) 0% (b) 5% (c) 2.5% (d) −5% (e) 1.25%

x = investment
after one year, investment $= 0.8x$
after second year, investment is $1.25(0.8x) = \frac{5}{4}\left(\frac{4}{5}x\right) = x$
$\Rightarrow$ we're back to where we started, i.e. 0 net gain

17. The number 2^{29} is a 9-digit number with distinct digits. Which digit is missing?

(a) 0 (b) 3 (c) 4 (d) 5 (e) 7

$2^{29} = 2^{10}\,2^{10}\,2^{9} = 1024\,(1024)(512)$

1024
1024
4096
2048
10240
1048576

1048576
× 512
2097152
1048576
5242880
536,870,912

18. If this pattern continues, where would the number 289 apppear?

1
3 5
7 9 11
13 15 17 19

(a) 8th element in row 16
(b) 9th element in row 17
(c) 9th element in row 18
(d) last element in row 17
(e) last element in row 18

row 1
1
row 2
3 5
row 3
7 9 11
row 4
13 15 17 19
row 5
21 23 25 27 29
row 6
31 33 35 37 39 41
row 7
43 45 47 49 51 53 55

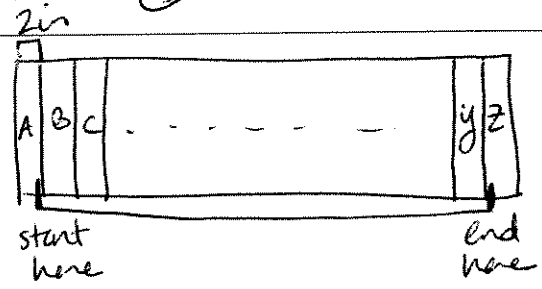
perfect squares

$289 = 17^2 \Rightarrow$ 17th row, in middle

$\Rightarrow$ 9th element

19. A set of 26 encyclopedias (one for each letter) is placed on a bookshelf in alphabetical order from left to right. Each encyclopedia is 2 inches thick including the front and back covers. Each cover (front or back) is 1/4 inch thick. A bookworm eats straight through the encyclopedias, beginning inside the front cover of volume A and ending after eating through the back cover of volume z. How many inches of book did the bookworm eat?

(a) 48 (b) 48.5 (c) 51.25 (d) 51.5 (e) 51.75



$2(26) - 1.75 - 1.75 = 52 - 3.5 = 48.5$

20. Solve for x. $2x^2 - 3x = 9 - 3x^2$

(a) $\frac{3\pm3\sqrt{7}}{10}$

(b) $\frac{-3\pm3\sqrt{21}}{10}$

(c) $\frac{3\pm3\sqrt{21}}{10}$

(d) $\frac{3\pm\sqrt{171}i}{10}$

(e) $\frac{-3\pm3\sqrt{7}}{10}$

$5x^2 - 3x - 9 = 0$

$a=5 \quad b=-3 \quad c=-9$

$x = \frac{-b\pm\sqrt{b^2-4ac}}{2a} = \frac{3\pm\sqrt{9-4(5)(-9)}}{2(5)}$

$= \frac{3\pm\sqrt{9(1+20)}}{10} = \frac{3\pm3\sqrt{21}}{10}$

21. If w, x, y, z are positive real numbers such that $w+x+y+z=2$, then

$$N=(w+x)(y+z)$$

satisfies

(a) $0 \le N \le 1$
(b) $1 \le N \le 2$
(c) $2 \le N \le 3$
(d) $3 \le N \le 4$
(e) $4 \le N \le 5$

Do first easy check $x=y=w=z=1/2$

$N=(\frac{1}{2}+\frac{1}{2})(\frac{1}{2}+\frac{1}{2})=1 \Rightarrow$ either (a) or (b)

Try $x=1/6$ $y=1/6$ $z=1/6$ $w=3/2$

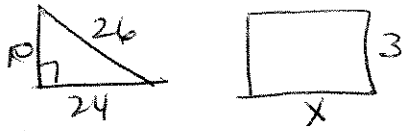
$\Rightarrow N=(\frac{3}{2}+\frac{1}{6})(\frac{1}{6}+\frac{1}{6})=(\frac{10}{6})(\frac{2}{6})=\frac{5}{3}(\frac{1}{3})=\frac{5}{9}<1$

$\Rightarrow$ (a)

22. Triangle ABC has sides 10, 24, and 26 cm long. A rectangle that has an area equal to that of the triangle has width 3 cm. Find the perimeter of the rectangle.

(a) 40 cm (b) 43 cm (c) 56 cm (d) 68 cm (e) 86 cm

We can tell it's a right Δ since the side lengths are multiple of 5-12-13 Pythagorean triple.



$A=\frac{1}{2}(10)(24)$
$=120$

$A=3x$

$\Rightarrow 120=3x \Leftarrow x=40$

Perimeter of rectangle
$=2(40)+2(3)$
$=80+6=86$

23. ${}_mP_n = {}_{10}C_7$ for what values of m and n?

(a) $m = 7, n = 10$
(b) $m = 10, n = 3$
(c) $m = 5, n = 4$
(d) $m = 5, n = 1$
(e) $m = 7, n = 5$

$${}_{10}C_7 = \frac{10!}{7!3!} = \frac{10 \cdot 9 \cdot 8}{3 \cdot 2 \cdot 1} = 10 \cdot 4 \cdot 3$$

$$= 5 \cdot 4 \cdot 3 \cdot 2 = 5!$$

$$= \frac{5!}{1!} = \frac{5!}{(5-4)!} = {}_5P_4$$

24. A recipe calls for $\frac{2}{3}$ of a cup of sugar. You find that you only have $\frac{1}{2}$ cup of sugar left. What fraction of the recipe can you make?

(a) $\frac{1}{6}$ (b) $\frac{1}{3}$ (c) $\frac{1}{2}$ (d) $\frac{3}{4}$ (e) $\frac{4}{3}$

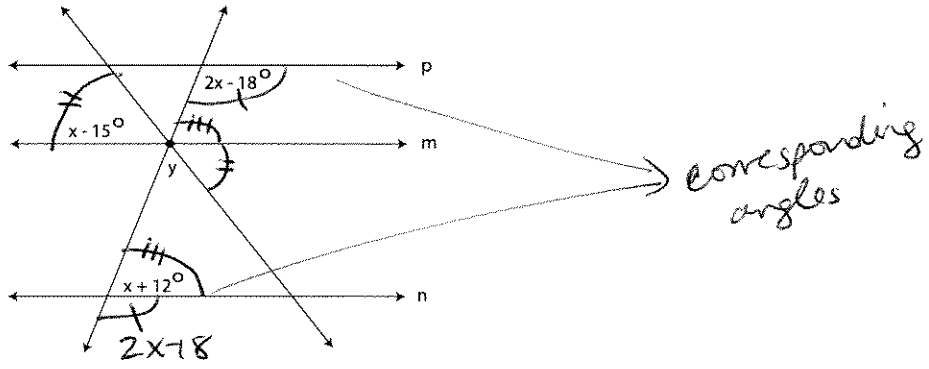
x = fraction of recipe I can make

$$\frac{1}{2} = x\left(\frac{2}{3}\right)$$

$$\frac{\frac{1}{2}}{\frac{2}{3}} = x$$

$$\frac{3}{2} \cdot \frac{1}{2} = x \iff x = 3/4$$

25. Find the measure of y given the following picture and the fact that $p||n||m$.



(a) 46^o (b) 62^o (c) 106^o (d) 74^o (e) 59^o

$2x-18+x+12=180$
$3x-6=180$
$3x=186$
$x=62^\circ$

$y+(x-15)+(x+12)=180$
$y+(62-15)+(62+12)=180$
$y+47+74=180$
$y+121=180$
$y=59^\circ$

26. In three years, Mary will be three times my present age, and I will be half as old as she. How old is Mary now?

(a) 6 yrs (b) 9 yrs (c) 15 yrs (d) 24 yrs (e) 27 yrs

x = my age now
m = Mary's age now

$m+3=3x$
$x+3=\frac{1}{2}(m+3)$

$\Leftrightarrow$

$x+3=\frac{1}{2}(3x)$
$x+3=\frac{3}{2}x$
$3=\frac{1}{2}x$
$6=x$ $\Rightarrow m=3(6)-3$
$=15$

27. Fred can mow the lawn in 3 hours. Joe can do it in 2 hours. If Fred, Joe and Susan work together to mow the lawn, they can do it in $\frac{12}{13}$ of an hour. How long does it take Susan to mow the lawn herself?

(a) 2 hrs (b) 3 hrs (c) 4 hrs (d) 5 hrs (e) 6 hrs

F 3 hrs
J 2 hrs
S x hrs
together 12/13 hr

$\frac{1}{3} + \frac{1}{2} + \frac{1}{x} = \frac{1}{12/13}$

$\frac{1}{3} + \frac{1}{2} + \frac{1}{x} = \frac{13}{12}$

$12x\left(\frac{1}{3} + \frac{1}{2} + \frac{1}{x}\right) = \frac{13}{12}(12x)$

$4x + 6x + 12 = 13x$

$12 = 3x$

$4 = x$

28. A young man spent $\frac{1}{4}$ of his allowance on a movie. He spent $\frac{11}{18}$ of the remainder on afterschool snacks. Then from the money remaining, he spent \$3 on a magazine, which left him with $\frac{1}{24}$ of his allowance to put into savings. How much of his allowance did he save?

(a) \$0.50 (b) \$1.00 (c) \$1.25 (d) \$12 (e) \$31

Let x = allowance

$\frac{1}{4}x + \frac{11}{18}\left(\frac{3}{4}x\right) + 3 + \frac{1}{24}x = x$

$\frac{1}{4}x + \frac{11}{24}x + 3 + \frac{1}{24}x = x$

$\frac{1}{4}x + \frac{12}{24}x - x = -3$

$\frac{3}{4}x - x = -3$

$-\frac{1}{4}x = -3 \iff x = 12$

$\Rightarrow$ saved $\frac{1}{24}(12) = \frac{\$1}{2}$ or \$0.50 or 50¢

29. Sally has 4 red flags, 3 green flags and 2 white flags. How many 9-flag signals can she run up a flagpole?

(a) 4!3!2! (b) $\frac{7!}{4}$ (c) 9! (d) $\frac{9!}{4!3!}$ (e) $\frac{7!}{4!3!}$

$${}_9C_4 \cdot {}_5C_3 \cdot {}_2C_2 = \left(\frac{9!}{4!5!}\right)\left(\frac{5!}{3!2!}\right)(1)$$

$$= \frac{9\cdot 8\cdot 7\cdot 6\cdot 5}{3\cdot 2\cdot 2} = 9\cdot 7\cdot 5\cdot 4 = 7\cdot 5\cdot 4\cdot 3(3)$$

$$= \frac{7\cdot 6\cdot 5\cdot 4\cdot 3\cdot 2}{4} = \frac{7!}{4}$$

30. At a party, 66 handshakes took place. Each person shook hands exactly once with each of the others present. How many people were at the party?

(a) 9 (b) 10 (c) 11 (d) 12 (e) 13

$1+2+3+\cdots+n$ = total # handshakes, assuming $n+1$ people there

$$1+2+3+\cdots+n = 66 = \frac{n(n+1)}{2}$$

$$132 = n^2+n$$

$$n^2+n-132=0$$

$$(n+12)(n-11)=0$$

$n = 11, -12$ ⇒ 12 people present @ party