On the Construction and Cohomology of a Self-Dual Perverse Sheaf Motivated by String Theory

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- Prof. R. MacPherson (IAS) for making the observation that a certain projective object in the category of perverse sheaves has the properties of the cohomology theory that I was looking for

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Overview

- Review and Motivation
- Hubsch Conjecture
- Mathematical Construction
- Perverse Sheaves and the Zig-Zag Category
- Statement of Main Result
- Duality
- Conclusion

Review

- *Witten 1982*: Close relationship between cohomology of real Riemannian space Y and the Hilbert space of a SUSY σ-model with target space Y
- Y is a real Riemannian manifold for N=1 SUSY model
- Correspondence derives from formal isomorphism (and associated complexes) between
  - exterior derivative algebra: $\{d, d^\dagger\} = \Delta$
  - SUSY algebra: $\{Q, \bar{Q}\} = H$
- With more than one SUSY and on complex manifolds, algebras are modified
  - exterior derivative (holomorphic) $\{\partial, \partial^\dagger\} = \Delta_\partial$ and $\{\bar{\partial}, \bar{\partial}^\dagger\} = \Delta_{\bar{\partial}}$
  - SUSY relation (real) $\{Q_\pm, \bar{Q}_\pm\} = H \pm p$
- Y Kähler : $\Delta_\partial = \Delta_{\bar{\partial}} = 1/2\,\Delta_d$ we can
  - Define: $d_\pm = \partial \pm \bar{\partial}$
  - Whereupon $\{d_\pm, d^\dagger_\pm\} = \Delta_d$

Review (cont'd)

- Zero modes (kernel) of
  - Δ_d correspond to deRham H*
  - $\Delta_{\bar{\partial}}$ correspond to Dolbeault H*
  - (H ± p) correspond to $\overline{Q}_{\pm}$ - H*

- Translationally invariant zero-modes are also annihilated by *H*, hence have zero energy

- Since <H> ≥ 0, zero modes are the 'ground' states, i.e. supersymmetric vacua (string spectrum)

- Example:
  - Twisted model where $\overline{Q}_{\pm}$ operators have spin 0 and generate BRST symmetry
  - BRST symmetry produces associated complex with cohomology that is in 1-1 correspondence with original SUSY

Motivation and Background

- Most of literature (1988-1995) focused on Y smooth (Green, Hubsch, Strominger, et al.)
- Interest developed in (conifolds) ‘mildly’ singular target spaces
- Want to explore zero-modes of *H*, i.e. $\overline{Q}_{\pm}$ – cohomology, in relation to (co)homology of singular varieties
- *hep-th/9612075*: T. Hubsch defines a working definition of ‘Stringy Singular Cohomology’
- *hep-th/0210394*: T. Hubsch and A. Rahman construct cohomology theory motivated by hep-th/9612075, but found “obstruction” in the middle dimension
- *math.AT/0704.3298*: A. Rahman constructs perverse sheaf that fulfills part of Kahler package and has necessary cohomology

Brief Overview of Model (hep-th/0210394)

- Spacetime is identified as the 'ground state variety' of a supersymmetric σ-model

- Massless fields/particles correspond to cohomology classes of this ground state variety

- Simplest physically interesting and non-trivial case spacetime is of the form $X^{9,1} = M^{3,1} \times Y$ where $M^{3,1}$ is the usual Minkowski space and Y is a Calabi-Yau manifold

Model (cont'd)

- Y was constructed as a projective hypersurface made up of the bosonic coordinates $\{p, s_0, ..., s_4\}$

  where Y admits a $\mathbb{C}^*$ action,

  $$\hat{\lambda}: \{p, s_0, \cdots, s_4\} \mapsto \{\lambda^{-5}p, \lambda s_0, \cdots, \lambda s_5\}\ , \qquad \lambda \in \mathbb{C}^*$$

- The invariant superpotential W=pG(s) where G(s) is a degree five homogeneous polynomial $G(\lambda s_0, ..., \lambda s_4) = \lambda^5 G(s_0, ..., s_4)$

Ground State Variety

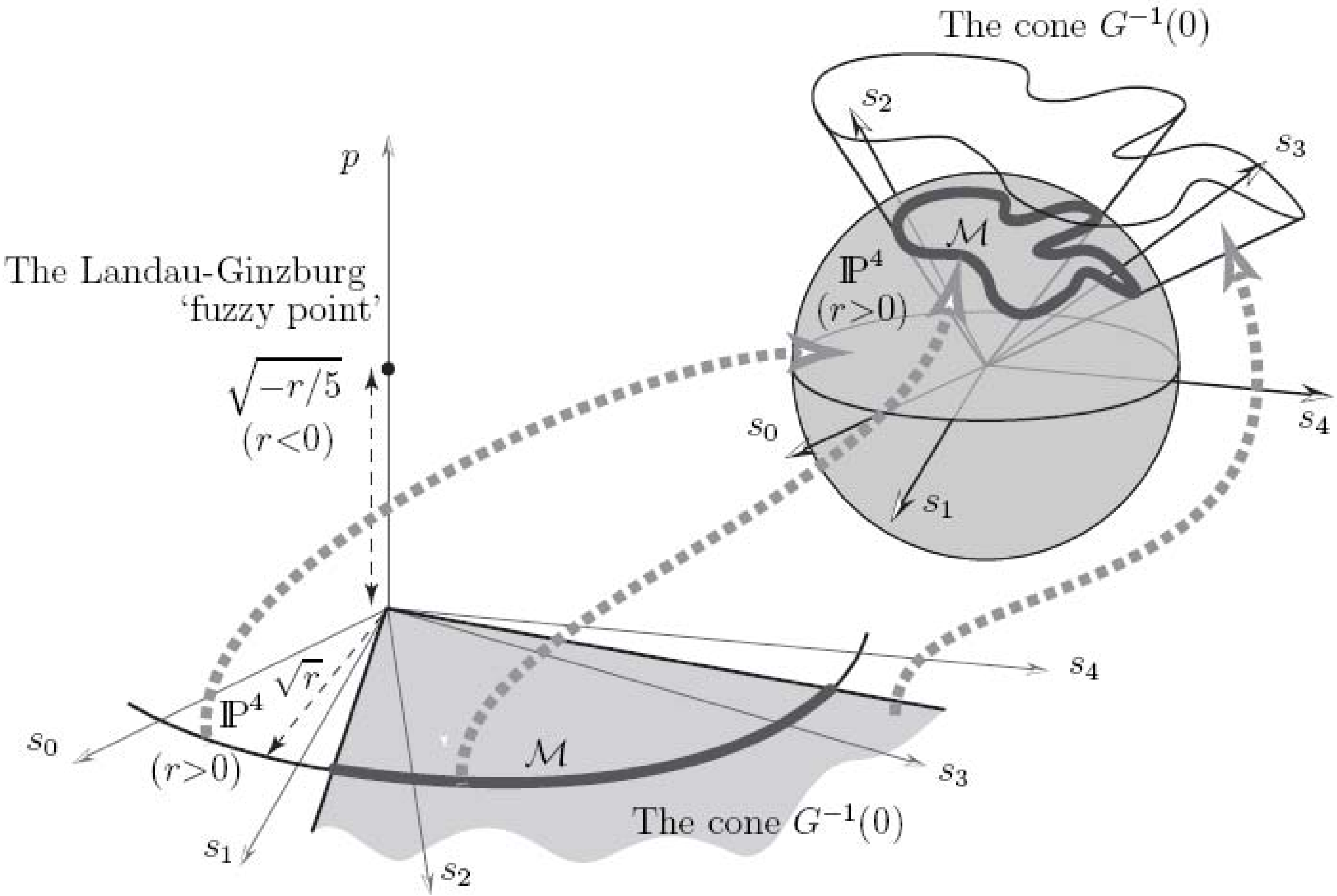
- Examine zero locus of the superpotential W $(\partial W)^{-1}(0) \;=\; G^{-1}(0) \;\cap\; (p{\cdot}\partial_s G)^{-1}(0)$

- Then we can define the ground state variety as (holomorphic form)

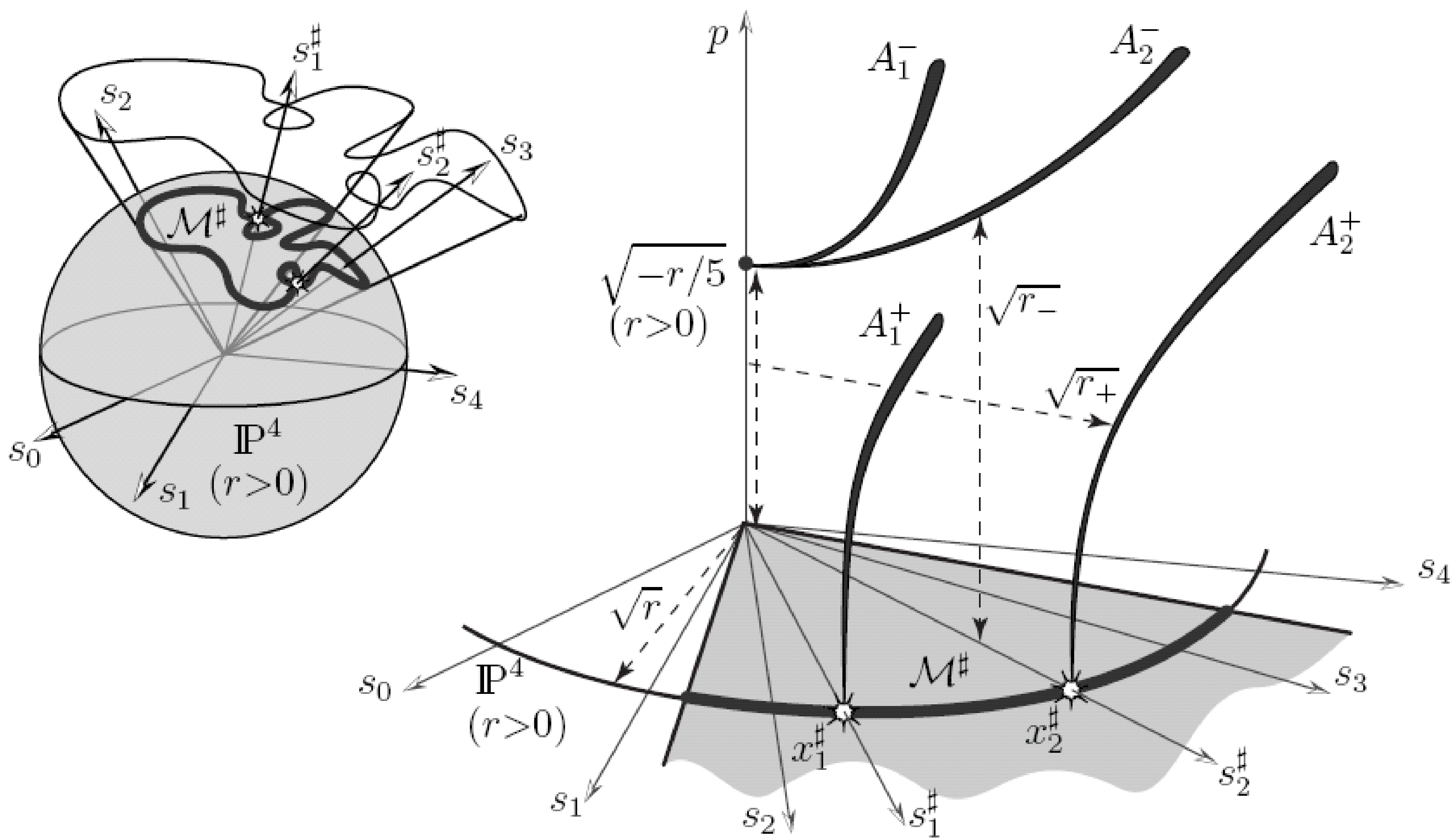
$$\mathcal{V} = \left[G^{-1}(0) \cap (p \cdot \partial G)^{-1}(0) - \text{f.p.}(\hat{\lambda})\right] / \hat{\lambda}$$

where f.p. are the set of fixed points of the action $\hat{\lambda}$

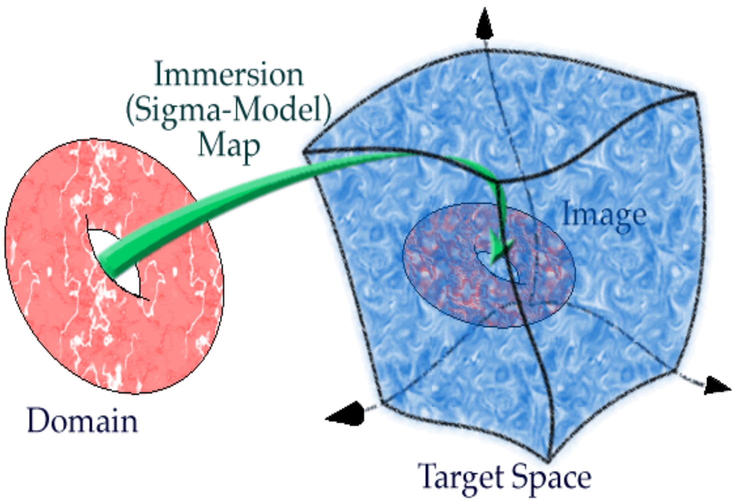
G(s) transversal: G=dG=0 only at s=0



G(s) non-transversal: $\partial_s G$ on finite $s_j = s^\#$

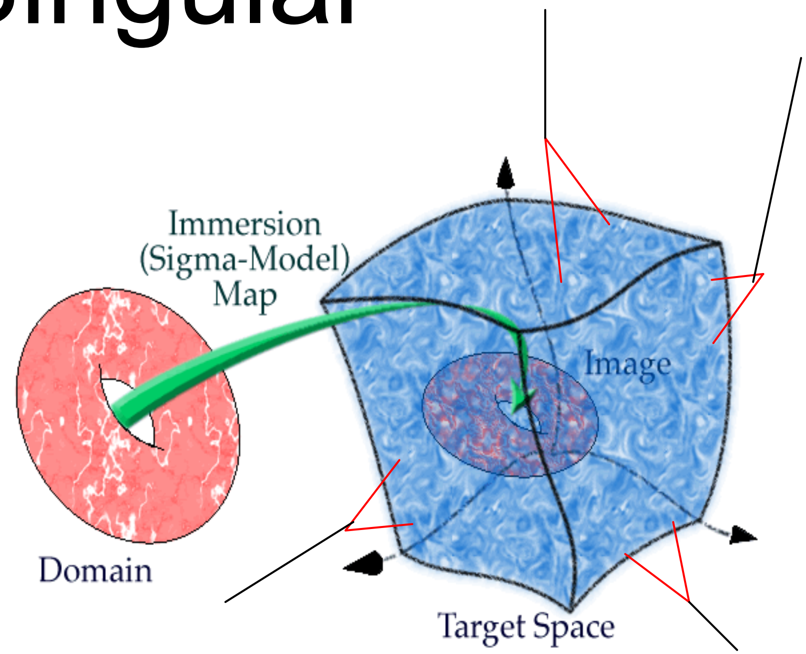


Smooth to Singular



Smooth target space

G(s) is transversal



Singular target space

G(s) is non-transversal

String spectrum

- Wave functionals in world-sheet field theory are the canonical coordinates in the effective space-time field theory

E. Witten, Supersymmetry and Morse Theory, *J. Diff. Geom.* **17** (1982) 661–692

- Wave functionals → Massless fields and particles → elements of $H^*(Y)$

- In order to determine string spectrum, we must understand $H^*(Y)$

Hubsch Conjecture

- T. Hubsch *(hep-th/9612075)* conjectured that the cohomology theory for string theory should be as follows:

$$SH_k(Y) = \begin{cases} H_k(Y), & k > n; \\ H_n(Y - y) \cup H_n(Y), & k = n; \\ H_k(Y - y), & k < n \end{cases}$$

- The middle dimension has more cycles due to details surrounding 'shrinking' of cycles and then subsequent counting of cycles embedded in machinery of conifold transition

Issues…

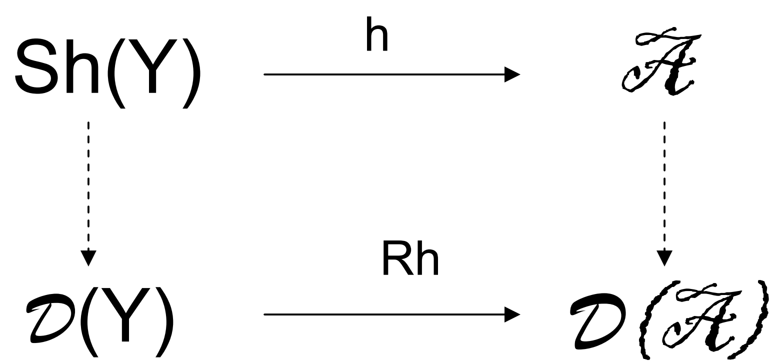
- Hubsch conjecture is not mathematically “correct”…He sought other cohomology theories that had correct properties like $\underline{\underline{IC^{\cdot}}}$
- ***Problem:*** $\underline{\underline{IC^{\cdot}}}$ does not have the correct rank of cohomology in the middle dimension. This does not fulfill the String theory requirement in the middle dimension.
- What is the cohomology theory for String Theory for singular and smooth target spaces?

Mathematical Construction

- We seek a complex of sheaves $\underline{\underline{\mathcal{S}_0}}^{\cdot}$ such that we have the same cohomology as $\underline{\underline{IC}}^{\cdot}$ but more cohomology in the middle dimension

$$\begin{array}{ccc} & H^n(Y;\underline{\underline{\mathcal{S}_0}}^{\cdot}) & \\ \overset{\xi}{\nearrow} & & \overset{\chi}{\searrow} \\ H^n(Y-y;\underline{\underline{\mathbb{Q}}}) & \xrightarrow{\quad\tau\quad} & H^n(Y;\underline{\underline{\mathbb{Q}}}) \end{array}$$

Derived Category



Given a functor h from h:Sh(Y) → A. There is a right derived functor Rh: D(Y) → D(A)

The motivation for this problem lies in the following construction. Consider the long exact sequence in the middle dimension for the pair (Y,Y°):

$$H^n(Y;\underline{\underline{\mathcal{S}_0^{\cdot}}})$$

$$\ldots \to H^n(Rj_!j^*\mathbb{Q}) \to H^n(\mathbb{Q}) \xrightarrow{\alpha} H^n(Ri_*i^*\mathbb{Q}) \to H^{n+1}(Rj_!j^*\mathbb{Q}) \to \ldots$$

$$\operatorname{Im}(\alpha) = H^n(\underline{\underline{IC^{\cdot}}})$$

Further Requirements for $\underline{\underline{\mathcal{S}_0^{\cdot}}}$

- Off-middle dimension k ≠ n

$$H^k(Y; \underline{\underline{\mathcal{S}_0^{\cdot}}}) \cong H^k(Y; \underline{\underline{IC^{\cdot}}})$$

- Meet other 'properties' of String theory: the Kähler package
  - Poincare Duality
  - Kunneth Formula
  - Complex Conjugation
  - Hodge Structure

Notation

- Assume Y has only one singular point {y}
- Y is 2n-dimensional Calabi-Yau manifold
- $Y^o = Y - \{y\}$ is the non-singular part of the space
- Define inclusions i: $Y^o \rightarrow Y$ and j: $\{y\} \rightarrow Y$
- For a complex of sheaves on Y, we have functors i^* , i_* , $i_!$,j^* ,j_* , $j^!$,and $j_!$
- Derived functors will be noted $R\, i_*$, $Ri_!$, $R\, j_*$, and $R\, j_!$
- Category of complexes of constructible sheaves of Q-vector spaces

What should this object be?

- R. MacPherson (IAS) suggested that there is a particular perverse sheaf with properties:
  - For k>n, cohomology of the whole space Y
  - For k<n, cohomology of the non-singular part of the space
  - For k=n, more cohomology than other degrees
- Solution: Perverse sheaves: Full subcategory of derived category $\mathcal{D}^b(Y)$ i.e. same morphisms, particular objects

Definition 3.1. *The category of perverse sheaves $\mathbb{P}(Y)$ is the full sub-category of $\mathcal{D}^b(Y)$ whose objects are complexes of sheaves $\underline{\underline{\mathcal{S}}}^{\cdot}$ which satisfy the following properties:*

1. *There exists $M \in \mathbb{Z}$ such that $\underline{\underline{\mathcal{S}}}^i = 0 \ \forall \ i < M$ (bounded below)*

2. *The sheaf $i^*\underline{\underline{\mathcal{S}}}^{\cdot}$ is quasi-isomorphic to a local system on Y^o (in degree 0). In other words,*

   (a) $\underline{\underline{H}}^k(i^*\underline{\underline{\mathcal{S}}}^{\cdot}) = 0$ *if* $k \neq 0$

   (b) $\underline{\underline{H}}^0(i^*\underline{\underline{\mathcal{S}}}^{\cdot})$ *is a local system*

3. $H^k(j^*\underline{\underline{\mathcal{S}}}^{\cdot}) = 0$ *for* $k > n$ *(support)*

4. $H^k(j^!\underline{\underline{\mathcal{S}}}^{\cdot}) = 0$ *for* $k < n$ *(cosupport)*

How do we construct it?

- Use theorem of MacPherson and Vilonen which requires understanding of the Zig-zag category…

Theorem 3.4. *(MacPherson-Vilonen [23])*

1. *The zig-zag functor* $\mu : \mathbb{P}(Y) \to Z(Y, y)$ *gives rise to a bijection from isomorphism classes of objects of* $\mathbb{P}(Y)$ *to isomorphism classes of objects of* $Z(Y, y)$,
2. *Given* $\underline{\underline{\mathcal{S}}}, \underline{\underline{\mathcal{S}'}} \in \mathbb{P}(Y)$. *Then* $\mu : Hom_{\mathbb{P}}(\underline{\underline{\mathcal{S}}}, \underline{\underline{\mathcal{S}'}}) \to Hom_{Z}(\mu(\underline{\underline{\mathcal{S}}}), \mu(\underline{\underline{\mathcal{S}'}}))$ *is a surjection.*

Zig-zag Category

Definition 0.1. *The zig-zag category $Z(Y, y)$ is defined as follows. An object in $Z(Y, y)$ is a sextuple $\Theta = (\mathcal{L}, K, C, \alpha, \beta, \gamma)$ where $\mathcal{L}$ is a local system on Y^o, and K and C are vector spaces on the singular point y together with an exact sequence:*

$$H^{n-1}(j^* i_* \mathcal{L}) \xrightarrow{\alpha} K \xrightarrow{\beta} C \xrightarrow{\gamma} H^n(j^* i_* \mathcal{L})$$

Remarks:

- An object of P(Y) can be constructed from an object in Z(Y,y).

- Note that Z(Y,y) requires the following:
  - Choose vector spaces K and C such that the sequence is exact
  - Define a local system on non-singular part

Definition 3.3. *The zig-zag functor $\mu : \mathbb{P}(Y) \to Z(Y, y)$ is defined by sending an object $\underline{\underline{Q}}^{\cdot} \in \mathbb{P}(Y)$ to the triple $(i^*\underline{\underline{Q}}^{\cdot},\ H^n(j^!\underline{\underline{Q}}^{\cdot}),\ H^n(j^*\underline{\underline{Q}}^{\cdot}))$ together with the exact sequence*

$$H^{n-1}(j^*i_*i^*\underline{\underline{Q}}^{\cdot}) \longrightarrow \overset{K}{H^n(j^!\underline{\underline{Q}}^{\cdot})} \longrightarrow \overset{C}{H^n(j^*\underline{\underline{Q}}^{\cdot})} \longrightarrow H^n(j^*i_*i^*\underline{\underline{Q}}^{\cdot}) \tag{3.3}$$

This sequence needs to be exact at K and C

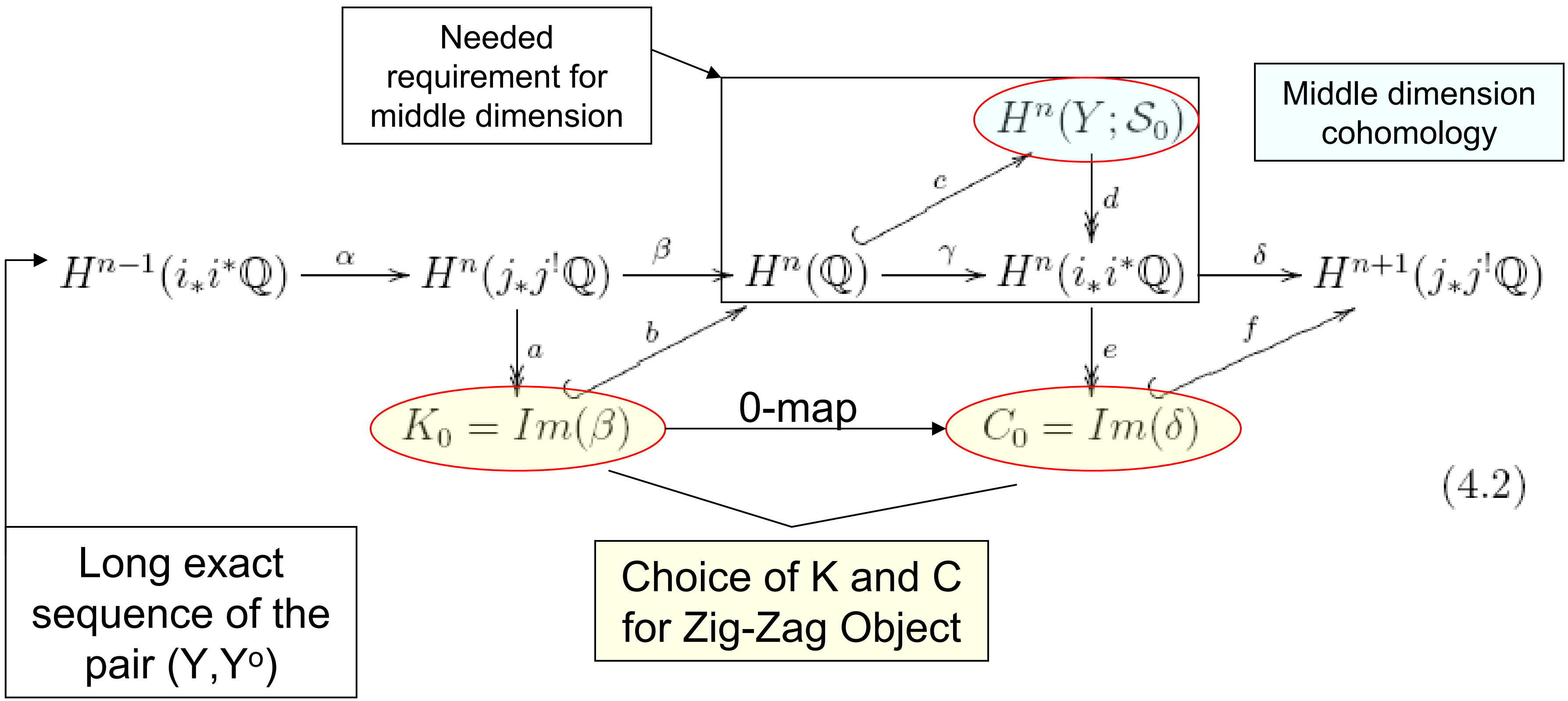
Maps come from distinguished triangle

Strategy…

With the cohomology properties in mind, construct a perverse sheaf by:

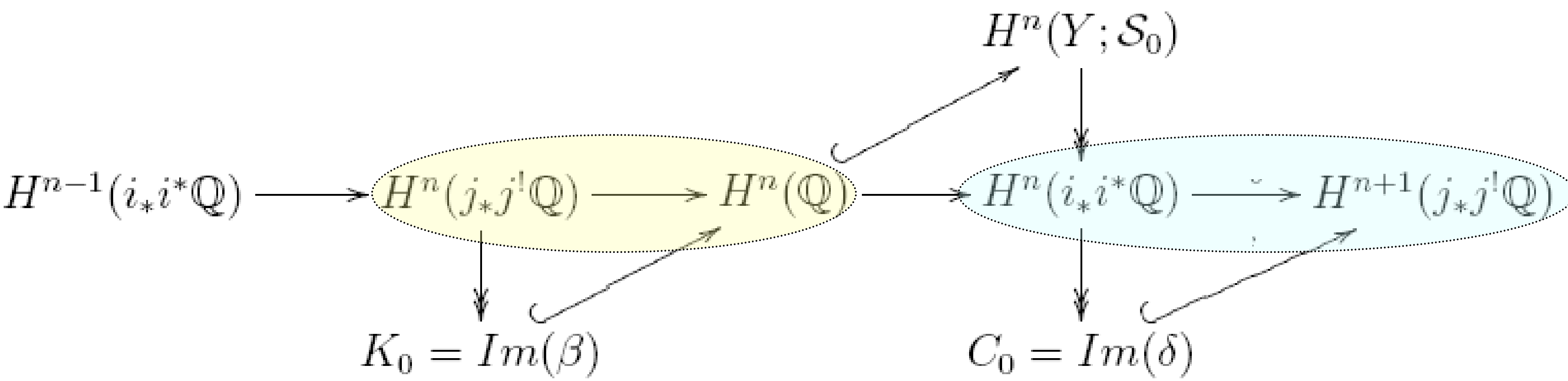
- Choosing a K and C
- Verify these choices give a Zig-zag object
- Use Theorem of MacPherson and Vilonen to show there is a perverse sheaf that corresponds to this object

The choice of K and C *we want* fit in the following diagram where the long exact sequence is expressed with derived functors



Zig-zag object

Proposition 3.7. *Define* $\Theta_0 = (\underline{\underline{\mathbb{Q}}}, K_0, C_0, \alpha_0, \beta_0, \gamma_0)$ *where* $K_0 = Im(H^n_c(c^o L) \to H^n_c(Y^o))$, $C_0 = Im(H^n(Y^0) \to H^{n+1}_c(c^o L))$, $\alpha_0 : H^n_c(c^o L) \to Im(H^n_c(c^o L) \to H^n_c(Y^o))$, β_0 *is the 0-map, and* $\gamma_0 : Im(H^n(Y^0) \to H^{n+1}_c(c^o L)) \to H^{n+1}_c(c^o L)$. *Then* $\Theta_0 \in Obj(Z_{\mathbb{Q}}(Y, y))$.



(4.2)

Statement of the Main Theorem

Theorem 4.1. *The perverse sheaf $\underline{\underline{\mathcal{S}_0}}$ has the following properties:*

1. $H^i(Y;\underline{\underline{\mathcal{S}_0}}) = \begin{cases} H^i(Y), & i > n, \\ H^i(Y^o), & i < n \end{cases}$

2. $H^n(Y;\underline{\underline{\mathcal{S}_0}})$ *is specified by the following two canonical short exact sequences:*

   (a) $0 \to K_0 \to H^n(Y;\underline{\underline{\mathcal{S}_0}}) \to H^n(Y^o) \to 0$

   (b) $0 \to H^n_c(Y^o) \to H^n(Y;\underline{\underline{\mathcal{S}_0}}) \to C_0 \to 0$

3. $\underline{\underline{\mathcal{S}_0}}$ *is self-dual.*

Duality

$$
\begin{array}{ccc}
\mathbb{P}(Y) & \xrightarrow{\ \mu\ } & Z(Y, y) \\
\mathcal{D}_V \Big\downarrow & & \Big\downarrow \mathcal{D}_Z \\
\mathbb{P}(Y) & \xrightarrow{\ \mu\ } & Z(Y, y)
\end{array}
$$

Duality Functor in Z(Y,y)

$$H^{n-1}(j^*i_*\mathcal{L}) \xrightarrow{\alpha} K \xrightarrow{\beta} C \xrightarrow{\gamma} H^n(j^*i_*\mathcal{L})$$

Lemma 5.6. *Let* $\Theta = (\mathcal{L}, K, C, \alpha, \beta, \gamma) \in Obj(Z(Y,y))$ *and* $\mathcal{D}_Z(\Theta) = (\mathcal{L}^*, C^*, K^*, \gamma^*, \beta^*, \alpha^*)$ *with the following maps,*

$$H^{n-1}(j^*i_*\mathcal{L}^*) \xrightarrow{\gamma^*} C^* \xrightarrow{\beta^*} K^* \xrightarrow{\alpha^*} H^n(j^*i_*\mathcal{L}^*) \tag{5.12}$$

where α^*, β^*, *and* γ^* *are the dual maps defined in Definition 5.5. Then the sequence (5.12) is exact and it follows that* $\mathcal{D}_Z(\Theta)$ *is an object of* $Z(Y,y)$.

Duality

Definition 5.11. *Let* $\Theta = (\mathcal{L}, K, C, \alpha, \beta, \gamma) \in Obj(Z(Y,y))$. *We can say that the object* Θ *is self-dual in* $Z(Y,y)$ *if there exists an isomorphism* $\Theta \to \mathcal{D}_Z(\Theta)$ *in* $Z(Y,y)$.

μ

Theorem 5.14. $\underline{\underline{\mathcal{S}_0}}$ *is self-dual in* $\mathbb{P}_{\mathbb{Q}}(Y)$.

Corollary 5.15. *(Poincare Duality) For degrees* $i \geq 0$, $H^i(Y; \underline{\underline{\mathcal{S}_0}}) \cong H^{2n-i}(Y; \underline{\underline{\mathcal{S}_0}})$.

Final Thoughts

- We have constructed a perverse sheaf with one part of the Kahler package, cohomology rank in all degrees. Remains to prove the remainder of Kähler package

- w/ T. Pantev:
  - Analyze S_0 through studying properties of nearby and vanishing cycles perverse sheaves.
  - Define exactly what S_0 is in terms of ‘well-known’ objects
  - Look at how it fits in with Orlov constructions

- w/ T. Hubsch:
  - String theory examples using degree five polynomials
  - Spaces with more degenerate singularities