GLSM's, gerbes, and Kuznetsov's homological projective duality

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T Pantev, ES, hepth/0502027, 0502044, 0502053

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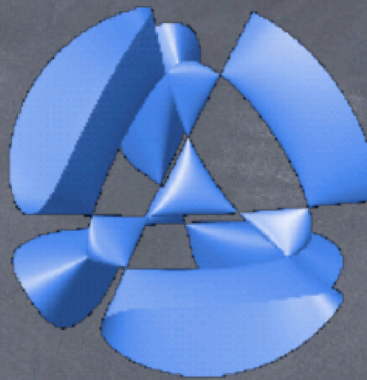
R Donagi, ES, arxiv: 0704.1761

A Caldararu, J Distler, S Hellerman, T Pantev, ES, to appear

Outline

- Basics of string compactifications on stacks
- Cluster decomposition conjecture for strings on gerbes:
  CFT(gerbe) = CFT(disjoint union of spaces)
- Application to GLSM's; realization of Kuznetsov's homological projective duality

Stacks



Stacks are a mild generalization of spaces.

One would like to understand strings on stacks:

-- to understand the most general possible string compactifications

-- they often appear physically inside various constructions

Stacks

How to make sense of strings on stacks concretely?

Every* (smooth, Deligne-Mumford) stack can be presented as a global quotient

$$[X/G]$$

for X a space and G a group.

To such a presentation, associate a G-gauged sigma model on X.

(* with minor caveats)

Stacks

If to [X/G] we associate ``G-gauged sigma model,"
then:

$$[\mathbf{C}^2/\mathbf{Z}_2]$$

defines a 2d theory with a symmetry called conformal invariance

=

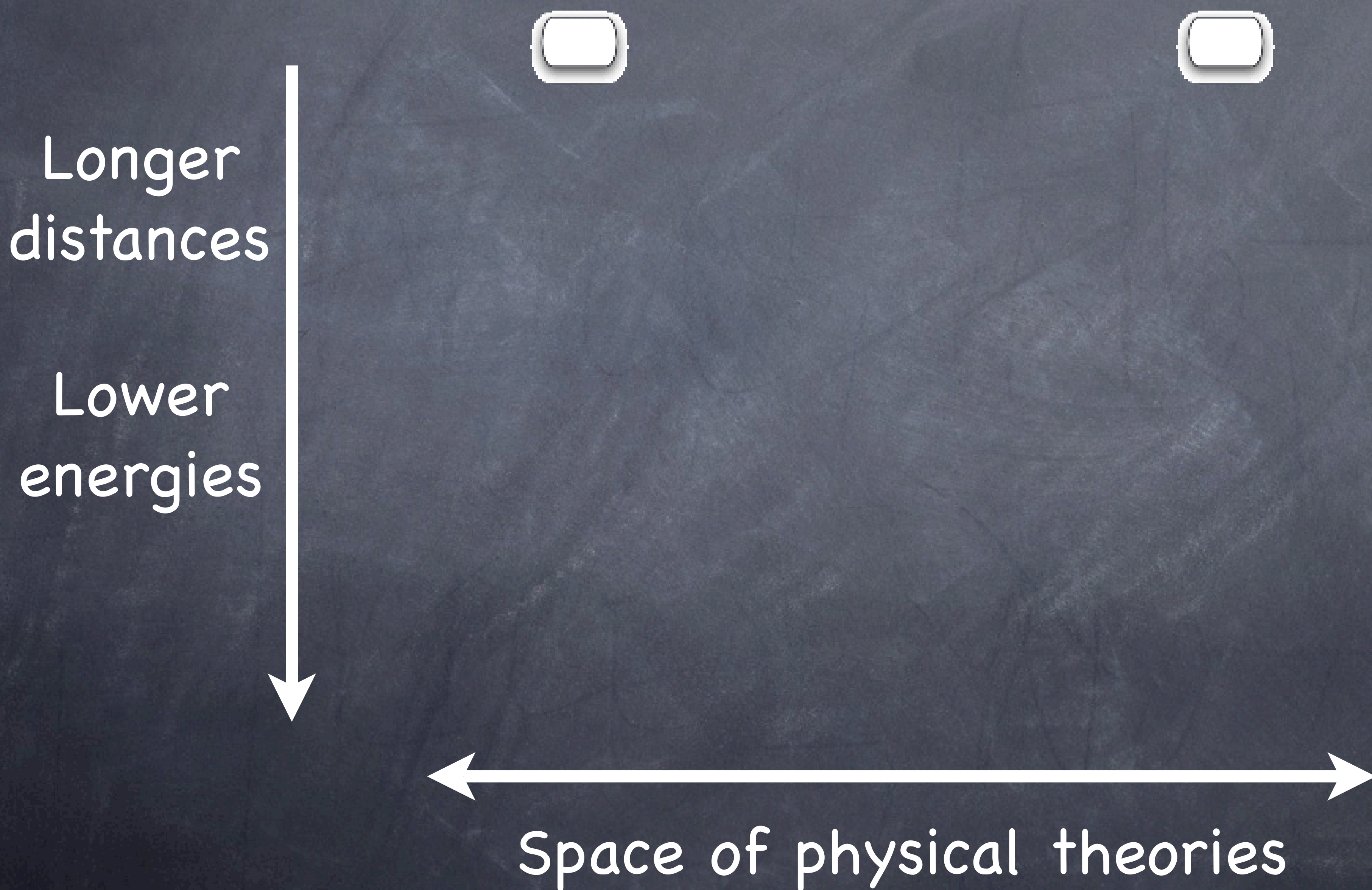
$$[X/\mathbf{C}^\times]$$

$$\left(X = \frac{\mathbf{C}^2 \times \mathbf{C}^\times}{\mathbf{Z}_2} \right)$$

defines a 2d theory w/o conformal invariance

Potential presentation-dependence problem:
fix with renormalization group flow

Renormalization group

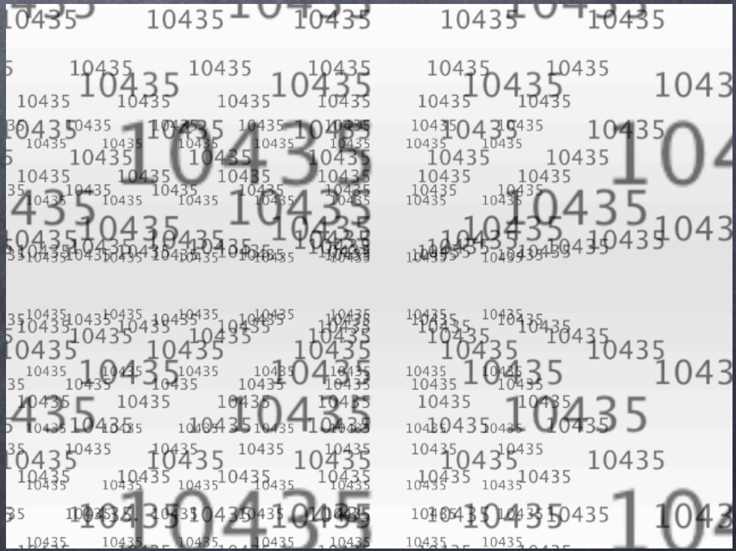


Renormalization group

-- is a powerful tool, but unfortunately we really can't follow it completely explicitly in general.

-- can't really prove in any sense that two theories will flow under renormalization group to same point.

Instead, we do lots of calculations,
perform lots of consistency tests,
and if all works out,
then we believe it.



The problems here are analogous to the derived-categories-in-physics program.

There, to a given object in a derived category, one picks a representative with a physical description (as branes/antibranes/tachyons).

Alas, such representatives are not unique.

It is conjectured that different representatives give rise to the same low-energy physics, via boundary renormalization group flow.

Only indirect tests possible, though.

Stacks

Potential problems / reasons to believe that presentation-independence fails:

* Deformations of stacks $\neq$ Deformations of physical theories
* Cluster decomposition issue for gerbes

These potential problems can be fixed. (ES, T Pantev)

Results include: mirror symmetry for stacks, new Landau-Ginzburg models, physical calculations of quantum cohomology for stacks, understanding of noneffective quotients in physics

General decomposition conjecture

Consider $[X/H]$ where

$$1 \longrightarrow G \longrightarrow H \longrightarrow K \longrightarrow 1$$

and G acts trivially.

We now believe, for (2,2) CFT's,

$$\mathrm{CFT}([X/H]) = \mathrm{CFT}\left(\left[(X \times \hat{G})/K \right] \right)$$

(together with some B field), where

$\hat{G}$ is the set of irreps of G

Decomposition conjecture

For banded gerbes, K acts trivially upon $\hat{G}$ so the decomposition conjecture reduces to

$$\mathrm{CFT}(G - \mathrm{gerbe\ on\ } X) \;=\; \mathrm{CFT}\left(\coprod_{\hat{G}} (X, B) \right)$$

where the B field is determined by the image of

$$H^2(X, Z(G)) \stackrel{Z(G) \to U(1)}{\longrightarrow} H^2(X, U(1))$$

Banded Example:

Consider $[X/D_4]$ where the center acts trivially.

$$1 \longrightarrow \mathbf{Z}_2 \longrightarrow D_4 \longrightarrow \mathbf{Z}_2 \times \mathbf{Z}_2 \longrightarrow 1$$

The decomposition conjecture predicts

$$\mathrm{CFT}\left([X/D_4]\right) = \mathrm{CFT}\left([X/\mathbf{Z}_2 \times \mathbf{Z}_2] \coprod [X/\mathbf{Z}_2 \times \mathbf{Z}_2] \right)$$

One of the effective orbifolds has vanishing discrete torsion, the other has nonvanishing discrete torsion.

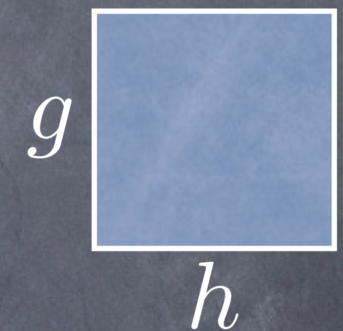
(Using the relationship between discrete torsion and B fields first worked out by ES, c. 2000.)

Check genus one partition functions:

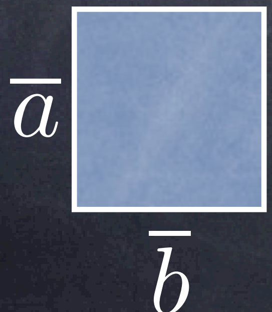
$$D_4 \;=\; \{1, z, a, b, az, bz, ab, ba = abz\}$$

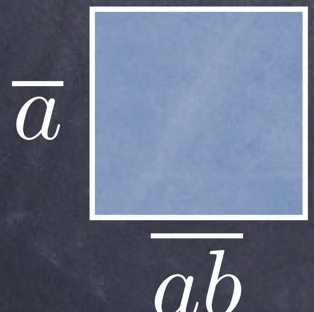
$$\mathbf{Z}_2 \times \mathbf{Z}_2 \;=\; \{1, \overline{a}, \overline{b}, \overline{ab}\}$$

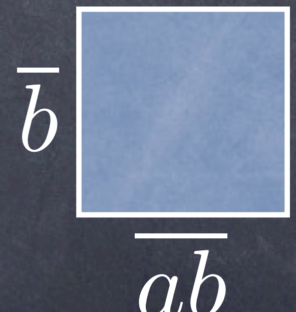
$$Z(D_4) \;=\; \frac{1}{|D_4|} \sum_{g,h \in D_4, gh=hg} Z_{g,h}$$



Each of the $Z_{g,h}$ twisted sectors that appears, is the same as a $\mathbf{Z}_2 \times \mathbf{Z}_2$ sector, appearing with multiplicity $|\mathbf{Z}_2|^2 = 4$ except for the







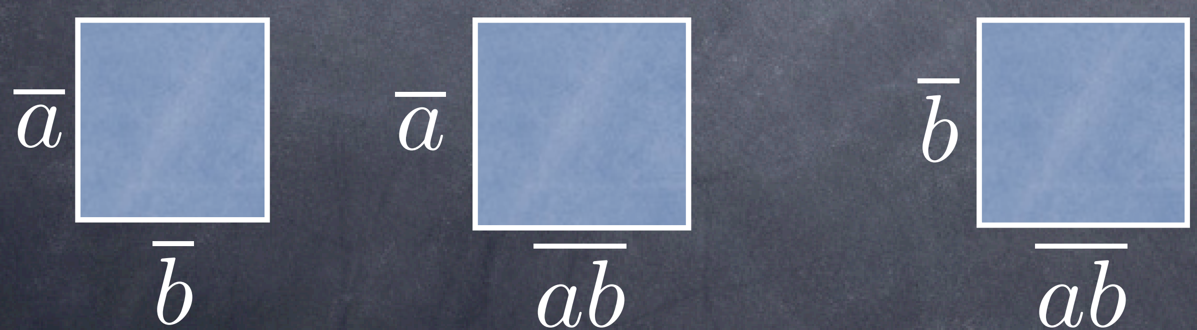
sectors.

Partition functions, cont'd

$$\begin{aligned} Z(D_4) &= \frac{|\mathbf{Z}_2 \times \mathbf{Z}_2|}{|D_4|} |\mathbf{Z}_2|^2 \left(Z(\mathbf{Z}_2 \times \mathbf{Z}_2) \,-\, (\text{some twisted sectors}) \right) \\ &= 2 \left(Z(\mathbf{Z}_2 \times \mathbf{Z}_2) \,-\, (\text{some twisted sectors}) \right) \end{aligned}$$

(In ordinary QFT, ignore multiplicative factors, but string theory is a 2d QFT coupled to gravity, and so numerical factors are important.)

Discrete torsion acts as a sign on the



twisted sectors

so we see that $Z([X/D_4]) = Z\left([X/\mathbf{Z}_2 \times \mathbf{Z}_2] \coprod [X/\mathbf{Z}_2 \times \mathbf{Z}_2] \right)$

with discrete torsion in one component.

A quick check of this example comes from comparing massless spectra:

Spectrum for $[T^6/D_4]$:

$$\begin{array}{ccccccc}
 & & & 2 & & & \\
 & & 0 & & 0 & & \\
 & 0 & & 54 & & 0 & \\
2 & & 54 & & 54 & & 2 \\
 & 0 & & 54 & & 0 & \\
 & & 0 & & 0 & & \\
 & & & 2 & & &
\end{array}$$

and for each $[T^6/\mathbf{Z}_2 \times \mathbf{Z}_2]$:

$$\begin{array}{ccccccc}
 & & & 1 & & & \\
 & & 0 & & 0 & & \\
 & 0 & & 3 & & 0 & \\
1 & & 51 & & 51 & & 1 \\
 & 0 & & 3 & & 0 & \\
 & & 0 & & 0 & & \\
 & & & 1 & & &
\end{array}
\qquad
\begin{array}{ccccccc}
 & & & 1 & & & \\
 & & 0 & & 0 & & \\
 & 0 & & 51 & & 0 & \\
1 & & 3 & & 3 & & 1 \\
 & 0 & & 51 & & 0 & \\
 & & 0 & & 0 & & \\
 & & & 1 & & &
\end{array}$$

Sum matches. ✓

Nonbanded example:

Consider $[X/\mathbf{H}]$ where $\mathbf{H}$ is the eight-element group of quaternions, and a $\mathbf{Z}_4$ acts trivially.

$$1 \longrightarrow < i > (\cong \mathbf{Z}_4) \longrightarrow \mathbf{H} \longrightarrow \mathbf{Z}_2 \longrightarrow 1$$

The decomposition conjecture predicts

$$\mathrm{CFT}([X/\mathbf{H}]) = \mathrm{CFT}\left([X/\mathbf{Z}_2] \coprod [X/\mathbf{Z}_2] \coprod X \right)$$

Straightforward to show that this is true at the level of partition functions, as before.

Another class of examples: global quotients by nonfinite groups

The banded $\mathbf{Z}_k$ gerbe over $\mathbf{P}^N$
with characteristic class $-1 \bmod k$
can be described mathematically as the quotient

$$\left[\frac{\mathbf{C}^{N+1} - \{0\}}{\mathbf{C}^\times} \right]$$

where the $\mathbf{C}^\times$ acts as rotations by k times

which physically can be described by a U(1) susy gauge theory with N+1 chiral fields, of charge k

How can this be different from ordinary $\mathbf{P}^N$ model?

The difference lies in nonperturbative effects.
(Perturbatively, having nonminimal charges makes no difference.)

Example: Anomalous global U(1)'s

$$\mathbf{P}^{N-1}: \quad U(1)_A \mapsto \mathbf{Z}_{2N}$$
$$\text{Here}: \quad U(1)_A \mapsto \mathbf{Z}_{2kN}$$

Example: A model correlation functions

$$\mathbf{P}^{N-1}: \quad < X^{N(d+1)-1} >= q^d$$
$$\text{Here}: \quad < X^{N(kd+1)-1} >= q^d$$

Example: quantum cohomology

$$\mathbf{P}^{N-1}: \quad \mathbf{C}[x]/(x^N - q)$$
$$\text{Here}: \quad \mathbf{C}[x]/(x^{kN} - q)$$

Different physics

General argument:

Compact worldsheet:

To specify Higgs fields completely, need to specify what bundle they couple to.

If the gauge field $\sim L$
then Φ charge Q implies

$$\Phi \in \Gamma(L^{\otimes Q})$$

Different bundles => different zero modes => different anomalies => different physics

For noncpt worldsheets, analogous argument exists.

(Distler, Plesser)

K theory implications

This equivalence of CFT's implies a statement about K theory (thanks to D-branes).

$$1 \longrightarrow G \longrightarrow H \longrightarrow K \longrightarrow 1$$

If G acts trivially on X
then the ordinary H-equivariant K theory of X

is the same as

twisted K-equivariant K theory of $X \times \hat{G}$

* Can be derived just within K theory
* Provides a check of the decomposition conjecture

D-branes and sheaves

D-branes in the topological B model can be described with sheaves and, more gen'ly, derived categories.

This also is consistent with the decomp' conjecture:

Math fact:

A sheaf on a banded G-gerbe

is the same thing as

a twisted sheaf on the underlying space,
twisted by image of an element of $H^2(X,Z(G))$

which is consistent with the way D-branes should behave according to the conjecture.

D-branes and sheaves

Similarly, massless states between D-branes should be counted by Ext groups between the corresponding sheaves.

Math fact:

Sheaves on a banded G-gerbe decompose according to irrep' of G,
and sheaves associated to distinct irreps have vanishing Ext groups between them.

Consistent w/ idea that sheaves associated to distinct reps should describe D-branes on different components of a disconnected space.

Gromov-Witten prediction

Notice that there is a prediction here for Gromov-Witten theory of gerbes:

GW of $[X/H]$

should match

GW of $\left[(X \times \hat{G})/K \right]$

Works in basic cases:
BG (T Graber), other exs (J Bryan)

Mirrors to stacks

There exist mirror constructions for any model realizable as a 2d abelian gauge theory.

For toric stacks (BCS '04), there is such a description.

Standard mirror constructions now produce character-valued fields, a new effect, which ties into the stacky fan description of (BCS '04).

(ES, T Pantev, '05)

Toda duals

Ex: The "Toda dual" of $\mathbf{P}^N$ is described by the holomorphic function

$$W = \exp(-Y_1) + \cdots + \exp(-Y_N) + \exp(Y_1 + \cdots + Y_N)$$

The analogous duals to $\mathbf{Z}_k$ gerbes over $\mathbf{P}^N$ are described by

$$W = \exp(-Y_1) + \cdots + \exp(-Y_N) + \Upsilon^n \exp(Y_1 + \cdots + Y_N)$$

where Υ is a character-valued field

(discrete Fourier transform of components in decomp' conjecture)

(ES, T Pantev, '05)

Summary so far:

string compactifications on stacks exist

CFT(string on gerbe) = CFT(string on spaces)

GLSM's

This result can be applied to understand GLSM's.

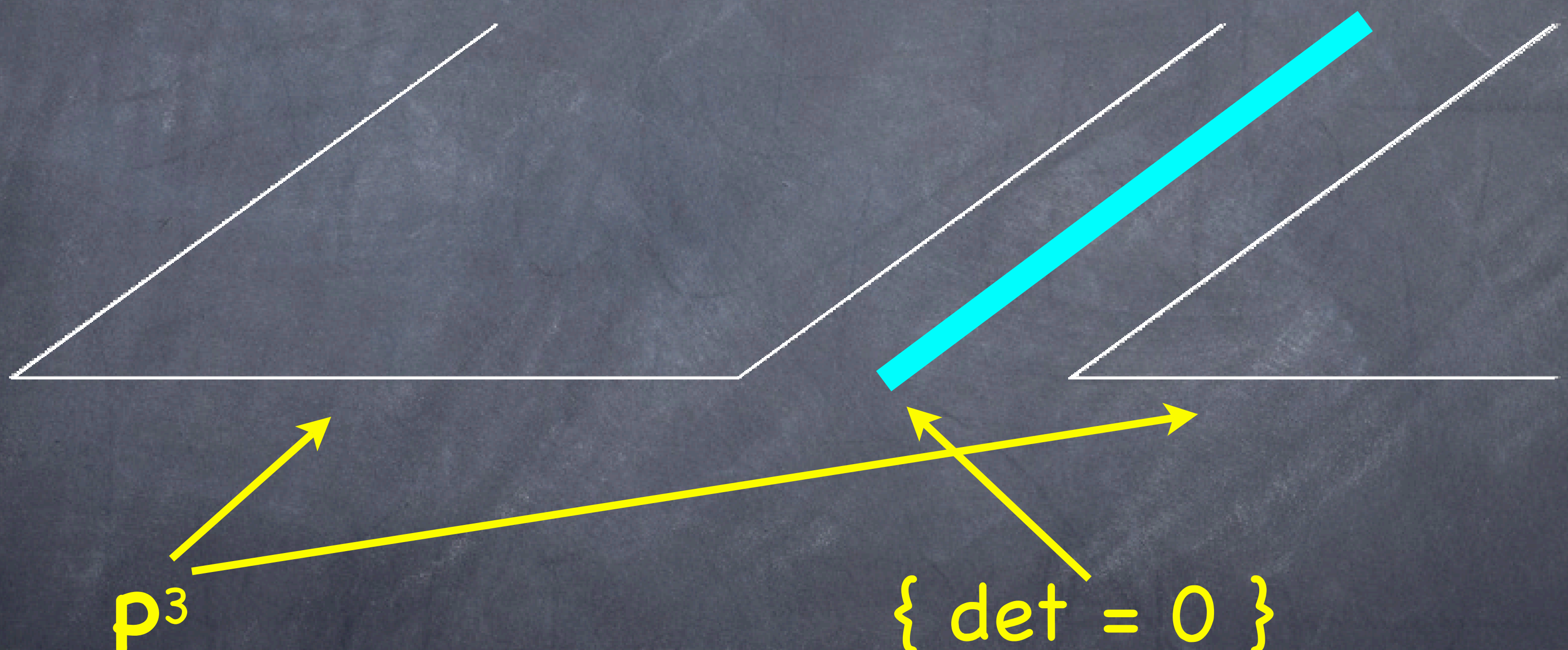
Example: $\mathbf{P}^7[2,2,2,2]$

At the Landau-Ginzburg point, have superpotential

$$\sum_a p_a G_a(\phi) = \sum_{ij} \phi_i A^{ij}(p) \phi_j$$

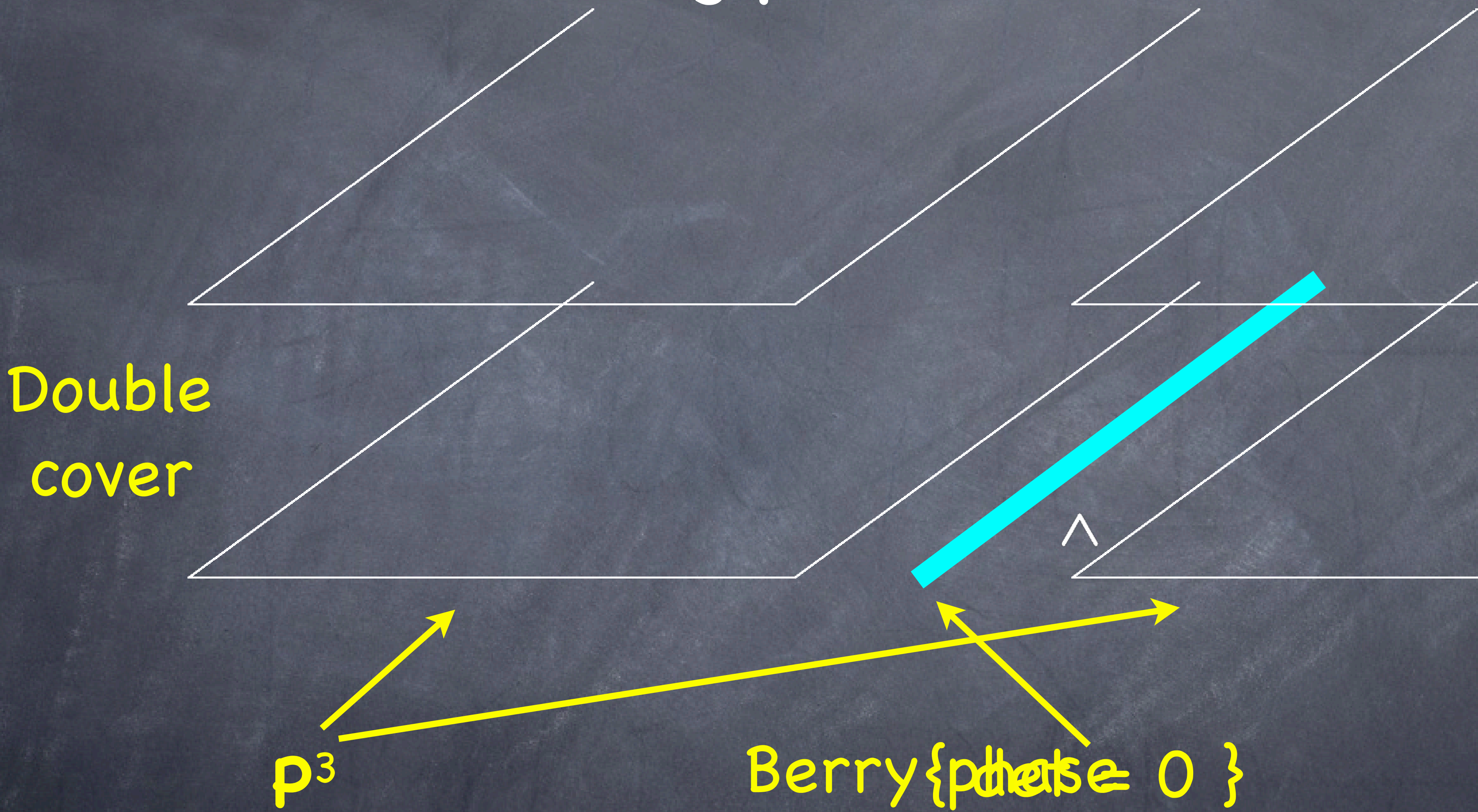
* mass terms for the ϕ_i, away from locus $\{\det A = 0\}$.
* leaves just the p fields, of charge -2
* $\mathbf{Z}_2$ gerbe, hence double cover

The Landau-Ginzburg point:



Because we have a $\mathbf{Z}_2$ gerbe over $\mathbf{P}^3$....

The Landau-Ginzburg point:



Result: branched double cover of $\mathbf{P}^3$

So far:

The GLSM realizes:

$P^7[2,2,2,2]$ $\xleftrightarrow{\text{Kahler}}$ branched double cover of $\mathbf{P}_3$

(Clemens' octic double solid)

where RHS realized at LG point via local $\mathbf{Z}_2$ gerbe structure + Berry phase.

(S. Hellerman, A. Henriques, T. Pantev, ES, M Ando, '06; R Donagi, ES, '07; A. Caldararu, J. Distler, S. Hellerman, T. Pantev, E.S., in progress)

Non-birational twisted derived equivalence

Some puzzles:

* the branched double cover will be singular, but the GLSM is smooth at those singularities.

* monodromy about LG point not consistent with large-radius geometric interpretation

Solution?....

Solution to these puzzles:

We believe the GLSM is actually describing a `noncommutative resolution' of the branched double cover worked out by Kuznetsov.

Kuznetsov has defined `homological projective duality' that relates $\mathbf{P}^7[2,2,2,2]$ to the noncommutative resolution above.

Check that we are seeing K's noncomm' resolution:

K defines a `noncommutative space' via its sheaves
-- so for example, a Landau-Ginzburg model can be a
noncommutative space via matrix factorizations.

Here, K's noncomm' res'n is defined by ($\mathbf{P}^3$,B)
where B is the sheaf of even parts of Clifford
algebras associated with the universal quadric over $\mathbf{P}^3$
defined by the GLSM superpotential.

B plays the role of structure sheaf;
other sheaves are B-modules.

Physics?......

Physics picture of K's noncomm' space:

Matrix factorization for a quadratic superpotential:
even though the bulk theory is massive, one still has
D0-branes with a Clifford algebra structure.

(Kapustin, Li)

Here: a `hybrid LG model' fibered over $\mathbf{P}^3$,
gives sheaves of Clifford algebras (determined by the
universal quadric / GLSM superpotential)
and modules thereof.

So: open string sector duplicates Kuznetsov's def'n.

Note we have a physical
realization of nontrivial examples of Kontsevich's
`noncommutative spaces'
realized in gauged linear sigma models.

Summary so far:

The GLSM realizes:

$P^7[2,2,2,2]$ $\overset{\text{Kahler}}{\longleftrightarrow}$ branched double cover of $\mathbf{P}_3$

where RHS realized at LG point via local $\mathbf{Z}_2$ gerbe structure + Berry phase.

(A. Caldararu, J. Distler, S. Hellerman, T. Pantev, E.S., to appear)

Non-birational twisted derived equivalence

Physical realization of Kuznetsov's homological projective duality

More examples:

CI of n quadrics in $\mathbf{P}^{2n-1}$ $\xleftrightarrow{\text{Kahler}}$ branched double cover of $\mathbf{P}^{n-1}$, branched over deg 2n locus

Both sides CY

Homologically projective dual

More examples:

CI of 2 quadrics in the total space of

$$\mathbf{P}\left(\mathcal{O}(-1,0)^{\oplus 2} \oplus \mathcal{O}(0,-1)^{\oplus 2}\right) \longrightarrow \mathbf{P}^1 \times \mathbf{P}^1$$

↔ Kahler

branched double cover of $\mathbf{P}^1 \times \mathbf{P}^1 \times \mathbf{P}^1$,
branched over deg (4,4,4) locus

* In fact, the GLSM has 8 Kahler phases, 4 of each of the above.

* Related to an example of Vafa-Witten involving discrete torsion

(Caldararu, Borisov)

* Believed to be homologically projective dual

A non-CY example:

CI 2 quadrics in $\mathbf{P}^{2g+1}$ $\xleftrightarrow{\text{Kahler}}$ branched double cover of $\mathbf{P}^1$, over deg 2g+2 (= genus g curve)

Homologically projective dual.

Here, r flows -- not a parameter.

Semiclassically, Kahler moduli space falls apart into 2 chunks.

Positively curved | Negatively curved

r flows: $\dashrightarrow$ $\dashrightarrow$ $\dashrightarrow$

Another non-CY example:

CI 2 quadrics in $\mathbf{P}^4$ (= deg 4 del Pezzo) $\xleftrightarrow{\text{Kahler}}$ Disjoint union of 2 copies of $\mathbf{P}^1$ w/ 5 $\mathbf{Z}_2$ singularities

Why a disjoint union instead of a double cover?
Answer: different Berry phase

Homologically projective dual

Analogous results for $\mathbf{P}^6[2,2,2]$, $\mathbf{P}^6[2,2,2,2]$

So far, we have only considered complete intersections of **quadrics**.

However, part of the analysis applies more generally.

Ex: $\mathbf{P}^5[3,3]$

The LG point of the GLSM is a hybrid LG model, with base a $\mathbf{Z}_3$ gerbe over $\mathbf{P}^1$, and fibers LG models for K3's.

Matches Kuznetsov's homological projective duality.

Aside:

One of the lessons of this analysis is that
gerbe structures are commonplace,
even generic,
in the hybrid LG models arising in GLSM's.

To understand the LG points of typical GLSM's,
requires understanding gerbes in physics.

So far we have discussed several GLSM's s.t.:

* the LG point realizes geometry in an unusual way

* the geometric phases are not birational

* instead, related by Kuznetsov's homological projective duality

We conjecture that Kuznetsov's homological projective duality applies much more generally to GLSM's.....

More Kuznetsov duals:

Another class of examples, also realizing Kuznetsov's h.p.d., were realized in GLSM's by Hori-Tong.

$G(2,7)[1^7]$ $\xleftrightarrow{\text{Kahler}}$ Pfaffian CY

(Rodland, Kuznetsov, Borisov-Caldararu, Hori-Tong)

* unusual geometric realization (via strong coupling effects in nonabelian GLSM)

* non-birational

More Kuznetsov duals:

$G(2,5)[1^4]$ (= deg 5 del Pezzo) ⟷ (Kahler) Vanishing locus in $\mathbf{P}^3$ of Pfaffians

= (left) and = (right)

Vanishing locus in $\mathbf{P}^5$ of Pfaffians ⟷ (Kahler) $G(2,5)[1^6]$

Positively curved | Negatively curved

r flows: ⇢ ⇢ ⇢

More Kuznetsov duals:

$G(2,N)[1^m]$ (N odd) $\xleftrightarrow{\text{Kahler}}$ vanishing locus in P^{m-1} of Pfaffians

Check r flow:

$K = O(m-N)$ $\qquad\qquad$ $K = O(N-m)$

Opp sign, as desired,
so all flows in same direction.

More Kuznetsov duals:

So far we have discussed how Kuznetsov's h.p.d. realizes Kahler phases of several GLSM's with exotic physics.

We conjecture it also applies to ordinary GLSM's.

Ex: flops

Some flops are already known to be related by h.p.d.; K is working on the general case.

UNDER CONSTRUCTION

New heterotic CFT's

Although (2,2) models decompose into a disjoint union, (0,2) models do not seem to in general.

Prototype: $\mathcal{O}(1) \longrightarrow \mathbf{P}_{[k,k,\cdots,k]}$ "$\mathcal{O}(1/k)$"

-- understanding of some of the 2d (0,4) theories appearing in geometric Langlands program

-- genuinely new string compactifications

A lesson for the landscape:
many more string vacua may exist than previously enumerated.

Summary

- Basics of string compactifications on stacks
- Cluster decomposition conjecture for strings on gerbes:
  CFT(gerbe) = CFT(disjoint union of spaces)
- Application to GLSM's; realization of Kuznetsov's homological projective duality
- Future directions