Explorations into Knot Theory: Colorability

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Abstract

We explore properties of a generalized "coloring" of a knot including existence, changes in regards to satellite knots and specifically composition, non-colorable knots. We also explore homomorphisms from the knot group to the dihedral group of a knot and provide computer programs developed in our research of knots. We also provide a database and online resource for future investigations into these arenas.

1 Introduction

The idea of coloration probably has roots as far back as the idea of strands, since it is but a simple extension of the idea. Since the trefoil knot is tricolorable, undoubtedly it was this discovery that prompted the study of tricolorability of other knots and as an invariant. Of course, it was not long after this that a generalized idea of colorability was concocted. This general colorability, or *n-Coloration*, is the topic of this study.

Since the n-Coloration of a knot can be found using its crossing matrix, it is relatively easy to calculate. This at first might seem like a gold mine of information on all knots, but colorability is not an invariant for all knots, as we shall see. In fact, its distinguishing power is not comparable to any of the other common invariants, but its usefulness lies in both its simplicity and its ease of application. Also, as we shall see, the theory of n-Coloration yields some interesting relations to existing theory concerning knots.

We first provide some definitions to help us maneuver in the language of knots, colorations, and operations on these:

Definition 1 *A knot is a closed, one dimensional, and non-intersecting curve in three dimensional space. From a more set-theoretic standpoint, a knot is a homeomorphism that maps a circle into three dimensional space.*

Definition 2 *A* homeomorphism *is generally considered to be an additive and continuous function. Additive refers to the following property:*

$$g(A + B) = g(A) + g(B)$$

Also, two objects are considered homeomorphic *if there is a homeomorphism between them.*

Definition 3 *An* ambient isotopy*, also known as an* isotopy*, is a continuous deformation of a knot or link. This represents the "rubber-sheet geometry" aspect of topology, where the knot or link may be bent, twisted, stretched, or pulled. Under no circumstances, however, may the curve be allowed to intersect itself or be cut.*

Definition 4 *Any group knots or links are considered* ambiently isotopic *or isotopic if there exists an ambient isotopy between them. Such a group is called an isotopic class. All members of an isotopic class, called projections, are considered to be the same knot or link.*

Definition 5 *An* isotopic class *is a group of knots or links that are ambiently isotopic, or can be continuously deformed into each other. For example, the unknot and figure-eight below belong to the same isotopic class because they can be deformed into each other through a type I Reidemeister move.*

Definition 6 *A* projection *is a specific member of an isotopic class. For example, the unknot and figure-eight below are two different projections of the same isotopic class.*

Definition 7 *The* unknot*, also known as the "trivial knot", is simply a circle embedded in three-dimensional space with no crossings and all other projections of the same isotopic class. The circle and the figure-eight are both considered unknots.*

Definition 8 *A* link *is a group of knots or unknots embedded in three dimensional space. Each knot or unknot embedded in the link is called a component. A link with only one component is usually just referred to as a knot or unknot. A link in which another member of the link's isotopic class has components that can each be bounded with non-overlapping spheres, or otherwise separated components, is called the trivial link.*

Definition 9 Tricolorability *deals with the ability to use three different "colors" to color a knot. A knot is colored by individual strands, where a strand is the part of a two-dimensional representation of a knot between undercrossings. A knot is tricolorable if:*
Rule 1: At every crossing, either all three strands are of a different color, or the same color.
Rule 2: All three colors are used in coloring the knot.

2 Existence of Colorings

In this section it is shown that the existence of an ℓ-coloration of a given projection of a knot implies the (non)existence of an ℓ-coloration of any projection of that knot up to Reidemeister moves I, II, and III. Because any ambient isotopy of a knot compliment space is equivalent to a series of Reidemeister moves and planar isotopies of a given projection of the knot, existence of an ℓ-coloration, or ℓ-colorability, is a knot invariant.

Given a projection of a knot to the plane, label its n strands $a_1, \ldots, a_n$. Each crossing then corresponds to a linear expression given by: $a_i + a_j \Leftrightarrow 2a_k$, where a_i and a_j are the two understrands of the crossing and a_k is the overcrossing.

As each strand has exactly two ends, and each crossing has exactly two understrands or ends, there are always n crossings for given n strands. Thus, a given knot projection and a given labelling of its n strands determines a so-called 'crossing matrix' of $n \times n$ dimensions.

Important: Because a permutation of the labelling corresponds to a series of column transpositions, many important properties (i.e. null space, determinant of the matrix (minors) up to sign) of the matrix are invariant of labelling.

To be ℓ-colorable, a crossing matrix of a projection must have a null space of dimension two or greater when taken mod ℓ. This means that there is at least one solution which is fundamentally different from the trivial coloration $a, \ldots, a$.

Suppose a projection is ℓ-colorable:

Reidemeister Case 1

In the first Reidemeister Case, a strand is given a single twist or a single twist is untwisted. The expression for the crossing of a single twist is $a_1 + a_2 \Leftrightarrow 2a_1$. Whatever ℓ is, a_1 must be congruent to a_2, mod ℓ. Thus, if a projection contains a single twist and is ℓ-colorable, the two strands must indeed be considered as one, 'colorwise', and so a pair of strands which compose a single twist is equivalent to a single strand in our coloration system. This shows that colorability is invariant under a type I Reidemeister move.

Reidemeister Case 2

For the second Reidemeister Case, a local neighborhood of the projection containing only two disjoint sections of strands which do not cross each other is transformed such that the piece of the knot corresponding to a strand a_1 dips underneath the piece of the knot corresponding to a strand a_2. There are now four strands $a_1, a_2, a_3, a_4 = a_1$ (see picture) and two crossings which (both) must satisfy the equation $a_1 + a_3 \Leftrightarrow 2a_2 \equiv 0 \bmod \ell$ if n-colorability is to be met. Since there is no other restriction on a_3 (as a_3 is entirely within the local neighborhood), there is no potential change in the rest of the projection if we designate $a_3 = 2a_2 \Leftrightarrow a_1$. Because nothing is changed outside the local section, colorability is preserved. Conversely, suppose we have a local section of a projection (which is ℓ-colorable) which contains three sections of strands with numbers a_1, a_2, a_3 and a single strand a_4, and two crossings, $a_1 + a_4 \Leftrightarrow 2a_2 \equiv 0 \mod \ell$ and $a_3 + a_4 \Leftrightarrow 2a_2 \equiv 0 \mod \ell$. Combining the two equations gives $a_4 \equiv 2a_2 \Leftrightarrow a_1 \mod \ell \Rightarrow a_3 + 2a_2 \Leftrightarrow a_1 \Leftrightarrow 2a_2 \equiv a_3 \Leftrightarrow a_1 \equiv 0 \mod \ell$. Therefore a_1 must be congruent to a_3 for the projection to be ℓ-colorable. If we transform this local section so that a_4 is deleted and the sections with numbers a_1 and a_3 become one section of strand, the inverse of the Reidemeister move II, we do not alter anything outside the local neighborhood, and so colorability is preserved under the second Reidemeister Case.

Reidemeister Case 3

In the Reidemeister Case 3, there is a local section of the projection with one stationary intersection $a_1 + a_3 \Leftrightarrow 2a_2$ and three strands corresponding to a section of the knot passing 'underneath' the local section. If the three strands a_4, a_5, a_6 pass on the side of the crossing such that $a_4 + a_5 \Leftrightarrow 2a_1 \equiv 0 \mod \ell$ and $a_5 + a_6 \Leftrightarrow 2a_2 \equiv 0 \mod \ell$, then 'shifting' to the other side should not change colorability, i.e. $a_4 + a_5' \Leftrightarrow 2a_2 \equiv 0 \mod \ell$ and $a_5' + a_6 \Leftrightarrow 2a_3 \equiv 0 \mod \ell$. This is because, if we combine both sets of equations such that the a_5's are eliminated, we see that $(a_4 \Leftrightarrow a_6) + (2a_2 \Leftrightarrow 2a_1) \equiv (a_4 \Leftrightarrow a_6) + (2a_3 \Leftrightarrow 2a_2) \mod \ell$, which implies that $a_1 + a_3 \Leftrightarrow 2a_2 \equiv 0 \mod \ell$. So the side which a_4, a_5, a_6 passes under (see picture) is trivial iff the stationary intersection is consistent with ℓ-colorability. Thus colorability is preserved under a Reidemeister move III.

3 Operations on Knots and Their Various Effects on Coloring

There exist various ways of constructing knots from preexisting ones. "Mutating" a knot and forming satellite knots are two of the simplest. We are interested in the coloration properties modified knots have and their relation to the coloration properties of their "parent" knots.

The easiest way to get a new knot from a pair of known knots is to compose them. This is done by cutting each of the "factor" knots open and attaching the resulting loose strands in such a way as to create a single knot.

Composition of knots can be generalized in a way that gives rise to the so-called satellite knots. Given two knots, K_1 and K_2, choose two tori, T_1, the boundary of an $\epsilon \Leftrightarrow$*neighborhood* of K_1 and T_2, such that $K_2 \subset S^3 \Leftrightarrow T_2$. Now remove the component of $S^3 \Leftrightarrow T_1$ containing the knot and glue in its place the portion of $S^3 \Leftrightarrow T_2$ containing K_2 such that the meridians and longitudes of the T_i are mapped, respectively, to each other. K_1 and K_2 are known as the companion and the satellite, respectively. A longitude on a torus embedded in S^3 is simply a curve that forms the boundary of an orientate surface in the complement of said torus.

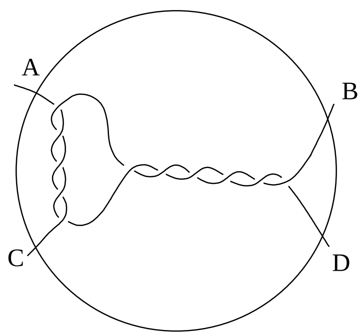


Figure 1: A (rational) tangle.

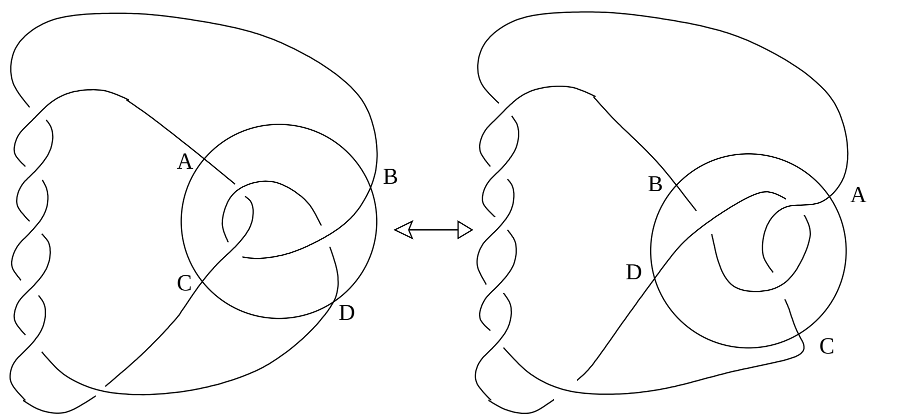


Figure 2: A knot and a mutant. These two are actually equivalent as knots too!

A satellite of two knots is the same as their sum if the intersection $K_2 \cap M$ of K_2 and some meridional disc M of T_2 contains only one point. The composition of K_1 and K_2 will be denoted $K_1 \# K_2$.

Mutating a knot is simpler than the formation of satellite knots. That simplicity has its price though: if a knot can be p-colored, so can any of its mutants.

A tangle is like a knot but, instead of being closed, has four points with neighborhoods homeomorphic to $[0, 1)$, the endpoints of the tangle corresponding to points on the boundary of a closed ball $B \subset S^3$ that contains the tangle. See Figure 1.

Note that a subset of a knot can be a tangle. Simply try to surround a portion of the knot by a sphere such that it is pierced by the knot exactly four times. The portions of the knot "inside" and "outside" of such a sphere are tangles. Referring to the figure, a mutant of a knot is a new knot obtained by rotating the sphere, along with it's contents, 180° so that A and B, C and D, are exchanged.

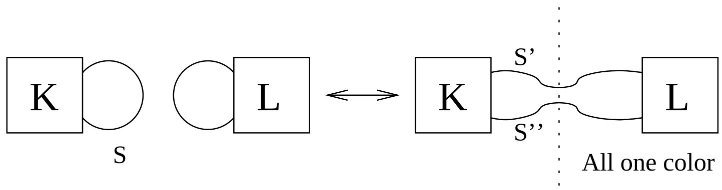


Figure 3: Coloring the sum of two knots.

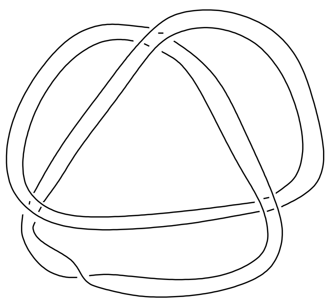


Figure 4: A satellite knot with trefoil as companion and torus knot (2,5) as satellite

Satellites

One property $K\#L$ has is that if K is p colorable and L is q colorable, then $K\#L$ is both p and q colorable. To p color the sum, simply color all of the strands inherited from K as if coloring K. Suppose the strand s of K was cut in the process of forming the sum. If it would have had color c, color the pieces, s', s'', of s, along with all the strands from L, c. See Figure 3. This suggests the following hypothesis: If a knot is prime, i.e. isn't the sum of two knots, then it cannot be colored in more than one way, for multiple colorings should correspond to factor knots. Alas, this is not true, as will be demonstrated in Section 4.

How do satellite knots behave under coloration? First, let's look at an example.

Example 1 *Figure 4 shows a satellite knot with trefoil as companion, and the (2,5) torus knot as satellite. (Watch where those longitudes and meridians go!) A simple (but very time consuming, if done by hand!) calculation reveals that this knot is* $5\Leftrightarrow$*colorable, like its satellite. Note that it isn't* $3\Leftrightarrow$*colorable, like its companion, the trefoil.*

When considering a satellite knot, it may be of help to consider any obvious colorings it may possess. Setting K_i and T_i as above, it is safe to assume that K_2 sits in T_2 as in Figure 5. Using this projection simplifies the construction of the satellite knot. What happens when we make the satellite knot? If the

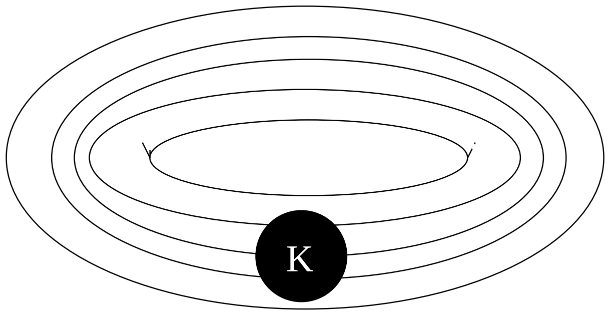


Figure 5: A satellite can sit in its torus like this.

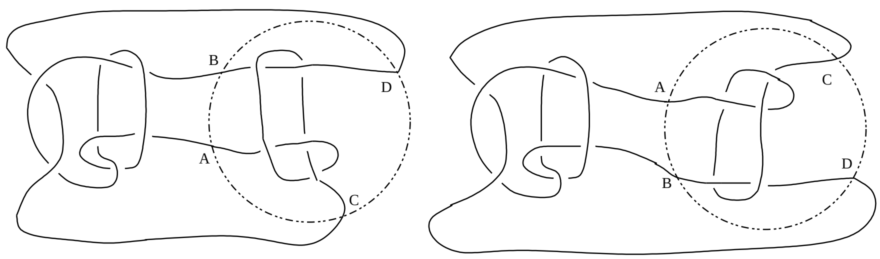
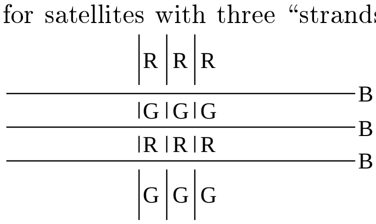


Figure 6: Kinoshita-Terasaka mutants.

number of strands that wrap around T_2 is odd, we're in business: the satellite knot is colorable as the companion is. The next picture indicates the situation for satellites with three "strands."The principle generalizes easily. vspace.5cm



If the number of strands is even we can't say anything. Figure 4. shows that satellite knots don't, in general, share colorabilities with their companions.

Mutants

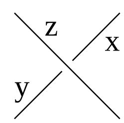
Any mutant of a knot has the same colorabilities its parent knot has. I hope the following example will induce the reader to find out more.

Example 2 *The two knots pictured in Figure 6 have the same colorabilities. In fact, they can't even be distinguished from the unknot using coloration techniques!*

4 Multicolorability and Colorability of Higher Nullities

From section 3 we learned that if the knot K is formed from the composition of the knots K_1 and K_2 which are p and q colorable respectively, then K is both p and q colorable. Thus any composite knot is at least bi-colorable if its components have different colorabilities. As we shall see below the converse is not true. It is natural to ask whether there are any prime knots (knots which are not the sum of two other knots) which are multicolorable? If the components of a knot both have the same colorability can anything be said about the composite colorability? The definition of ℓ-colorability is that the crossing matrix has nullity of at least two modulo ℓ, are there any knots which have nullity three or higher? To answer these questions we will first need to clarify some definitions and use our linear algebra skills.

We say the knot projection K, with n crossings, is ℓ colorable if and only if the crossing matrix C_K mod ℓ has nullity β where $\ell, \beta \in \mathbf{N}$ and $\beta \geq 2$. Well this is the same thing as saying there exists β distinct strands such that: to each of those β strands any integer from 0 to $\ell \Leftrightarrow 1$ can be assigned to the strand and there exist numberings (colorings) of the other $n \Leftrightarrow \beta$ strands (which are dependent on the numberings of the β strands, but not necessarily uniquely dependent) such that at each of the n crossings: $x + y \equiv 2\,z \bmod \ell$ where x, y represent the numbers assigned to the understrands and z the number assigned to the overstrand. See margin.



C_K is the $n \times n$ matrix whose rows correspond to the equations $x + y \Leftrightarrow 2\,z = 0$ for the n different crossings and the columns are the variables which correspond to the n strands. Let the $n \times 1$ column vector v represent a coloration of the knot projection, where the i^{th} component of v is v_i and represents the coloring of the i^{th} strand. In symbolic notation K is ℓ colorable $\Leftrightarrow \exists\, \beta \geq 2\ \exists$ distinct strands $i_1, i_2, \cdots, i_\beta$, such that $\forall$ colorings $q_1, q_2, \cdots, q_\beta \in Z_\ell$, there is a coloration $\vec{v} \in Z_\ell^n$ with those colorings $(\forall\, j \in \{1, \cdots, \beta\}\ v_{i_j} = q_j)$, such that $\vec{v}$ is a valid coloration modulo ℓ $(C_K \cdot \vec{v} \equiv \vec{0} \bmod \ell)$. Thus there are at least ℓ^β different ℓ-colorations. For example the simplest case K is ℓ-colorable with nullity two then $\exists\, i, j \mid i \neq j\ \forall\, q, p \in Z_\ell\ \exists\, \vec{v} \in Z_\ell^n$ with $v_i = q\ v_j = p$ and $C_K \cdot \vec{v} \equiv \vec{0} \bmod \ell$

Since there is no privileged frame of reference for the ordering of the strands in the knot projection, the columns of the crossing matrix can be permuted (reordered) without affecting the colorability of the knot. Thus there exits an $n \times n$ permutation matrix P such that the β distinct strands, whose colorings can be chosen freely, correspond to the last β columns of the matrix $C_K \cdot P$. Using this new permuted crossing matrix we can find β linearly independent coloration vectors $v_1, v_2, \cdots, v_\beta$ of K. These β column vectors can be placed in a $n \times \beta$ matrix V, where the columns of V are these β coloration vectors. The span of these vectors is in the modular null-space of $C_K \cdot P$, however these vectors are not necessarily a basis for the null-space, but if ℓ is a prime then they are. Since any integral linear combination of valid colorations is a valid coloration, we can let $\vec{d}$ be $\beta \times 1$ vector that represents a linear combination with integer components, then the $n \times 1$ column vector $V \cdot \vec{d}$ is a valid coloration of $C_K \cdot P$. Knowing that the colorings of the last β strands can be freely chosen and a little linear algebra, we can require the vectors $v_1, v_2, \cdots, v_\beta$ to take on the form that the i^{th} vector's $n \Leftrightarrow i+1$ component is one and all the components beneath the one are zero. Thus there is a V of the form

$$\begin{pmatrix} ? & ? & \cdots & ? & ? & ? & 1 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots \\ ? & ? & \cdots & ? & ? & ? & 1 \\ 1 & 0 & \cdots & 0 & 0 & 0 & 1 \\ 0 & 1 & \ddots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \ddots & 0 & 0 & 0 & 1 \\ 0 & 0 & \ddots & 1 & 0 & 0 & 1 \\ 0 & 0 & \ddots & 0 & 1 & 0 & 1 \\ \vdots & \vdots & \ddots & 0 & 0 & 1 & 1 \\ 0 & 0 & \cdots & 0 & 0 & 0 & 1 \end{pmatrix}$$

if $\beta = 3$ then V is of the form

$$V = \begin{pmatrix} ? & ? & 1 \\ \vdots & \vdots & \vdots \\ ? & ? & 1 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}$$

and the simplest case if $\beta = 2$ then

$$V = \begin{pmatrix} ? & 1 \\ \vdots & \vdots \\ ? & 1 \\ 1 & 1 \\ 0 & 1 \end{pmatrix}$$

K is ℓ-colorable with nullity $\beta \geq 2$ if and only if there exists a permutation matrix P and a matrix V of the preceding form such that for all linear combination vectors $\vec{d} \in Z_\ell^\beta$ the congruence $C_K \frac{n}{\times} n \cdot P \frac{n}{\times} n \cdot V \frac{n}{\times} \beta \cdot \vec{d}_1 \frac{\beta}{\times} 1 \equiv \vec{0} \frac{n}{\times} 1 \mod \ell$ is satisfied.

Ok now that you have explicitly defined colorability with nullity β. How does one find out if a knot is ℓ-colorable, and what value the nullity of C_K takes on modulo ℓ? And once one knows that, how can one find these β vectors whose span is in the modular null-space? To answer those questions we simply look at the Smith and Hermite normal forms of the crossing matrix C_K. So lets just refresh our memories of what these forms are.

The Hermite normal form of a matrix A is an upper triangular matrix H with $rank(A)$ = the number of nonzero rows of H. The Hermite normal form is obtained by doing elementary row operations on A. This includes interchanging rows, multiplying through a row by -1, and adding an integral multiple of one row to another. The Smith normal form S, of an $n \times m$ rectangular matrix A of integers, is a diagonal matrix where $rank(A)$ = number of nonzero rows (columns) of S; $sign(S_{i,i}) = 1\ \forall i$; $S_{i,i}$ divides $S_{i+1,i+1} \forall i \leq rank(A)$; and $\prod_{i=1}^{r} S_{i,i}$ divides $det(M)$ for all minors M of rank $0 < r \leq rank(A)$. Hence if $n = m$ and $rank(A) = n$ then $|det(A)| = \prod_{i=1}^{n} S_{i,i}$. The Smith normal form is obtained by doing elementary row and column operations on A. This includes interchanging rows (columns), multiplying through a row (column) by -1, and adding integral multiples of one row (column) to another.

Since the algorithm for the Hermite normal form does not include column swapping, all of the colorabilities of K are not necessarily visible along the diagonal of the matrix H. In fact there exist crossing matrixes such that H modulo ℓ only has one row of zeros, despite the fact that H and C_K both have nullity two modulo ℓ. For example if take the following crossing matrix for 7_4, we attain the following Hermite form.

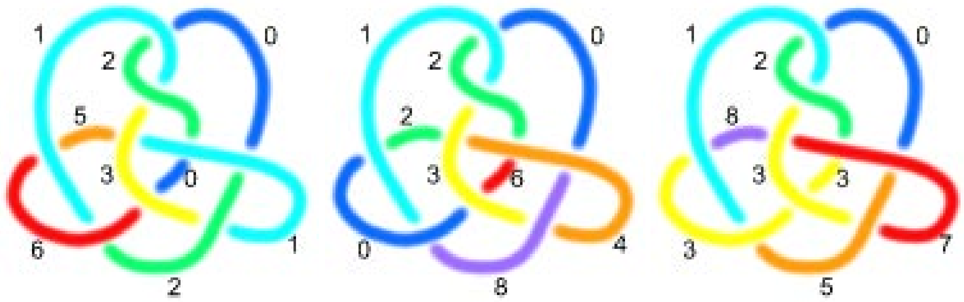
$$C_K = \begin{pmatrix} 1 & 0 & 0 & \Leftrightarrow 2 & 0 & 0 & 1 \\ 1 & 1 & 0 & 0 & 0 & \Leftrightarrow 2 & 0 \\ 0 & 1 & 1 & 0 & \Leftrightarrow 2 & 0 & 0 \\ \Leftrightarrow 2 & 0 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 & \Leftrightarrow 2 \\ 0 & \Leftrightarrow 2 & 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & \Leftrightarrow 2 & 0 & 0 & 1 & 1 \end{pmatrix}$$

$$H = \begin{pmatrix} 1 & 0 & 0 & 0 & 2 & 0 & \Leftrightarrow 3 \\ 0 & 1 & 0 & 0 & 1 & 1 & \Leftrightarrow 3 \\ 0 & 0 & 1 & 0 & 0 & 2 & \Leftrightarrow 3 \\ 0 & 0 & 0 & 1 & 1 & 0 & \Leftrightarrow 2 \\ 0 & 0 & 0 & 0 & 3 & 3 & \Leftrightarrow 6 \\ 0 & 0 & 0 & 0 & 0 & 5 & \Leftrightarrow 5 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

But if the fifth and sixth column of C_K were to be interchanged then the diagonal of H would be the same as the Smith form, namely five one's, a fifteen and a zero. Thus the original Hermite form doesn't explicitly reveal 7_4's 15-colorability.

Fortunately the Smith normal form does take advantage of column operations so therefor the colorabilities of a knot K along with their corresponding nullities are readably visible along the diagonal of the Smith normal form S of a crossing matrix C_K. Namely the knot K is ℓ-colorable if and only if ℓ divides the last/greatest non-zero diagonal element of the Smith normal form S. This can be extended to K is ℓ-colorable with nullity β if and only if ℓ divides all components of the last β rows and columns of the Smith normal form S.

Once the colorabilities of a knot are known the colorations of the projection K are easily determined by back substituting the colorings of the freely choissable strands into the Hermite normal form H of the crossing matrix C_K and solving the corresponding congruences modulo ℓ. Although the strands can be chosen freely, they do not necessarily define a unique coloration. For example the knot 9_{48} has nullity $2 \bmod 9$ but as the following pictures show, the colorings of the top two strands were fixed but the colorings of the lower strands are not uniquely defined.



Well one may think that this implies that the knot 9_{48} has nullity $3 \bmod 9$ but this is not the case since these three colorations are the only valid colorations with the top two strands having these values of one and zero. If it had nullity 3 then there would be nine different colorations with the top two strands fixed.

This phenomena occurs when a nontrivial divisor p of ℓ has

WHAT GOES HERE??

Theorem 1 *If K is ℓ-colorable with nullity β then $\forall p \mid 1 < p < \ell$ and $p \mid \ell$ then K is p-colorable with nullity of at least β.*

Proof 1 *After row and column reduction over the integers of the crossing matrix C_K, the Smith normal form has β rows which are congruent to the zero row modulo ℓ, if p divides ℓ then these same β rows are also the zero row modulo p. If $\vec{x} \equiv \vec{0} \mod \ell$ and $p \mid \ell$ then $\vec{x} \equiv \vec{0} \mod p$.*

Theorem 2 *If K is p-colorable then for all k there exist valid non-trivial kp-colorations even if K is not kp-colorable.*

Proof 2 *Let v be vector corresponding to any valid p-coloration. To obtain a valid kp-coloration, just multiply v by the scalar k and add to that any integral multiple of the vector of all ones v_1. If $C_K \cdot v \equiv \vec{0} \mod p$ and $C_K \cdot v_1 \equiv \vec{0} \mod kp$ then $\forall\, m\ C_K \cdot (kv + mv_1) \equiv \vec{0} \mod kp$. Let u and v be different p-colorations then ku and kv are not necessarily different kp-colorations, because $\vec{x} \not\equiv \vec{u} \mod p \nLeftrightarrow k\vec{x} \not\equiv k\vec{u} \mod kp$ Therefor there is no implication that K is kp-colorable.*

Theorem 3 *If K is p-colorable and q-colorable and p and q are relatively prime then K is also pq-colorable.*

Proof 3 *Let s be the greatest non-zero diagonal element of the Smith normal form of C_K then $p \mid s$ and $q \mid s$ and $gcd\,(p,q) = 1 \;\Rightarrow\; pq \mid s \;\Rightarrow\; K$ is pq-colorable.*

Let $L := \prod_{i=1}^{n-1} S_{i,i}$ $\forall\, i,j$ $|det(M_{i,j})| = L$ where $M_{i,j}$ is the i^{th} j^{th} minor of C_K

K is not L-colorable $\Leftrightarrow \exists \ell$ such that K is ℓ-colorable with nullity of at least 3.

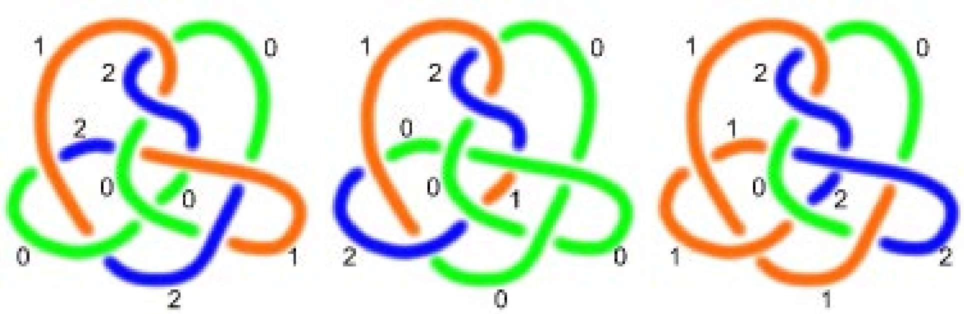
These last theorems put to rest a couple of questions.

If a knot is both p and q colorable and $p \neq q$ then is the knot a composite knot? No. 8_{20} and 9_1 are both 3 and 9 colorable; $7_4, 8_{21}, 9_2$, and 9_{37} are all $3, 5$, and 15 colorable; and $7_7, 8_5$, and 9_4 are all $3, 7$, and 21 colorable and all of these knots are prime.

Can you construct a knot that is q-colorable? Yes, if q can be factored into a finite set of integers which are all relatively prime to each other and each integer is less than or equal to 61 then this q-colorable knot can be constructed from the composition of prime knots with nine or less crossings, seeing how $\forall p \leq 61$ there exists a prime knot with crossing number less than ten which is p-colorable. If q can not be factored in such a way then the prime knots of ten and higher crossings have to be implemented in order to construct this knot.

Can you construct a knot that is q-colorable with nullity $\beta \geq 3$? Yes if q satisfies the previous condition then take the knot K which is q-colorable and compose it with itself $\beta \Leftrightarrow 1$ times.

Well those were composite knots, are there any prime knots with nullity $\beta \geq 3$? Yes $8_{18}, 9_{35}, 9_{46}, 9_{47}$, and 9_{48} all have nullity 3 modulo 3; 9_{40}, and 9_{49} both have nullity 3 modulo 5; and 9_{41} has nullity 3 modulo 7. Here are three pictures of the 27 different 3-colorations of 9_{48}, showing how the the colorings of the upper two strands were fixed and the coloring of the lower strand was freely varied, demonstrating how it takes the colorings of three strands to uniquely determine a valid coloration.



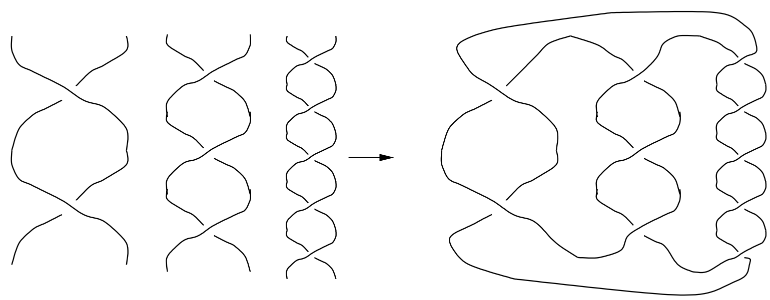


Figure 7: A $\{\Leftrightarrow 2, 3, 5\}$ pretzel knot.

5 Visible vs. Invisible Knots

Previous sections have been devoted to finding colorations for knots and determining properties of those colorations. Here, the goal is to investigate what kinds of knots are colorable and when. Specifically we will explore colorations for pretzel knots and (p,p-1) torus knots.

Now we will attempt to construct a coloring for the {i,j,k} pretzle knot. For simplicity we will assume that $i, j, k > 0$, however, none of the following arguments depend on this and we will arive at the same conclusions if we allow the signs to vary.

Definition 10 *A knot $\mathcal{K}$ is called invisible if there does not exist n such that $\mathcal{K}$ is colorable* $\pmod{n}$*, and is therefore indistinguishable from the unknot through the scope of colorability.*

Pretzel Knots

Recall that a pretzle knot $\{i, j, k\}$ is constructed from the rational tangles i,j,k by the formula $i * 0 + j * 0 + k * 0$ and then connecting the top two strands together and the bottom two strands. An example of a $\{\Leftrightarrow 2, 3, 5\}$ prezle knot is depicted in Fig. 7.

In figure 8, we have colored 5 of the strands (a, b, p, q, r). From the properties of colorations (as described in sec. Arron or mabey Allen) we know that to any coloration scheme we may perform the operations of adding any integer or multiplying by any integer and the result will itself be a coloration. ie, if...

$$2a \Leftrightarrow b \Leftrightarrow c = 0 \textit{(where a is the over strand and b and c are the understrands)} \tag{1}$$

then...

$2(a + d) \Leftrightarrow (b + d) \Leftrightarrow (c + d) = 0$

and...

$2af \Leftrightarrow bf \Leftrightarrow cf = 0$

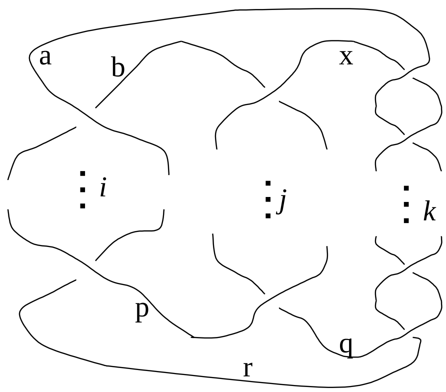


Figure 8: A coloring of a general 3-component pretzel knot.

Because of this fact, we can, without loss of generality, let $a = 0$ and $b = 1$, and then use our coloration formula to propagate these colors down the i twists. We quickly see that $r = i$ and $p = i + 1$.

Knowing that the incoming strands for the second component of our knot are labled 1 and x, we may follow the crossing formula down the j twists and find that p=jx-j+1=i+1 and q=(j+1)x-j. Like wise, the third component has incoming strands x and 0 yielding the colors $q = (1 \Leftrightarrow k)x$ and $r = \Leftrightarrow kx$ as shown in Figure 9.

This gives us three linearly dependent equations(placing restrictions on our variables), two of which are listed below.

$$\begin{aligned} i &= \Leftrightarrow kx \\ jx &= j + i \end{aligned}$$

If we multiply these equations together our x's fall out and we get the simple equation

$$ji + jk + ik = 0 (mod\ n) \tag{2}$$

and the only time an appropriate n cannot be chosen to satisfy this equation is when

$$ji + jk + ik = \pm 1$$

because whenever $|ji + jk + ik| \neq 1$ we can always choose an appropriate n, such that it will equal 0 mod n.

Corolary 1 *if i,j, and k share the same sign, then i,j,k is always colorable.*

proof: $|ij + ik + jk| > 1$.

Corrolary 1 *The {-i,2i,2i} pretzel knot is always colorable mod any n.*

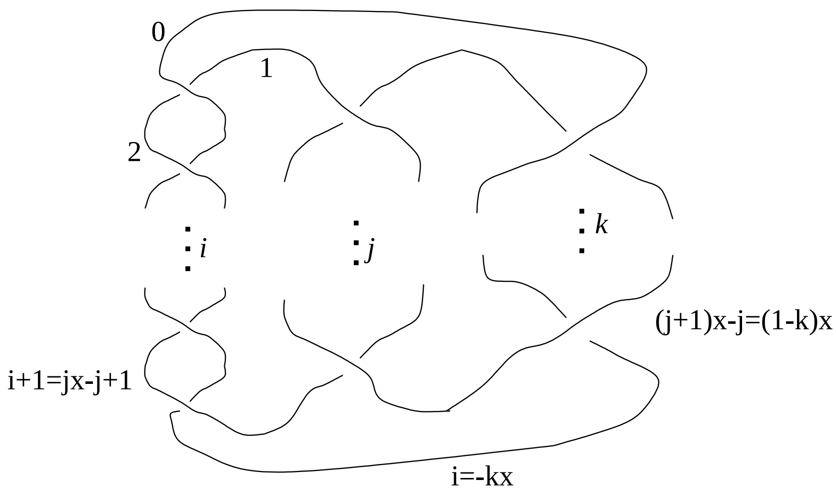


Figure 9: Some simple relations for the coloring of a general 3 component pretzel knot.

PROOF:

$$\Leftrightarrow i * 2i + \Leftrightarrow i * 2i + 2i * 2i = 0 \pmod{n} \ \forall \ n$$

Corollary 1 *If the 3 component pretzel knot $\{i,j,k\}$ is invisible, then i, j, and k must be relatively prime. Proof: if n devides i and j, then n must devide 1 since $i * j + i * k + j * k = \pm 1$*

Coloring A General Pretzle Knot

We now generalize this to a k-component pretzle knot $\mathcal{K} = \{C_0, C_1, ..., C_{k-1}\}$.

Theorem 1

$$\sum_{i=0}^{k-1} C_0 * C_1 * C_2 * ... * \widehat{C_i} * ... * C_{k-1} = \pm 1 \Leftrightarrow \mathcal{K} \ \textbf{\textit{invisible}} \tag{3}$$

Proof 4 *We shall color the strands of our knot according to the scheme in figure 10, where C_j's represent the number of crossings in the j'th tangle component, a_j's are the colors assigned to the strands at the top of those components, and P_j's represent the colors of the strands coming out of the bottom of the pretzle components.*

Using the equation for colors at a crossing $(2a \Leftrightarrow b \Leftrightarrow c = 0)$ as before we find that...

$$P_j = (C_{j-1} + 1)a_j \Leftrightarrow C_{j-1}a_{j-1} = C_j a_{j+1} \Leftrightarrow (C_j \Leftrightarrow 1)a_j \tag{4}$$

for j $\in \mathcal{Z}$/k Which can be re-writen as

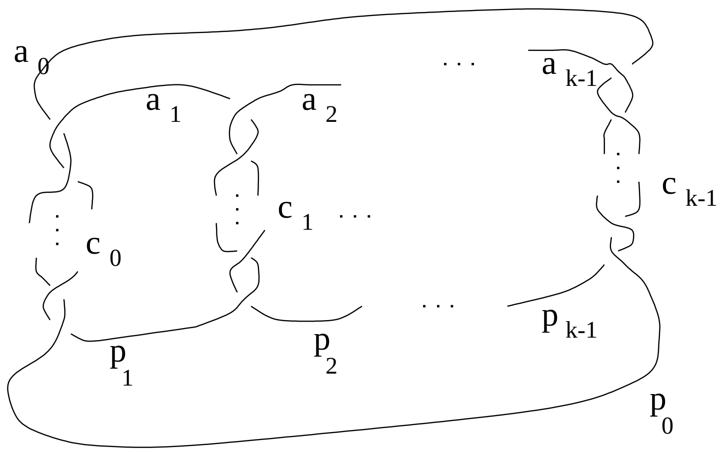


Figure 10: A gerneral coloring for a k component pretzel knot.

$$a_{j+1} = \frac{(C_{j-1} + C_j)a_j \Leftrightarrow C_{j-1}a_{j-1}}{C_j} \tag{5}$$

$$\text{or} \qquad a_j = \frac{(C_{j-2} + C_{j-1})a_{j-1} \Leftrightarrow C_{j-2}a_{j-2}}{C_{j-1}} \tag{6}$$

We say that $a_j(k)$ is the j'th color of a pretzel knot of k components (0...k-1). If we let $a_0 = 0$, $a_1 = 1$ and then apply equation 6 we get

$$a_0(i) = 0 = \frac{(C_{k-1} + C_k)a_i \Leftrightarrow C_{k-1}a_{k-1}}{C_k} \tag{7}$$

It turns out that there is a more simple equation for $a_0(k)$, namely

$$a_0(k) = \frac{C_0}{C_1} + \frac{C_0}{C_2} + ... + \frac{C_0}{C_{k-1}} + 1 \tag{8}$$

which can be shown by induction and the proof is omited. ¿From this we get that $\mathcal{K}$ is colorable implies that

$$\sum_{i=0}^{k-1} C_0 * C_1 * ... * \widehat{C_i} * ... * C_{k-1} \equiv 0 \pmod{n}$$

$\Rightarrow \mathcal{K}$ invisible if

$$\sum_{i=0}^{k-1} C_0 * C_1 * ... * \widehat{C_i} * ... * C_{k-1} = \pm 1$$

which ends our proof.

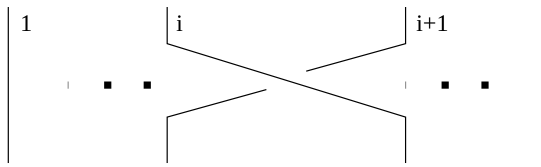


Figure 11: This shows a typical generator for the braid group on q strands.

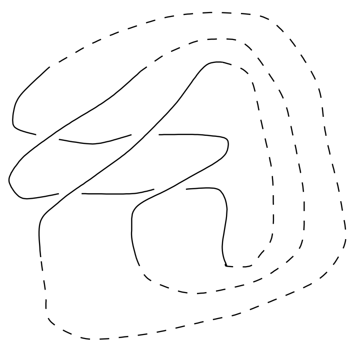


Figure 12: This shows how to close the ends of a braid knot.

Torus Knots

Definition 1 *A Torus knot $\mathcal{T}$ is one that is embedible on the surface of a torus, such that it never intersects itslef. Such a knot can be represented by to integers (p,q) where p is the number of times the knot loops around the meridian of the torus and q is the number of times it loops around the longitude.*

Theorem 1 *A (p,p-1) torus knot is always colorable* $\pmod n$. *Moreover, if p is odd then $n = p$ (or any factor of p) and if p is even then $n = p \Leftrightarrow 1$ (or any factor of p-1).*

It is easy to show that all torus knots can be represented by braid knots (described in section **??**). They are always of the form

$$\prod_{i=1}^{p} \sigma_1 * \sigma_2 * ... * \sigma_{q-1} = (p, q) \tag{9}$$

where σ_i's are the q-1 generators (see figure 11) of the braid group on q strands. The knot is then formed by connecting the strands at the top of the braid to those at the bottom in the cannonical manner depicted in figure 12

In this papper we will refer to the word $\sigma_1 * \sigma_2 * ... * \sigma_{q-1}$ as a "gword" and then represent a torus knot (p,q) as $(\text{gword}_q)^p$ and from now on we will only look at knots of the form (p,p-1).

Now let us try to color this (p,p-1) torus knot. We start by coloring the strands at the bottom of our braid knot $a_{0,1}$,$a_{0,2}$,...,$a_{0,p-1}$ and then labeling the

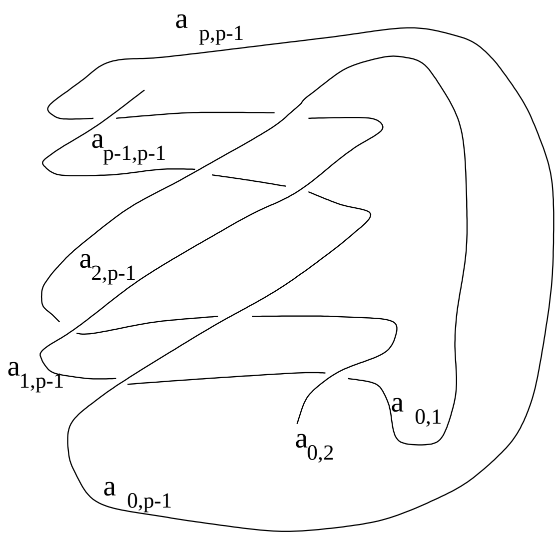


Figure 13: This is a general coloration of the (p,p-1) torus knot

top strands comming out of the i'th gword as $a_{i,1}$,$a_{i,2}$,..,$a_{i,p-1}$ however many of these colors will be repeats, so only the relevent colors are drawn in Figure 13.

From the crossing equation we can see that if $i < p$ then

$$a_{i,p-1} = \sum_{k=1}^{p-1} [2(\Leftrightarrow 1)^{k-p+1} + (\Leftrightarrow 1)^{k-p+1} \delta_i(k)] a_{0,k} \tag{10}$$

where δ is the kroneker delta function

$$\delta_i(k) = \begin{cases} 1 & \text{if } i = k \\ 0 & \text{if } i \neq k \end{cases}$$

and if $i > 1$ then

$$a_{i,p-1} = a_{0,i-1} \tag{11}$$

and for $i = p$ we get

$$a_{0,p-1} = a_{p,p-1} = \sum_{k=1}^{p-1} [2(\Leftrightarrow 1)^{k-p+1} + (\Leftrightarrow 1)^{p-1} \delta_1(k)] a_{k,p-1} \tag{12}$$

Let $a_{0,1} = 0$ & $a_{0,2} = 1$
and now take a look at the case when p is odd.

P odd

If p is odd then we can show that $a_{0,i} = i \Leftrightarrow 1$. First we know that it is true for $a_{0,1}$ and $a_{o,2}$ since we colored them 0 and 1 respectively. Then assume that it

is true for $a_{0,k}$ and prove for $a_{0,k+1}$.
So, if k even we get

$$\begin{aligned}
a_{0,k} \Leftrightarrow a_{0,1} &= k \Leftrightarrow 1 \qquad (13)\\
&= (2a0, p \Leftrightarrow 1 \Leftrightarrow 2a_{0,p-2} + ... \Leftrightarrow a_{0,k+1} + 2a_{0,k} \Leftrightarrow ... + 2a_{0,2})\\
&\quad \Leftrightarrow (2a0, p \Leftrightarrow 1 \Leftrightarrow 2a_{0,p-2} + ... \Leftrightarrow 3a_{0,k+1} + 2a_{0,k} \Leftrightarrow ... + 3a_{0,2})\\
&= a_{0,k+1} \Leftrightarrow 1
\end{aligned}$$

$$\Rightarrow a_{0,k+1} = k$$

and if k is odd we get

$$\begin{aligned}
a_{0,k} \Leftrightarrow a_{0,1} &= k \Leftrightarrow 1 \qquad (14)\\
&= (2a0, p \Leftrightarrow 1 \Leftrightarrow 2a_{0,p-2} + ... + 3a_{0,k+1} + 2a_{0,k} \Leftrightarrow ... + 2a_{0,2})\\
&\quad \Leftrightarrow (2a0, p \Leftrightarrow 1 \Leftrightarrow 2a_{0,p-2} + ... + 2a_{0,k+1} + 2a_{0,k} \Leftrightarrow ... + 3a_{0,2})\\
&= a_{0,k+1} \Leftrightarrow 1
\end{aligned}$$

and we get the same thing for $a_{0,k+1}$ so we have proven that $a_{0,k}$ must equal $k \Leftrightarrow 1$, and our equation for $a_{0,p-1}$ then simplifies to the following.

$$\begin{aligned}
a_{0,p-1} &= \\
a_{p,p-1} &= (\sum_{k=2}^{p-1} 2(\Leftrightarrow 1)^k a_{k,p-1}) \Leftrightarrow a_{1,p-1}\\
&= (\sum_{k=2}^{p-1} 2(\Leftrightarrow 1)^k a_{0,k-1}) \Leftrightarrow (\sum_{k=1}^{p-1} [2(\Leftrightarrow 1)^k + \delta_i(k)] a_{0,k-1})\\
&= 2a_{0,1} \Leftrightarrow 2a_{0,2} + 2a_{0,3} + ... + 2a_{0,p-2}\\
&\quad \Leftrightarrow (\Leftrightarrow a_{0,1} + 2a_{0,2} \Leftrightarrow 2a_{0,3} + ... \Leftrightarrow 2a_{0,p-2} + 2a_{0,p-1})\\
&= 3a_{0,1} \Leftrightarrow 4a_{0,2} + 4a_{0,3} \Leftrightarrow ... + 4a0, p \Leftrightarrow 2 \Leftrightarrow 2a_{0,p-1}
\end{aligned}$$

Using the fact that $a_{0,i} = i \Leftrightarrow 1$ we now obtain

$$\begin{aligned}
p \Leftrightarrow 2 &= 3(0) \Leftrightarrow 4(1) + 4(2) \Leftrightarrow ... + 4(p \Leftrightarrow 3) \Leftrightarrow 2(p \Leftrightarrow 2)\\
3(p \Leftrightarrow 2) &= \frac{4(p \Leftrightarrow 3)}{2}\\
3p \Leftrightarrow 6 &= 2p \Leftrightarrow 6
\end{aligned} \qquad (15)$$

$$\Rightarrow p \equiv 0 \pmod{n}$$

$\Rightarrow$ choosing $n = p$ (or any factor of p) will yield a coloration for the knot. Now lets look at the case when p is even.

P even

We saw that when p is odd we get $a_{0,k} = k \Leftrightarrow 1$. If we let p even then we will see that $a_{0,k} \equiv k \Leftrightarrow 1 \pmod 2$. We know that this is true for $i = 1$ and 2, so assume that it is true for k.
If k is even then $a_{0,k} = 1$ and

$$\begin{aligned}
a_{0,k} \Leftrightarrow a_{0,1} &= 1 \\
&= (2a0, p \Leftrightarrow 1 \Leftrightarrow 2a_{0,p-2} + \ldots \Leftrightarrow 3a_{0,k+1} + 2a_{0,k} \Leftrightarrow \ldots \Leftrightarrow 2a_{0,2}) \\
&\quad \Leftrightarrow (2a0, p \Leftrightarrow 1 \Leftrightarrow 2a_{0,p-2} + \ldots \Leftrightarrow 2a_{0,k+1} + 2a_{0,k} \Leftrightarrow \ldots \Leftrightarrow 3a_{0,2}) \\
&= \Leftrightarrow a_{0,k+1} + 1
\end{aligned}$$

$$\Rightarrow a_{0,k+1} = 0$$

similarly, if k is odd then

$$\begin{aligned}
a_{0,k} \Leftrightarrow a_{0,1} &= 0 \qquad (16) \\
&= (2a0, p \Leftrightarrow 1 \Leftrightarrow 2a_{0,p-2} + \ldots + a_{0,k+1} + 2a_{0,k} \Leftrightarrow \ldots \Leftrightarrow 2a_{0,2}) \\
&\quad \Leftrightarrow (2a0, p \Leftrightarrow 1 \Leftrightarrow 2a_{0,p-2} + \ldots + 2a_{0,k+1} + 2a_{0,k} \Leftrightarrow \ldots \Leftrightarrow 3a_{0,2}) \\
&= \Leftrightarrow a_{0,k+1} + 1
\end{aligned}$$

$$\Rightarrow a_{0,k+1} = 1$$

Now pluging this into our equation for $a_{i,p-1}$ we get

$$\begin{aligned}
a_{0,1} &= a_{2,p-1} = 0 \qquad (17) \\
&= \left(\sum_{k=1}^{p-1} [2(\Leftrightarrow 1)^{k-1} \Leftrightarrow \delta_2(k)] a_{0,k}\right) \\
&= 2a_{0,1} \Leftrightarrow 3a_{0,2} + \sum_{k=3}^{p-1} 2(\Leftrightarrow 1)^{k-1} a_{0,k} \\
&= \Leftrightarrow 3 + 2a_{0,3} \Leftrightarrow 2a_{0,4} + 2a_{0,5} \Leftrightarrow \ldots \Leftrightarrow 2a_{0,p-2} + 2a_{0,p-1} \\
&= \Leftrightarrow 3 \Leftrightarrow 2\left(\frac{p \Leftrightarrow 4}{2}\right) \\
&= \Leftrightarrow p + 1 \\
\Rightarrow p \Leftrightarrow 1 &\equiv 0 \pmod n
\end{aligned}$$

$\Rightarrow$ choosing $n = p \Leftrightarrow 1$ (or any factor of p-1) will yield a coloration for the knot.

Corolary 1

$$(p, q) \cong (q, p)$$

Corolary 1 *A (p-1,p) torus knot is always colorable* $\pmod n$ *and* $n = p$ *if* p *odd,* $n = p \Leftrightarrow 1$ *if* p *even.*

Corolary 1 *A (p,p-1) Torus knot with nulity* $\mathcal{N} \pmod n$ *can be constructed by letting* p *be the product of* $\mathcal{N}$ *distinct primes.*

6 n-Coloration as a Property of the Knot Group

The fundamental problem of knot theory [1] is the following: Given knots A and B, determine whether they are equivalent under ambient isotopy. The difficulty lies in the following: As we compare the properties of knot A with the properties knot B, we might in fact be comparing properties which are not inherent in knottedness itself, but rather in certain representations of knottedness. We might discover something interesting in the process, but we would not be engaged in knot theory. The only way to approach this difficulty is to use the very useful idea of an invariant. So, for example, in knot theory we are interested in all properties that remain invariant ambient isotopy, just as in Euclidean geometry we are interested in all properties which remain invariant under Euclidean transformation, etc...

The first and rather weak invariant is the notion of n-colorability. At the opposite end of the scale lies an extremely strong invariant, the fundamental group of the complement of a knot, which we shall from now on refer to as simply the knot group. Since the fundamental group is such a strong invariant, it is natural to ask how it relates to the weakest of invariants, n-colorability. In this section we show that n-coloration is directly related to a simple property of the knot group by showing that a knot is n-colorable if and only if a certain map defines a homomorphism from the knot group to the dihedral group of order 2n. In order to show this equivalence, we first give a brief explanation of the dihedral and knot groups.

First, the dihedral group. Consider a regular n-sided polygon in the Euclidian plane. This figure has two obvious types of symmetry: reflections and rotations. These n reflections and n rotations form a group of order $2n$ denoted by $\mathbb{D}_n$. Proving these transformations form a group is a simple exercise after realizing the following: Euclidean transformations are functions and therefore associative under composition, a rotation of angle 0 acts as an identity, each reflection is it's own inverse and each rotation has an inverse rotation. This group has the following properties: all reflections are of order 2, there exists a minimal rotation of order n which generates all other rotations, and reflections conjugate rotations to their inverses. With this information, we have the following presentation of $\mathbb{D}_n$.

$$< x, y \mid x^n = 1,\ y^2 = 1,\ y^{-1}xy = x^{-1} >$$

Various dihedral groups have as subgroups smaller dihedral groups (Ex: Choosing every-other vertex of an octagon yields a square: $\mathbb{D}_4 < \mathbb{D}_8$.)

To visualize the knot group, we begin with an explanation of the fundamental group of a space[3]. Given a topological space X consider continuous paths in X starting and ending at a certain base point' $x_0 \in X$. Notice that we can equate paths that are virtually the same in the following respect: each can be continuously 'stretched' into the other without leaving the space. We can also compose paths end to end, invert any path, etc... After checking that there is an 'identity' path and that path composition is associative, we obtain a group whose elements are the equivalence classes of paths in the space X under what topologists call homotopy. This group is denoted by $\pi_0(X, x_0)$, or simply $\pi_0(X)$ if appropriate, and is called the fundamental group of a space X with basepoint x_0.

Such groups are important in the study of topological spaces. However, the fundamental group $\pi_0(K)$ of a knot itself is rather uninteresting, as the essence of knottedness lies not in the 1-dimensional properties of the knot, but in the way this 1-dimensional closed curve is embedded in 3-dimensions. Therefore we consider instead the fundamental group of the ambient space minus the image of the knot, or $\pi_0(S^3 \Leftrightarrow K)$. Around 1923 a mathematician by the name of Wilhelm Wirtinger discovered a method of calculating the knot group of a knot from it's (oriented) diagram. The method is as follows[2]. Given an oriented knot diagram D with strands $s_1, s_2, \ldots, s_m \in Q$ let T be the set of triples $(a, b, c) \in Q^3$, each corresponding to a particular crossing, where a is the overstrand, b is the incoming understrand, and c is the outgoing understrand. $\pi_0(S^3 \Leftrightarrow K)$ is now given by the presentation:

$$< s_1, s_2, \ldots, s_m \in Q \mid a^{-1}ba = c \ for\ all\ (a, b, c) \in T \subset Q^3 >$$

Verbally, this could be stated as follows: the overstrand conjugates the incoming understrand to the outgoing understrand, with each strand associated to the path that leaves the base point, loops around the (oriented) strand according to the right hand rule, and returns to the base point. It is a rather interesting visual exercise to show that the 'Wirtinger Relation,' namely $a^{-1}ba = c$, holds.

We now attempt to define a homomorphism from $\pi_0(S^3 \Leftrightarrow K)$ to $\mathbb{D}_n$ by defining it's action on the generators of $\pi_0(S^3 \Leftrightarrow K)$. Using the conjugacy relation in the dihedral presentation as a hint, we try $\phi : s_i \to yx^{k(i)}$ where $k : Q \to \mathbb{Z}$. This leads to the following:

Theorem 4 *Let D be an oriented knot diagram, and let K be the knot it represents. D is n-colorable if and only if there exists a homomorphism ϕ from the knot group $\pi_0(S^3 \Leftrightarrow K)$ to $\mathbb{D}_n$ where ϕ sends each s_i to $yx^{k(i)}$*

Proof 5 *To check the validity of the homomorphism, we determine under which conditions the image of $\pi_0(S^3 \Leftrightarrow K)$ under ϕ satisfies the Wirtinger relation for all $(a, b, c) \in T$, which leads us to the following equivalent equations:*

$$\begin{array}{rcl}
\phi(a^{-1}ba) & = & \phi(c) \\
(yx^{k(a)})^{-1}(yx^{k(b)})(yx^{k(a)}) & = & yx^{k(c)} \\
(x^{-k(a)}y^{-1})(yx^{k(b)})(yx^{k(a)}) & = & yx^{k(c)} \\
(x^{k(b)-k(a)})(yx^{k(a)}) & = & yx^{k(c)} \\
(x^{k(b)-k(a)})(yx^{k(a)}y) & = & (yx^{k(c)}y) \\
(x^{k(b)-2k(a)}) & = & x^{-k(c)} \\
2k(a) \Leftrightarrow k(b) \Leftrightarrow k(c) & = & 0 \bmod n
\end{array}$$

Notice the last equation is, with our notation, the condition for the n-colorability of a knot, with the coloration function $v(q) = k(q) \bmod n$. $\square$

We have now shown that the concept of n-colorability is encapsulated in the fundamental group of a knot. Couldn't analogous results hold for other knot-theoretic concepts? For instance, suppose for some strange reason history took a different course and we had the concepts of the knot polynomial and the knot group but not the concept of n-colorability. Then, while studying the knot group, we ask "With which knots, if any, does there exist a homomorphism to a dihedral group?" After a little experimentation, we obtain the homomorphism defined by ϕ, and derive the concept of colorability from group theoretic properties of the knot group. Likewise, we can now ask ourselves, given a class of groups G, "With which knots, if any, does there exist a homomorphism from the knot group to a group in G?" By determining such conditions, we can perhaps derive new knots-theoretic concepts from group theoretic properties. The first natural question to ask is: Which concepts correspond to the existence of homomorphisms from the knot group to the class of symmetric groups?

7 Summary of work

This paper has discussed the idea of a "coloration" of a knot which shows that a knot is different from the unknot. We have shown that an example of this is "tri-colorability" which extends naturally to "n-colorability."

We have shown the existence of n-colorations of knots through computation by inspection. Also we have used the crossing matrix to compute the number of colors needed to compute the coloration of a given knot. This results in a matrix with a nullity of two modulo some integer which is the number of colors.

There are prime knots and composite knots. Like the integers, a knot is either composite or prime. And like the integers a composite knot is the unique composition of primes.

We have observed that some crossing matrices has a nullity greater than two. Many of these knots are pretzel knots.

Unfortunately n-colorability of knots is not a universal detector of the unknot. There are some knots which have a Jones polynomial which is different from the unknot, which are not colorable. Hence in these situations n-colorability fails as a test for the unknot.

We have explored representations of the fundamental group into the dihedral group.

Finally we have computed the crossing matrices for all of the knots appearing in our text. These were included in the preceding section "Tables of matrices." This demonstrates the wide range of usability of this technique.

8 Conclusions

This paper represents a sample of the work done in eight short weeks understanding the theory of knots. Additional time was spent in an overview of groups and hyperbolic geometry and the application of each to the theory of knots. All were given opportunity to teach and share the things that they had learned.

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[4] K. Reidemeister, *Knotentheorie*. Ergebnisse der Mathematik und ihrer Grenzgebiete, (Alte Folge), Band 1, Heft 1.

A Tables of Matrices

The following crossing matrices are in 1-1 correspondence with the table of knots in the appendix of **??**. Somebody please tell me how to get the two-column stuff to end up on this page too!

3-1

$$\begin{pmatrix} 1 & -2 & 1 \\ 1 & 1 & -2 \\ -2 & 1 & 1 \end{pmatrix}$$

4-1

$$\begin{pmatrix} 1 & 0 & -2 & 1 \\ 1 & 1 & 0 & -2 \\ -2 & 1 & 1 & 0 \\ 0 & -2 & 1 & 1 \end{pmatrix}$$

5-1

$$\begin{pmatrix} 1 & 0 & -2 & 0 & 1 \\ 1 & 1 & 0 & -2 & 0 \\ 0 & 1 & 1 & 0 & -2 \\ -2 & 0 & 1 & 1 & 0 \\ 0 & -2 & 0 & 1 & 1 \end{pmatrix}$$

5-2

$$\begin{pmatrix} 1 & 0 & -2 & 0 & 1 \\ 1 & 1 & 0 & -2 & 0 \\ -2 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & -2 \\ 0 & -2 & 0 & 1 & 1 \end{pmatrix}$$

6-1

$$\begin{pmatrix} 1 & 0 & -2 & 0 & 0 & 1 \\ 1 & 1 & 0 & 0 & -2 & 0 \\ -2 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & -2 \\ 0 & -2 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & -2 & 1 & 1 \end{pmatrix}$$

6-2

$$\begin{pmatrix} 1 & 0 & -2 & 0 & 0 & 1 \\ 1 & 1 & 0 & -2 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & -2 \\ 0 & 0 & 1 & 1 & -2 & 0 \\ -2 & 0 & 0 & 1 & 1 & 0 \\ 0 & -2 & 0 & 0 & 1 & 1 \end{pmatrix}$$

6-3

$$\begin{pmatrix} 1 & 0 & 0 & -2 & 0 & 1 \\ 1 & 1 & -2 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & -2 \\ 0 & 0 & 1 & 1 & -2 & 0 \\ -2 & 0 & 0 & 1 & 1 & 0 \\ 0 & -2 & 0 & 0 & 1 & 1 \end{pmatrix}$$

7-1

$$\begin{pmatrix} 1 & 0 & 0 & -2 & 0 & 0 & 1 \\ 1 & 1 & 0 & 0 & -2 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & -2 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 & -2 \\ -2 & 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & -2 & 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & -2 & 0 & 0 & 1 & 1 \end{pmatrix}$$

7-2

$$\begin{pmatrix} 1 & 0 & -2 & 0 & 0 & 0 & 1 \\ 1 & 1 & 0 & 0 & -2 & 0 & 0 \\ -2 & 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 & -2 \\ 0 & 0 & 0 & 1 & 1 & -2 & 0 \\ 0 & -2 & 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & -2 & 0 & 1 & 1 \end{pmatrix}$$

7-3

$$\begin{pmatrix} 1 & 0 & 0 & -2 & 0 & 0 & 1 \\ 1 & 1 & 0 & 0 & 0 & -2 & 0 \\ 0 & 1 & 1 & 0 & -2 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 & -2 \\ -2 & 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & -2 & 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & -2 & 0 & 0 & 1 & 1 \end{pmatrix}$$

7-4

$$\begin{pmatrix} 1 & 0 & 0 & -2 & 0 & 0 & 1 \\ 1 & 1 & 0 & 0 & -2 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & -2 & 0 \\ -2 & 0 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 & -2 \\ 0 & 0 & -2 & 0 & 1 & 1 & 0 \\ 0 & -2 & 0 & 0 & 0 & 1 & 1 \end{pmatrix}$$

7-5

$$\begin{pmatrix} 1 & 0 & -2 & 0 & 0 & 0 & 1 \\ 1 & 1 & 0 & 0 & -2 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & 0 & -2 \\ 0 & 0 & 1 & 1 & 0 & -2 & 0 \\ -2 & 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & -2 & 1 & 1 & 0 \\ 0 & -2 & 0 & 0 & 0 & 1 & 1 \end{pmatrix}$$

7-6

$$\begin{pmatrix} 1 & -2 & 0 & 0 & 0 & 0 & 1 \\ 1 & 1 & 0 & 0 & -2 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & 0 & -2 \\ 0 & 0 & 1 & 1 & 0 & -2 & 0 \\ 0 & 0 & -2 & 1 & 1 & 0 & 0 \\ -2 & 0 & 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & -2 & 0 & 1 & 1 \end{pmatrix}$$

7-7

$$\begin{pmatrix} 1 & 0 & 0 & -2 & 0 & 0 & 1 \\ 1 & 1 & -2 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & -2 & 0 \\ 0 & 0 & 1 & 1 & -2 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 & -2 \\ -2 & 0 & 0 & 0 & 1 & 1 & 0 \\ 0 & -2 & 0 & 0 & 0 & 1 & 1 \end{pmatrix}$$

8-1

$$\begin{pmatrix} 1 & 0 & -2 & 0 & 0 & 0 & 0 & 1 \\ 1 & 1 & 0 & 0 & 0 & -2 & 0 & 0 \\ -2 & 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 & 0 & -2 \\ 0 & 0 & 0 & 1 & 1 & 0 & -2 & 0 \\ 0 & -2 & 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & -2 & 1 & 1 & 0 \\ 0 & 0 & 0 & -2 & 0 & 0 & 1 & 1 \end{pmatrix}$$

8-2

$$\begin{pmatrix} 1 & 0 & 0 & -2 & 0 & 0 & 0 & 1 \\ 1 & 1 & 0 & 0 & -2 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & 0 & -2 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 & 0 & -2 \\ 0 & 0 & 0 & 1 & 1 & -2 & 0 & 0 \\ -2 & 0 & 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & -2 & 0 & 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & -2 & 0 & 0 & 0 & 1 & 1 \end{pmatrix}$$

8-3

$$\begin{pmatrix} 1 & 0 & 0 & -2 & 0 & 0 & 0 & 1 \\ 1 & 1 & 0 & 0 & 0 & 0 & -2 & 0 \\ 0 & 1 & 1 & 0 & 0 & -2 & 0 & 0 \\ -2 & 0 & 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 & 0 & -2 \\ 0 & 0 & -2 & 0 & 1 & 1 & 0 & 0 \\ 0 & -2 & 0 & 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & -2 & 0 & 1 & 1 \end{pmatrix}$$

8-4

$$\begin{pmatrix} 1 & 0 & 0 & 0 & -2 & 0 & 0 & 1 \\ 1 & 1 & 0 & -2 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & 0 & -2 & 0 \\ 0 & 0 & 1 & 1 & 0 & -2 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 & 0 & -2 \\ -2 & 0 & 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & -2 & 0 & 0 & 1 & 1 & 0 \\ 0 & -2 & 0 & 0 & 0 & 0 & 1 & 1 \end{pmatrix}$$

8-5

$$\begin{pmatrix} 1 & 0 & -2 & 0 & 0 & 0 & 0 & 1 \\ 1 & 1 & 0 & 0 & 0 & -2 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & 0 & 0 & -2 \\ -2 & 0 & 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 & -2 & 0 \\ 0 & -2 & 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & -2 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & -2 & 0 & 1 & 1 \end{pmatrix}$$

8-6

$$\begin{pmatrix} 1 & 0 & -2 & 0 & 0 & 0 & 0 & 1 \\ 1 & 1 & 0 & 0 & -2 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & 0 & 0 & -2 \\ 0 & 0 & 1 & 1 & 0 & 0 & -2 & 0 \\ 0 & 0 & 0 & 1 & 1 & -2 & 0 & 0 \\ -2 & 0 & 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & -2 & 0 & 1 & 1 & 0 \\ 0 & -2 & 0 & 0 & 0 & 0 & 1 & 1 \end{pmatrix}$$

8-7

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & -2 & 0 & 1 \\ 1 & 1 & 0 & 0 & -2 & 0 & 0 & 0 \\ -2 & 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 & -2 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 & 0 & -2 \\ 0 & -2 & 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & -2 & 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & -2 & 0 & 0 & 1 & 1 \end{pmatrix}$$

8-8

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & -2 & 1 \\ 1 & 1 & 0 & 0 & 0 & -2 & 0 & 0 \\ 0 & 1 & 1 & 0 & -2 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 & 0 & -2 \\ 0 & -2 & 0 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -2 & 1 & 1 & 0 & 0 \\ -2 & 0 & 0 & 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & -2 & 0 & 0 & 0 & 1 & 1 \end{pmatrix}$$

8-9

$$\begin{pmatrix} 1 & 0 & 0 & -2 & 0 & 0 & 0 & 1 \\ 1 & 1 & 0 & 0 & 0 & 0 & -2 & 0 \\ 0 & 1 & 1 & 0 & 0 & -2 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 & 0 & -2 \\ -2 & 0 & 0 & 1 & 1 & 0 & 0 & 0 \\ 0 & -2 & 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & -2 & 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & -2 & 0 & 1 & 1 \end{pmatrix}$$

8-10

$$\begin{pmatrix} 1 & 0 & -2 & 0 & 0 & 0 & 0 & 1 \\ 1 & 1 & 0 & 0 & -2 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & 0 & 0 & -2 \\ -2 & 0 & 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 & -2 & 0 \\ 0 & -2 & 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & -2 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & -2 & 1 & 1 \end{pmatrix}$$

8-11

$$\begin{pmatrix} 1 & 0 & 0 & -2 & 0 & 0 & 0 & 1 \\ 1 & 1 & 0 & 0 & 0 & -2 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & 0 & -2 & 0 \\ 0 & 0 & 1 & 1 & -2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 & 0 & -2 \\ 0 & -2 & 0 & 0 & 1 & 1 & 0 & 0 \\ -2 & 0 & 0 & 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & -2 & 0 & 0 & 0 & 1 & 1 \end{pmatrix}$$

8-12

$$\begin{pmatrix} 1 & 0 & 0 & -2 & 0 & 0 & 0 & 1 \\ 1 & 1 & 0 & 0 & 0 & 0 & -2 & 0 \\ 0 & 1 & 1 & 0 & -2 & 0 & 0 & 0 \\ -2 & 0 & 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & -2 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & 0 & -2 \\ 0 & -2 & 0 & 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & -2 & 1 & 1 \end{pmatrix}$$

8-13

$$\begin{pmatrix} 1 & 0 & -2 & 0 & 0 & 0 & 0 & 1 \\ 1 & 1 & 0 & 0 & 0 & -2 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & 0 & -2 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 & 0 & -2 \\ 0 & -2 & 0 & 1 & 1 & 0 & 0 & 0 \\ -2 & 0 & 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & -2 & 1 & 1 & 0 \\ 0 & 0 & 0 & -2 & 0 & 0 & 1 & 1 \end{pmatrix}$$

8-14

$$\begin{pmatrix} 1 & 0 & 0 & -2 & 0 & 0 & 0 & 1 \\ 1 & 1 & 0 & 0 & 0 & -2 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & 0 & 0 & -2 \\ -2 & 0 & 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & -2 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & -2 & 0 \\ 0 & -2 & 0 & 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & -2 & 0 & 1 & 1 \end{pmatrix}$$

8-15

$$\begin{pmatrix} 1 & 0 & -2 & 0 & 0 & 0 & 0 & 1 \\ 1 & 1 & 0 & 0 & 0 & -2 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & 0 & 0 & -2 \\ 0 & -2 & 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 & -2 & 0 \\ 0 & 0 & 0 & -2 & 1 & 1 & 0 & 0 \\ -2 & 0 & 0 & 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & -2 & 0 & 1 & 1 \end{pmatrix}$$

8-16

$$\begin{pmatrix} 1 & 0 & -2 & 0 & 0 & 0 & 0 & 1 \\ 1 & 1 & 0 & -2 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & 0 & 0 & -2 \\ 0 & 0 & 1 & 1 & 0 & -2 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 & -2 & 0 \\ 0 & -2 & 0 & 0 & 1 & 1 & 0 & 0 \\ -2 & 0 & 0 & 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & -2 & 0 & 1 & 1 \end{pmatrix}$$

8-17

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & -2 & 0 & 1 \\ 1 & 1 & 0 & 0 & 0 & 0 & -2 & 0 \\ 0 & 1 & 1 & 0 & -2 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 & 0 & -2 \\ -2 & 0 & 0 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & -2 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & -2 & 0 & 1 & 1 & 0 \\ 0 & -2 & 0 & 0 & 0 & 0 & 1 & 1 \end{pmatrix}$$

8-18

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & -2 & 0 & 1 \\ 1 & 1 & 0 & 0 & 0 & 0 & -2 & 0 \\ 0 & 1 & 1 & 0 & 0 & 0 & 0 & -2 \\ -2 & 0 & 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & -2 & 0 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & -2 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & -2 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & -2 & 0 & 1 & 1 \end{pmatrix}$$

8-19

$$\begin{pmatrix} 1 & 0 & -2 & 0 & 0 & 0 & 0 & 1 \\ 1 & 1 & 0 & 0 & -2 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & 0 & 0 & -2 \\ 0 & 0 & 1 & 1 & 0 & -2 & 0 & 0 \\ -2 & 0 & 0 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & -2 & 0 & 1 & 1 & 0 & 0 \\ -2 & 0 & 0 & 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & -2 & 0 & 1 & 1 \end{pmatrix}$$

8-20

$$\begin{pmatrix} 1 & 0 & -2 & 0 & 0 & 0 & 0 & 1 \\ 1 & 1 & 0 & 0 & 0 & 0 & -2 & 0 \\ 0 & 1 & 1 & 0 & -2 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 & -2 & 0 \\ -2 & 0 & 0 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & -2 & 0 & 1 & 1 & 0 & 0 \\ -2 & 0 & 0 & 0 & 0 & 1 & 1 & 0 \\ 0 & -2 & 0 & 0 & 0 & 0 & 1 & 1 \end{pmatrix}$$

8-21

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & -2 & 1 \\ 1 & 1 & 0 & 0 & 0 & -2 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & 0 & 0 & -2 \\ 0 & 0 & 1 & 1 & 0 & -2 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 & 0 & -2 \\ 0 & 0 & 0 & 0 & 1 & 1 & -2 & 0 \\ 0 & 0 & 0 & -2 & 0 & 1 & 1 & 0 \\ 0 & -2 & 0 & 0 & 0 & 0 & 1 & 1 \end{pmatrix}$$

9-1

$$\begin{pmatrix} 1 & 0 & 0 & 0 & -2 & 0 & 0 & 0 & 1 \\ 1 & 1 & 0 & 0 & 0 & -2 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & 0 & -2 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 & 0 & -2 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & -2 \\ -2 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 \\ 0 & -2 & 0 & 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & -2 & 0 & 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & -2 & 0 & 0 & 0 & 1 & 1 \end{pmatrix}$$

9-2

$$\begin{pmatrix} 1 & 0 & -2 & 0 & 0 & 0 & 0 & 0 & 1 \\ 1 & 1 & 0 & 0 & 0 & -2 & 0 & 0 & 0 \\ -2 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & -2 \\ 0 & 0 & 0 & 1 & 1 & 0 & 0 & -2 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & -2 & 0 & 0 \\ 0 & -2 & 0 & 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & -2 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & -2 & 0 & 0 & 0 & 1 & 1 \end{pmatrix}$$

9-3

$$\begin{pmatrix} 1 & 0 & 0 & 0 & -2 & 0 & 0 & 0 & 1 \\ 1 & 1 & 0 & 0 & 0 & 0 & -2 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & -2 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 & 0 & -2 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & -2 \\ -2 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 \\ 0 & -2 & 0 & 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & -2 & 0 & 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & -2 & 0 & 0 & 0 & 1 & 1 \end{pmatrix}$$

9-4

$$\begin{pmatrix} 1 & 0 & 0 & -2 & 0 & 0 & 0 & 0 & 1 \\ 1 & 1 & 0 & 0 & 0 & -2 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & 0 & -2 & 0 & 0 \\ -2 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & -2 \\ 0 & 0 & 0 & 0 & 1 & 1 & 0 & -2 & 0 \\ 0 & -2 & 0 & 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & -2 & 0 & 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & -2 & 0 & 0 & 1 & 1 \end{pmatrix}$$

9-5

$$\begin{pmatrix} 1 & 0 & 0 & 0 & -2 & 0 & 0 & 0 & 1 \\ 1 & 1 & 0 & 0 & 0 & -2 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & 0 & 0 & -2 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 & -2 & 0 & 0 \\ -2 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & -2 \\ 0 & 0 & 0 & -2 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & -2 & 0 & 0 & 0 & 1 & 1 & 0 \\ 0 & -2 & 0 & 0 & 0 & 0 & 0 & 1 & 1 \end{pmatrix}$$

9-6

$$\begin{pmatrix} 1 & 0 & 0 & -2 & 0 & 0 & 0 & 0 & 1 \\ 1 & 1 & 0 & 0 & 0 & -2 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & 0 & 0 & -2 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & -2 \\ 0 & 0 & 0 & 1 & 1 & 0 & -2 & 0 & 0 \\ -2 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -2 & 1 & 1 & 0 & 0 \\ 0 & -2 & 0 & 0 & 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & -2 & 0 & 0 & 0 & 0 & 1 & 1 \end{pmatrix}$$

9-7

$$\begin{pmatrix} 1 & 0 & -2 & 0 & 0 & 0 & 0 & 0 & 1 \\ 1 & 1 & 0 & 0 & 0 & -2 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & -2 \\ 0 & 0 & 1 & 1 & 0 & 0 & 0 & -2 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 & -2 & 0 & 0 \\ -2 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -2 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & -2 & 0 & 0 & 1 & 1 & 0 \\ 0 & -2 & 0 & 0 & 0 & 0 & 0 & 1 & 1 \end{pmatrix}$$

9-8

$$\begin{pmatrix} 1 & -2 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 1 & 1 & 0 & 0 & 0 & -2 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & -2 \\ 0 & 0 & 1 & 1 & 0 & 0 & 0 & -2 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 & -2 & 0 & 0 \\ 0 & 0 & 0 & -2 & 1 & 1 & 0 & 0 & 0 \\ -2 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & -2 & 0 & 1 & 1 & 0 \\ 0 & 0 & -2 & 0 & 0 & 0 & 0 & 1 & 1 \end{pmatrix}$$

9-9

$$\begin{pmatrix} 1 & 0 & 0 & -2 & 0 & 0 & 0 & 0 & 1 \\ 1 & 1 & 0 & 0 & -2 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & 0 & -2 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 & 0 & -2 & 0 \\ -2 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & -2 \\ 0 & -2 & 0 & 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & -2 & 0 & 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & -2 & 0 & 1 & 1 \end{pmatrix}$$

9-10

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & -2 & 0 & 0 & 1 \\ 1 & 1 & 0 & 0 & -2 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & 0 & -2 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 & 0 & -2 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & -2 \\ -2 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & -2 & 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & -2 & 0 & 0 & 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & -2 & 0 & 0 & 0 & 1 & 1 \end{pmatrix}$$

9-11

$$\begin{pmatrix} 1 & 0 & -2 & 0 & 0 & 0 & 0 & 0 & 1 \\ 1 & 1 & 0 & 0 & 0 & 0 & -2 & 0 & 0 \\ -2 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & -2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 & 0 & -2 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & -2 \\ 0 & -2 & 0 & 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & -2 & 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & -2 & 0 & 0 & 1 & 1 \end{pmatrix}$$

9-12

$$\begin{pmatrix} 1 & 0 & -2 & 0 & 0 & 0 & 0 & 0 & 1 \\ 1 & 1 & 0 & 0 & 0 & 0 & -2 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & -2 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & -2 \\ 0 & 0 & 0 & 1 & 1 & 0 & 0 & -2 & 0 \\ 0 & 0 & 0 & -2 & 1 & 1 & 0 & 0 & 0 \\ 0 & -2 & 0 & 0 & 0 & 1 & 1 & 0 & 0 \\ -2 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & -2 & 0 & 0 & 1 & 1 \end{pmatrix}$$

9-13

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & -2 & 0 & 0 & 1 \\ 1 & 1 & 0 & 0 & -2 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & 0 & -2 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 & 0 & -2 & 0 \\ -2 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & -2 & 0 & 0 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & -2 \\ 0 & 0 & 0 & -2 & 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & -2 & 0 & 0 & 0 & 0 & 1 & 1 \end{pmatrix}$$

9-14

$$\begin{pmatrix} 1 & 0 & 0 & -2 & 0 & 0 & 0 & 0 & 1 \\ 1 & 1 & 0 & 0 & -2 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & 0 & 0 & -2 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 & -2 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & -2 & 0 & 0 & 0 \\ -2 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & -2 \\ 0 & 0 & -2 & 0 & 0 & 0 & 1 & 1 & 0 \\ 0 & -2 & 0 & 0 & 0 & 0 & 0 & 1 & 1 \end{pmatrix}$$

9-15

$$\begin{pmatrix} 1 & 0 & -2 & 0 & 0 & 0 & 0 & 0 & 1 \\ 1 & 1 & 0 & 0 & 0 & -2 & 0 & 0 & 0 \\ 0 & 1 & 1 & -2 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & -2 \\ 0 & 0 & 0 & 1 & 1 & 0 & 0 & -2 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & -2 & 0 & 0 \\ 0 & -2 & 0 & 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & -2 & 0 & 1 & 1 & 0 \\ -2 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 \end{pmatrix}$$

9-16

$$\begin{pmatrix} 1 & 0 & 0 & 0 & -2 & 0 & 0 & 0 & 1 \\ 1 & 1 & 0 & -2 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & 0 & -2 & 0 & 0 \\ -2 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & -2 & 0 & 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & 0 & -2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & -2 \\ 0 & 0 & -2 & 0 & 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & -2 & 0 & 1 & 1 \end{pmatrix}$$

9-17

$$\begin{pmatrix} 1 & 0 & 0 & 0 & -2 & 0 & 0 & 0 & 1 \\ 1 & 1 & 0 & 0 & 0 & 0 & -2 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & 0 & 0 & -2 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & -2 \\ -2 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -2 & 1 & 1 & 0 & 0 & 0 \\ 0 & -2 & 0 & 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -2 & 1 & 1 & 0 \\ 0 & 0 & -2 & 0 & 0 & 0 & 0 & 1 & 1 \end{pmatrix}$$

9-18

$$\begin{pmatrix} 1 & 0 & -2 & 0 & 0 & 0 & 0 & 0 & 1 \\ 1 & 1 & 0 & 0 & -2 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & 0 & -2 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & -2 \\ -2 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & 0 & -2 & 0 \\ 0 & -2 & 0 & 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -2 & 1 & 1 & 0 \\ 0 & 0 & 0 & -2 & 0 & 0 & 0 & 1 & 1 \end{pmatrix}$$

9-19

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & -2 & 0 & 1 \\ 1 & 1 & 0 & 0 & 0 & 0 & 0 & -2 & 0 \\ -2 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & -2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & -2 \\ 0 & 0 & 0 & -2 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & -2 & 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & -2 & 0 & 0 & 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & -2 & 0 & 0 & 1 & 1 \end{pmatrix}$$

9-20

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & -2 & 0 & 0 & 1 \\ 1 & 1 & 0 & 0 & -2 & 0 & 0 & 0 & 0 \\ -2 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 & -2 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 & 0 & -2 & 0 \\ 0 & -2 & 0 & 0 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & -2 \\ 0 & 0 & 0 & -2 & 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & -2 & 0 & 0 & 0 & 0 & 1 & 1 \end{pmatrix}$$

9-21

$$\begin{pmatrix} 1 & 0 & -2 & 0 & 0 & 0 & 0 & 0 & 1 \\ 1 & 1 & 0 & 0 & 0 & 0 & 0 & -2 & 0 \\ 0 & 1 & 1 & 0 & 0 & 0 & -2 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & -2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & -2 \\ 0 & 0 & 0 & -2 & 1 & 1 & 0 & 0 & 0 \\ -2 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & -2 & 0 & 0 & 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & -2 & 0 & 0 & 1 & 1 \end{pmatrix}$$

9-22

$$\begin{pmatrix} 1 & 0 & 0 & 0 & -2 & 0 & 0 & 0 & 1 \\ 1 & 1 & 0 & 0 & 0 & 0 & 0 & -2 & 0 \\ 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & -2 \\ 0 & 0 & 1 & 1 & 0 & -2 & 0 & 0 & 0 \\ -2 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & -2 & 0 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -2 & 0 & 1 & 1 & 0 & 0 \\ 0 & -2 & 0 & 0 & 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -2 & 1 & 1 \end{pmatrix}$$

9-23

$$\begin{pmatrix} 1 & 0 & 0 & -2 & 0 & 0 & 0 & 0 & 1 \\ 1 & 1 & 0 & 0 & 0 & 0 & 0 & -2 & 0 \\ 0 & 1 & 1 & 0 & 0 & -2 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & -2 \\ 0 & -2 & 0 & 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & -2 & 0 & 0 \\ 0 & 0 & -2 & 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & -2 & 0 & 1 & 1 & 0 \\ -2 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 \end{pmatrix}$$

9-24

$$\begin{pmatrix} 1 & 0 & 0 & -2 & 0 & 0 & 0 & 0 & 1 \\ 1 & 1 & 0 & 0 & 0 & 0 & 0 & -2 & 0 \\ 0 & 1 & 1 & 0 & 0 & -2 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & -2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & -2 \\ 0 & 0 & 0 & 0 & 1 & 1 & -2 & 0 & 0 \\ 0 & 0 & -2 & 0 & 0 & 1 & 1 & 0 & 0 \\ -2 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 \\ 0 & -2 & 0 & 0 & 0 & 0 & 0 & 1 & 1 \end{pmatrix}$$

9-25

$$\begin{pmatrix} 1 & 0 & -2 & 0 & 0 & 0 & 0 & 0 & 1 \\ 1 & 1 & 0 & 0 & 0 & 0 & -2 & 0 & 0 \\ 0 & 1 & 1 & 0 & -2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & -2 \\ 0 & -2 & 0 & 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -2 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 1 & -2 & 0 \\ -2 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & -2 & 0 & 1 & 1 \end{pmatrix}$$

9-26

$$\begin{pmatrix} 1 & 0 & 0 & -2 & 0 & 0 & 0 & 0 & 1 \\ 1 & 1 & -2 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & -2 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 & -2 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 & 0 & -2 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & -2 \\ 0 & -2 & 0 & 0 & 0 & 1 & 1 & 0 & 0 \\ -2 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & -2 & 0 & 0 & 1 & 1 \end{pmatrix}$$

9-27

$$\begin{pmatrix} 1 & -2 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 1 & 1 & 0 & 0 & 0 & -2 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & 0 & 0 & -2 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 & -2 & 0 & 0 \\ 0 & 0 & -2 & 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & -2 \\ -2 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & -2 & 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & -2 & 0 & 0 & 1 & 1 \end{pmatrix}$$

9-28

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & -2 & 0 & 1 \\ 1 & 1 & -2 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & -2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 & 0 & -2 & 0 \\ 0 & -2 & 0 & 1 & 1 & 0 & 0 & 0 & 0 \\ -2 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & -2 \\ 0 & 0 & 0 & 0 & 0 & -2 & 1 & 1 & 0 \\ 0 & 0 & 0 & -2 & 0 & 0 & 0 & 1 & 1 \end{pmatrix}$$

9-29

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & -2 & 0 & 0 & 1 \\ 1 & 1 & 0 & -2 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & 0 & -2 & 0 & 0 \\ -2 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 & 0 & -2 & 0 \\ 0 & 0 & -2 & 0 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & -2 \\ 0 & 0 & 0 & 0 & -2 & 0 & 1 & 1 & 0 \\ 0 & -2 & 0 & 0 & 0 & 0 & 0 & 1 & 1 \end{pmatrix}$$

9-30

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & -2 & 0 & 0 & 1 \\ 1 & 1 & 0 & 0 & 0 & 0 & 0 & -2 & 0 \\ -2 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & -2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 & -2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & -2 \\ 0 & 0 & 0 & -2 & 0 & 1 & 1 & 0 & 0 \\ 0 & -2 & 0 & 0 & 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & -2 & 0 & 0 & 0 & 0 & 1 & 1 \end{pmatrix}$$

9-31

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & -2 & 0 & 0 & 1 \\ 1 & 1 & 0 & 0 & 0 & 0 & -2 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & 0 & 0 & -2 & 0 \\ 0 & -2 & 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ -2 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & -2 \\ 0 & 0 & 0 & 0 & -2 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & -2 & 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & -2 & 0 & 0 & 0 & 0 & 1 & 1 \end{pmatrix}$$

9-32

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & -2 & 0 & 0 & 1 \\ 1 & 1 & 0 & -2 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & -2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 & -2 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 & 0 & -2 & 0 \\ 0 & 0 & -2 & 0 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & -2 \\ 0 & -2 & 0 & 0 & 0 & 0 & 1 & 1 & 0 \\ -2 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 \end{pmatrix}$$

9-33

$$\begin{pmatrix} 1 & 0 & 0 & -2 & 0 & 0 & 0 & 0 & 1 \\ 1 & 1 & 0 & 0 & 0 & 0 & 0 & -2 & 0 \\ -2 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & -2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 & -2 & 0 & 0 \\ 0 & 0 & -2 & 0 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & -2 \\ 0 & -2 & 0 & 0 & 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & -2 & 0 & 0 & 1 & 1 \end{pmatrix}$$

9-34

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & -2 & 0 & 0 & 1 \\ 1 & 1 & 0 & 0 & 0 & 0 & 0 & -2 & 0 \\ 0 & 1 & 1 & 0 & 0 & 0 & -2 & 0 & 0 \\ -2 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & -2 & 0 & 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & -2 & 0 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & -2 \\ 0 & 0 & 0 & 0 & -2 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & -2 & 0 & 0 & 0 & 1 & 1 \end{pmatrix}$$

9-35

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & -2 & 0 & 0 & 1 \\ 1 & 1 & 0 & 0 & -2 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & 0 & -2 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & -2 \\ 0 & 0 & 0 & 1 & 1 & 0 & 0 & -2 & 0 \\ -2 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & -2 & 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & -2 & 0 & 0 & 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & -2 & 0 & 0 & 0 & 1 & 1 \end{pmatrix}$$

9-36

$$\begin{pmatrix} 1 & 0 & -2 & 0 & 0 & 0 & 0 & 0 & 1 \\ 1 & 1 & 0 & 0 & -2 & 0 & 0 & 0 & 0 \\ -2 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 & -2 & 0 & 0 \\ 0 & -2 & 0 & 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & 0 & -2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & -2 \\ 0 & 0 & 0 & -2 & 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & -2 & 0 & 1 & 1 \end{pmatrix}$$

9-37

$$\begin{pmatrix} 1 & 0 & 0 & 0 & -2 & 0 & 0 & 0 & 1 \\ 1 & 1 & 0 & 0 & 0 & 0 & -2 & 0 & 0 \\ 0 & 1 & 1 & -2 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 & 0 & -2 & 0 \\ 0 & 0 & 0 & 1 & 1 & -2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & -2 \\ 0 & -2 & 0 & 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & -2 & 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & -2 & 0 & 0 & 0 & 0 & 1 & 1 \end{pmatrix}$$

9-38

$$\begin{pmatrix} 1 & 0 & -2 & 0 & 0 & 0 & 0 & 0 & 1 \\ 1 & 1 & 0 & 0 & 0 & -2 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & -2 \\ 0 & 0 & 1 & 1 & 0 & 0 & -2 & 0 & 0 \\ -2 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & 0 & -2 & 0 \\ 0 & 0 & 0 & -2 & 0 & 1 & 1 & 0 & 0 \\ 0 & -2 & 0 & 0 & 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & -2 & 0 & 0 & 1 & 1 \end{pmatrix}$$

9-39

$$\begin{pmatrix} 1 & 0 & 0 & 0 & -2 & 0 & 0 & 0 & 1 \\ 1 & 1 & 0 & 0 & 0 & 0 & -2 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & -2 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 & 0 & -2 & 0 \\ 0 & -2 & 0 & 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & -2 \\ 0 & 0 & 0 & -2 & 0 & 1 & 1 & 0 & 0 \\ -2 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & -2 & 0 & 0 & 0 & 0 & 1 & 1 \end{pmatrix}$$

9-40

$$\begin{pmatrix} 1 & 0 & 0 & 0 & -2 & 0 & 0 & 0 & 1 \\ 1 & 1 & 0 & -2 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & -2 \\ 0 & 0 & 1 & 1 & 0 & 0 & 0 & -2 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 & -2 & 0 & 0 \\ 0 & 0 & -2 & 0 & 1 & 1 & 0 & 0 & 0 \\ 0 & -2 & 0 & 0 & 0 & 1 & 1 & 0 & 0 \\ -2 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & -2 & 0 & 1 & 1 \end{pmatrix}$$

9-41

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & -2 & 0 & 0 & 1 \\ 1 & 1 & 0 & 0 & 0 & 0 & 0 & -2 & 0 \\ 0 & 1 & 1 & 0 & 0 & 0 & -2 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & -2 \\ 0 & -2 & 0 & 1 & 1 & 0 & 0 & 0 & 0 \\ -2 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & -2 & 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & -2 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & -2 & 0 & 0 & 0 & 1 & 1 \end{pmatrix}$$

9-42

$$\begin{pmatrix} 1 & 0 & 0 & 0 & -2 & 0 & 0 & 0 & 1 \\ 1 & 1 & 0 & 0 & 0 & 0 & 0 & -2 & 0 \\ 0 & 1 & 1 & 0 & -2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 & -2 & 0 & 0 \\ 0 & 0 & -2 & 1 & 1 & 0 & 0 & 0 & 0 \\ -2 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -2 & 0 & 1 & 1 & 0 & 0 \\ -2 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 \\ 0 & -2 & 0 & 0 & 0 & 0 & 0 & 1 & 1 \end{pmatrix}$$

9-43

$$\begin{pmatrix} 1 & 0 & 0 & -2 & 0 & 0 & 0 & 0 & 1 \\ 1 & 1 & 0 & 0 & 0 & 0 & -2 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & -2 \\ 0 & -2 & 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 & -2 & 0 & 0 \\ 0 & 0 & 0 & -2 & 1 & 1 & 0 & 0 & 0 \\ -2 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & -2 & 0 & 1 & 1 & 0 \\ 0 & 0 & -2 & 0 & 0 & 0 & 0 & 1 & 1 \end{pmatrix}$$

9-44

$$\begin{pmatrix} 1 & 0 & -2 & 0 & 0 & 0 & 0 & 0 & 1 \\ 1 & 1 & 0 & 0 & 0 & 0 & 0 & -2 & 0 \\ 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & -2 \\ 0 & -2 & 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & -2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & 0 & -2 & 0 \\ 0 & -2 & 0 & 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & -2 & 0 & 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & -2 & 0 & 0 & 1 & 1 \end{pmatrix}$$

9-45

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & -2 & 0 & 1 \\ 1 & 1 & 0 & 0 & -2 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & -2 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 & -2 & 0 & 0 \\ 0 & -2 & 0 & 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -2 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & -2 \\ 0 & 0 & -2 & 0 & 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & -2 & 0 & 1 & 1 \end{pmatrix}$$

9-46

$$\begin{pmatrix} 1 & 0 & 0 & 0 & -2 & 0 & 0 & 0 & 1 \\ 1 & 1 & 0 & -2 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & -2 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 & 0 & -2 & 0 \\ -2 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & -2 & 0 & 1 & 1 & 0 & 0 & 0 \\ 0 & -2 & 0 & 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & -2 & 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & -2 & 0 & 0 & 0 & 0 & 1 & 1 \end{pmatrix}$$

9-47

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & -2 & 0 & 1 \\ 1 & 1 & 0 & 0 & 0 & 0 & 0 & -2 & 0 \\ 0 & 1 & 1 & 0 & 0 & 0 & -2 & 0 & 0 \\ -2 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & -2 & 0 & 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & -2 & 0 & 1 & 1 & 0 & 0 & 0 \\ -2 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & -2 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & -2 & 0 & 0 & 0 & 1 & 1 \end{pmatrix}$$

9-48

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & -2 & 0 & 0 & 1 \\ 1 & 1 & 0 & 0 & 0 & 0 & 0 & -2 & 0 \\ 0 & 1 & 1 & 0 & 0 & -2 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 & -2 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 & 0 & -2 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & -2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & -2 \\ 0 & -2 & 0 & 0 & 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & -2 & 0 & 0 & 0 & 1 & 1 \end{pmatrix}$$

9-49

$$\begin{pmatrix} 1 & 0 & 0 & 0 & -2 & 0 & 0 & 0 & 1 \\ 1 & 1 & 0 & 0 & 0 & 0 & 0 & -2 & 0 \\ 0 & 1 & 1 & 0 & 0 & -2 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 & 0 & -2 & 0 \\ 0 & -2 & 0 & 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & -2 \\ 0 & 0 & 0 & -2 & 0 & 1 & 1 & 0 & 0 \\ 0 & -2 & 0 & 0 & 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & -2 & 0 & 0 & 0 & 0 & 1 & 1 \end{pmatrix}$$