**What is Algebraic Geometry?**

The University of Utah well known for its specialization in the field of Algebraic Geometry. It has attracted many professors and graduate students to come research or attend one of the numerous conferences on the subject. For these reasons it is important for first year graduate students to learn some of the basic principles of Algebraic Geometry. On January 19, 26 and February 2, 2007 professors Aaron Bertram, Christopher Hacon, and Tommaso de Fernex presented introductory concepts and general insights into their research specialties for the pilot course of Early Research Directions. Transcribing these lectures were (in order), Erika Meucci, Peter Marcy, and Liang Zhang.

**Aaron Bertram**

Algebraic Geometry is the study of the solutions to systems of polynomial equations.

As for the algebraic aspects, there is a natural question which comes up: Which fields are to be used for coefficients and for the solutions? For example, we can consider fields like $\mathbb{Q}$, $\mathbb{F}_p$, $\mathbb{R}$, or $\mathbb{Q}(t)$ which are not algebraically closed; or we could take $\overline{\mathbb{Q}}$, $\overline{\mathbb{F}_p}$ or $\mathbb{C}$, which are, of course, algebraically closed. We can choose one field for the coefficients, say $\mathbb{Q}$, and another field for the solutions, say $\mathbb{C}$. In particular, $\mathbb{R}$ and $\mathbb{C}$ are called **geometric fields** because we are able to visualize them.

Let $k$ be the chosen field. $\mathbb{A}^n(k) = k^n$ is called the **affine space** over $k$. Now we consider the polynomials $f_1, \dots, f_r \in k[x_1, \dots, x_n]$. We want to study the set

$$V(f_1, \dots, f_r) = (f_1 = \cdots = f_r = 0) \subset \mathbb{A}^n(k).$$

Another space that is very important is the **projective space**. The projective space is a compactification of $\mathbb{A}^n(k)$. We can think, for example, $k = \mathbb{R}$ or $\mathbb{C}$. How does it work? We can think $\mathbb{P}^n(k)$ as the set of the lines through the origin in $\mathbb{A}^{n+1}(k)$. For example, $\mathbb{P}^1(\mathbb{R})$ is the set of the lines through the origin in $\mathbb{R}^2$. If we intersect these lines with the line $y = 1$, we have that the intersection is a point for each line except for the line $y = 0$. But if you consider the intersection between these lines and $x = 1$, we have that the intersection is a point for each line except for the line $x = 0$. Hence, if $U_{y=1}$ is the set of the lines through the origin which intersect $y = 1$ and $U_{x=1}$ is the set of the lines through the origin which intersect $x = 1$, then

$$S^1 = \mathbb{P}^1(\mathbb{R}) = \mathbb{A}^1(\mathbb{R}) \cup \{\infty\} = U_{x=1} \cup U_{y=1}.$$

This works in general, i.e.

$$\mathbb{P}^2(k) = \mathbb{A}^2(k) \cup \mathbb{P}^1(k) = U_{x=1} \cup U_{y=1} \cup U_{z=1},$$

where we think $\mathbb{P}^1(k)$ as the set of the points at infinity, that is the lines in the $xy$-plane.

The basic fact is that if we have some polynomial equations $(f_1 = \cdots = f_r = 0) \subset \mathbb{A}^n(k)$, then there exist homogenous polynomials $F_1, \ldots, F_s$ such that $(F_1 = \cdots = F_s = 0) \subset \mathbb{P}^n(k)$ is equal to the closure of $(f_1 = \cdots = f_r = 0)$. For example, the homogeneous polynomial associated to $y^2 = x + a$ is $y^2 = xz + az^2$.

Since $(F_1 = \cdots = F_s = 0)$ is a closed subset of the compact set $\mathbb{P}^n(k)$, $(F_1 = \cdots = F_s = 0)$ is compact, and this is a very important fact in Algebraic Geometry.

Important concepts come from the **conic sections**. We consider the cone $F = (x^2 + y^2 = z^2)$ and we are interested in the intersection between this cone and a plane. If we consider the plane $U_{z=1}$, we have

$$(F = 0) \cap U_{z=1} = (x^2 + y^2 = 1)$$

which is $S^1$. If we consider the plane $U_{y=1}$, we have

$$(F = 0) \cap U_{y=1} = (x^2 - z^2 = 1)$$

which is a hyperbola.

Given $f_1, \ldots, f_r \in k[x_1, \ldots, x_n]$, we can form the ideal generated by these polynomials:

$$\langle f_1, \ldots, f_r \rangle := \left\{ \sum_{i=1}^{r} g_i f_i \mid g_i \in k[x_1, \ldots, x_n] \right\}.$$

The following Theorems are among the most important in Algebraic Geometry.

**Theorem 1 (Hilbert Basis Theorem)** *Any ideal in $k[x_1, \ldots x_n]$ is of the form $< f_1, \ldots, f_r >$.*

**Theorem 2 (Hilbert Nullstellensatz)** *If $k$ is algebraically closed, then $(f_1 = \cdots = f_r = 0) = (h_1 = \cdots = h_s = 0)$ if and only if $\sqrt{(f_1 = \cdots = f_r = 0)} = \sqrt{(h_1 = \cdots = h_s = 0)}$.*

In the above theorem $\sqrt{(f_1 = \ldots = f_r = 0)}$ is the **radical ideal** of $(f_1 = \cdots = f_r = 0)$, and is defined to be $\{g \in k[x_1, \ldots, x_n] \mid \exists N : \ g^N \in \langle f_1, \ldots, f_r \rangle\}$ .

The building blocks of this theory are the prime ideals. An ideal $P \subset k[x_1, \ldots x_n]$ is **prime** if the quotient $k[x_1, \ldots x_n]/P$ is a domain. Another important notion is that of dimension. The **dimension**, denoted by $\dim(P = 0)$, is the transcendence degree of the field of fractions of $k[x_1, \ldots, x_n]/P$. We call $X := (P = 0)$ and $k(x)$ the field of fractions of $k[x_1, \ldots, x_n]/P$.

A very difficult problem in Algebraic Geometry is the following: given $P$, determine if $k(x) = k(t_1, \ldots, t_{\dim(P=0)})$. There are some results about this problem, but we don't have a general answer. If there are more than three equations or more variables than one, the problem is still open.

And now we move on to Geometry. We consider a homogeneous prime ideal $Q$. What are the (global) invariants of $X = (Q = 0) \subset \mathbb{P}^n(k)$?

What is the meaning of a function being smooth at one point? A function is smooth at one point $x_0$ if it is infinitely many differentiable in $x_0$. For example, $y^2 = x^2(x+1)$ and $y^2 = x^3$ are not smooth at the origin.

We will try to understand what happens in general. We define $(P = 0) := (Q = 0) \cup U_{x=1} \subset \mathbb{A}^n(k)$ and we think $(P = 0)$ as the same to see $(Q = 0)$ locally. One of the most powerful Theorems in Algebraic Geometry (and in many other branches of mathematics) is the Implicit Function Theorem:

**Theorem 3 (Implicit Function Theorem)** *If $k = \mathbb{R}$ or $\mathbb{C}$ and there are $f_1, \ldots, f_{n-d} \in P$ such that the matrix $\left(\frac{\partial f_i}{\partial x_j}\right)$ has rank $n-d$ at $p \in X = (P = 0)$, then $X$ has analytic local coordinates at $p$.*

For example, $x$ is a nice analytic local coordinate of $x^2 + y^2 = 1$ near $(0, 1)$. We do not want an ramified map so if we look at a neighborhood, we have an analytic function, but if we look at this function globally, we don't have an analytic function. Hence, we need to develop a new theory, that is Algebraic Geometry.

I would like to explain how I got into Algebraic Geometry. I wanted to solve the Fermat equation $x^n + y^n = z^n$. The statement of Fermat's Last Theorem is the following:

**Theorem 4 (Fermat's Last Theorem)** *There are not solutions in $\mathbb{Z}$ for $x^n + y^n = z^n$ when $n > 2$, except the obvious solutions.*

We can think of $(x^n + y^n - z^n = 0)$ in $\mathbb{P}^2(\mathbb{Q})$ or in $\mathbb{P}^2(\mathbb{C})$ and we can consider the graph

$$(x^n + y^n = 1) = (x^n + y^n = z^n) \cap U_{z=1}.$$

Obviously, if $n = 1$, we get a line and we have infinitely many solutions in $\mathbb{Q}$. If $n = 2$, we have infinitely many solutions and we can find the solutions by considering the parametrization

$$\left(\frac{2t}{t^2+1}, \frac{t^2-1}{t^2+1}\right), \quad t \in \mathbb{Q}.$$

Moreover, if $C_n(k) = (x^n + y^n - z^n = 0) \subset \mathbb{P}^2(\mathbb{C})$, we have a one-dimensional compact complex manifold (called a Riemann Surface). Given $Q \in \mathbb{Q}[x_1, \ldots, x_{n+1}]$

homogeneous, prime ideal we call $C(\mathbb{Q}) = (Q = 0) \subset \mathbb{P}^n(\mathbb{C})$ a Riemann Surface (i.e. a smooth projective algebraic curve).

One-dimensional compact complex manifolds are classified by an invariant called the genus $g$ (or the numbers of holes), which is equal to

$$g = \frac{(n-1)(n-2)}{2}.$$

For example, if $n = 1$, then $g = 0$, so we have a sphere. If $n = 2$, then $g = 0$. If $n = 3$, then $g = 1$; that is a torus. If $n = 4$, then $g = 3$, and so on. There are three fundamental Theorems about the genus of Riemann Surfaces:

**Theorem 5** *If $g = 0$, then $C(\mathbb{Q})$ is empty or there exists a $1-1$ correspondence between $C(\mathbb{Q})$ and $\mathbb{Q} \cup \{\infty\} = \mathbb{P}^1(\mathbb{Q})$.*

**Theorem 6 (Mordell)** *If $g = 1$, then $C(\mathbb{C})$ is an abelian group and $C(\mathbb{Q})$ is a finitely generated abelian group.*

**Theorem 7 (Faltings, $\sim$ 1980's)** *If $g \geq 2$, then $C(\mathbb{Q})$ is finite.*

A question of paramount importance is "How we can generalize these Theorems to higher dimensions?"

**Christopher Hacon**

My primary interest is the "Birational Classification of Complex Projective Algebraic Varieties". That is, $X \subset \mathbb{P}^N_{\mathbb{C}}$ is defined by homogeneous polynomials $P_1, ..., P_t$ and $\mathbb{P}^N_{\mathbb{C}}$ is the compactification of $\mathbb{C}^N = \mathbb{C}^N \cup \mathbb{P}^{N-1}_{\mathbb{C}}$. An example of this compactification is $\mathbb{P}^1_{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$.

Two varieties $X_1, X_2$ are birational if they have isomorphic non-empty open subsets; in other words a set of measure zero can be thrown away to get an isomorphism. Some examples of smooth 1-dimensional varieties, $X$ depend on the genus $g$ of a surface. For $g = 0$, there is only one variety: the Riemann sphere. For $g = 1$, there is only one surface topologically, but there is a 1-dimensional family of varieties to study. Similarly for $g \geq 1$ there is a $(3g - 3)$-dimensional family of such varieties.

One tool used frequently in study is the **canonical line bundle**, denoted $K_C$, which looks like $f(z)dz$ locally. Another commonly used symbol is $H^0$, which is the set of polynomials of $(\deg(f) - 2)$ on $\mathbb{C}$. For instance $H^0(K_{\mathbb{P}^1}) = 0$. In general $H^0(K_C) = \mathbb{C}^g$, where again $g$ is the genus number. For $g \geq 2$, $3K_C$ is "very ample"; that is, the sections of $H^0(3K_C)$ give embeddings of $C \hookrightarrow \mathbb{P}^N$ and can be written locally as $f(z)dz^3$. The Riemann-Roch Formula gives $H^0(3K_C) \cong \mathbb{C}^{5g-5} = \{f_0, ... f_{5g-6}\}$, and the map we get $C \hookrightarrow \mathbb{P}^N \supset \mathbb{C}^N$ sends

$$x \mapsto \left( \frac{f_1(x)}{f_0(x)}, \cdots, \frac{f_N(x)}{f_0(x)} \right).$$

I want to generalize all of this to higher dimensions, which is much harder. For dim $X = 2$, we start with $P \in \mathbb{P}^2$ which corresponds to $E \cong \mathbb{P}^1 \subset X$ and is called the "blow-up of a point". Blow-ups are essentially surgeries which take lines through a point and spread them out. Birational equivalence in dim $X = 2$ is generated by these blow-ups. In other words, $X_1$ and $X_2$ are birationally equivalent if there exists a series of blow-ups on each of them that will end with the same set. This contrasts dim $X = 1$ in which two varieties are birationally equivalent if and only they are isomorphic. For dimension 2, the canonical line bundle $K_X$ looks like $F(z_1, z_2)(dz_1 \wedge dz_2)$ and the sections are denoted $\wedge^2 T_X^\vee$. Four cases in this dimension can be explored.

Case -1: If $H^0(mK_X) = 0$ for all $m > 0$, then $X$ is birationally equivalent to $\mathbb{P}^1 \times C$, where $C$ is some curve.

Case 0: If max dim $H^0(mK_X) = 1$, then there are four possible scenarios. In the simplest case, $X$ looks like $E_1 \times E_2$ where $E_1$ and $E_2$ are genus 1 elliptic curves.

Case 1: If dim $H^0(mK_X) = \mathcal{O}(m)$, then the general fiber $F$ has genus 1 and $F \to X \to C$, for some curve $C$.

Case 2: If dim $H^0(mK_X) = \mathcal{O}(m^2)$, then "almost anything goes". It is understood that the characteristic

$$\chi(K_X) = \text{dim } H^0(K_X) - \text{dim } H^1(K_X) + \text{dim} H^2(K_X) \geq 1.$$

But even the case when $\chi(K_X) = 1$ is not completely understood.

This brings up another question: "Can one understand the case when dim $X = 3$ and max dim $H^0(mK_X) = 1$ (the analog of $g = 1$, dim $X$=1)?" It is not known whether there are finitely or infinitely many such families. This is very important in theoretical physics, namely it could say how many possible universes there are in the string theoretical framework.

There was a striking success by Bombieri about 35 years ago in the case of dim $X = 2$. The canonical line bundle $5K_X$ (which looks like $f(z_1, z_2)(dz_1 \wedge dz_2)^{\otimes 5}$ locally) gives an embedding $X \dashrightarrow \mathbb{P}^N$. A recent achievement of mine elaborates on these ideas:

**Theorem 8 (Tsuji, Hacon, McKernan, and Takayama)** *If dim*$(X) = n$ *and dim* $H^0(mK_X) = \mathcal{O}(m^n)$ *there exists an integer* $m_0$ *depending on* $n$ *such that* $H^0(m_0 K_X)$ *defines an embedding of* $X \dashrightarrow \mathbb{P}^N$.

However, even with this amazing result, "embarassingly little" is actually known about $m_0$. For instance in the $n = 3$ case, $43 \leq m_0(3) \leq 10^{100}$, but $m_0(4) \leq$???

**Tommaso de Fernex**

For this lecture, we will let $k$ denote a field such as $k = \mathbb{C}$.

We begin with Lüroth Problem (1861), which states: if $k \subsetneq L \subseteq k(x_1, x_2, \cdots, x_m)$, then is $L \cong k(\theta_1, \cdots, \theta_n)$?

Some positive answers to the Lüroth Problem have been obtained. Here is one of them:

**Theorem 9** *If $k \subsetneq L \subseteq k(x)$, then $L \cong k(\theta)$.*

Sketch of Proof:

Let $\lambda \in L \backslash k$. We then have $k \subset k(\lambda) \subset L \subset k(x)$ with $k(x)$ an algebraic extension of $L$ since the transcendence degrees of $k(\lambda)$ over $k$ and $k(x)$ over $k$ are both 1.

Let $p(t) = t^r + a_1 t^{r-1} + \cdots + a_r$ (with $a_i \in L \subset k(x)$) and $f(x,t) = b_0(x)t^r + b_1(x)t^{r-1} + \cdots + b_r(x) \in k[x,t]$ (with $a_i = \frac{b_i}{b_0}$). Also, let $\theta = a_i = \frac{g(x)}{h(x)}$ where $g$ and $h$ are coprime. Note that $a_i$ is not a constant.

Then we have $g(t) - \theta h(t) = g(t) - \frac{g(x)}{h(x)} h(t)$. This equation has a root $x = t$, which implies $p(t) | (g(t) - \theta h(t))$. Then we have $h(x)g(t) - g(x)h(t) = p(x,t)f(x,t)$. Here we assume $\deg_x f = s$ which implies that $\deg_x p \cdot f \geq s$. If $p$ is constant, then $\deg_x f = \deg_x p \cdot f$. ∎

Now we turn to geometry. Perhaps we want to find all rational solutions to $x^2 + t^2(x-1)^2 = 1$. This equation is equivalent to $(t^2+1)x^2 - 2t^2 x + t^2 - 1 = 0$. Thus, we obtain

$$x = \begin{cases} q = (1,0), \\ p_t = (\frac{t^2-1}{t^2+1}, \frac{-2t}{t^2+1}) \end{cases}$$

which is a rational parameterization.

Let us work inside $\mathbb{C}^2$( or $\mathbb{C}$). The complete picture is: $C = \{x^2 + y^2 = z^2\} \in \mathbb{CP}^2 = \mathbb{C}^3 \backslash \{0\} / \sim$. Here the quotient space is defined by the following equivalence relation: $v \sim v'$ if $v = \lambda v'$ for some $\lambda \in \mathbb{C}^*$. This is also a compactified parameter space. Here is another example:

$$\begin{aligned} \mathbb{C} \cup \{\infty\} = \mathbb{CP}^1 &\longrightarrow \mathbb{C} \\ t &\longmapsto p_t \\ \infty &\longmapsto q \end{aligned}$$

More generally, we can take $X \subset \mathbb{CP}^r$, a complex manifold defined by homogeneous polynomials. A typical example is $X = \{f(x_0, \cdots, x_r) = 0\} \subset$

$\mathbb{CP}^r$, where$f$ is a homogeneous polynomial. (In this talk, we only consider $f(x_0 \cdots, x_r) = x_r^d + \cdots + x_r^d$).

We say $X$ is **rational** if there exists a map:

$$\begin{array}{ccc} \mathbb{CP}^n & \rightsquigarrow & X \qquad (n = \dim_{\mathbb{C}} X) \\ \cup & & \cup \\ U & \xrightarrow{\cong} & V \quad \text{(birational map)} \end{array}$$

Similarly, $X$ is **unirational** if there exists a map:

$$\begin{array}{ccc} \mathbb{CP}^n & \rightsquigarrow & X \\ \cup & & \cup \\ U & \twoheadrightarrow & V \end{array}$$

A case of this is $X = \{x_0^2 + x_1^2 + x_2^2 + x_3^2 = 0\} \subseteq \mathbb{CP}^3$, in which case

$$\mathbb{CP}^2 = \{\text{lines in}\, \mathbb{CP}^3 \,\text{passing through}\, q\} \rightsquigarrow X$$
$$\text{“general line”}\ l \to p \in l \cap X$$
$$\mathbb{CP}^2 \backslash \{\text{tangent lines}\} \xrightarrow{\cong} X \backslash (l_1 \cup l_2).$$

We can consider a complex manifold $X \subset \mathbb{CP}^r \rightsquigarrow K(X) = \{\text{rational functions } f : X \to \mathbb{C}\}$. If $K(X) \subset \mathbb{C}$, then $K(X)$ are constant functions.
The idea is: {complex manifolds $X \subseteq \mathbb{CP}^r, r \geq 1$ } / {birational equivalence} $\to$ {finitely generated field extensions$K \supseteq \mathbb{C}$} / {isomorphisms}.

Two examples are $K(\mathbb{CP}^1) \cong \mathbb{C}(x)$ and $K(\mathbb{CP}^r) \cong \mathbb{C}(x_1, \cdots, x_r)$. Here $X$ is rational if and only if $K(X) \cong \mathbb{C}(x_1, \cdots, x_n)$, where $n = \dim X$. $X$is unirational if and only if $\mathbb{C} \subseteq K(X) \subseteq \mathbb{C}(x_1, \cdots, x_m)$, where $m \geq \dim X$.

The Lüroth Problem can be restated as: “Do we know whether $X$ is rational provided $X$ is unrational?” We divide this problem into different cases.

We first revisit $\dim X = 1$. Suppose $X$ is a complex complete curve. Then it is a Riemann surface and also a topologically orientable compact surface. Again, let $g$ be the genus of $X$. Then if $g = 0$, $X$ is rational. On the other hand, if $g \neq 0$, then $X$ is not rational

Now suppose $X$ is unirational and $g \geq 1$. Suppose there is a surjective map $\varphi : \mathbb{CP}^1 \to X$. Let $g_0$ be the genus of $\mathbb{CP}^1$, then $g_0 = 0$. By Riemann-Hurwitz formula, we have $2 = 2 - 2g_0 = \deg\varphi \cdot (2 - 2g) - \text{ramification number}$. Therefore, deg $\varphi = 0$.

We now turn to $\dim X = 2$; let $X$ be a complex surface (a 4-dimensional real manifold). In 1896, Castelnuovo gave the criterion for rationality of complex surfaces in terms of certain "genera". (If $\dim X = 1$ then $X$ is rational if and only if $g(X) = 0$.) One deduces that if $X$ is a complex surface contained in $\mathbb{CP}^r$, then $X$ is rational if and only if $X$ is unirational. This is the point to the Lüroth problem.

For $\dim X \geq 3$, the first counterexample to the Lüroth problem for $k = \mathbb{C}$ was constructed in the 1970's. In $\dim X = 3$ one counterexample is $X_d = \{x_0^d + \cdots + x_4^d\} \subseteq \mathbb{CP}^4$, although this is hard to prove.

Here are the known results for the Lüroth problem for hypersurfaces in $\mathbb{CP}^4$:

| $d$ | $X_d$ | unirational | rational |
|---|---|---|---|
| 1 | $\mathbb{CP}^3$ | $\surd$ | $\surd$ |
| 2 | quadratic | $\surd$ | $\surd$ |
| 3 | cubic | $\surd$ | $\times$ |
| 4 | quartic | $\surd$ | $\times$ |
| $d \geqslant 5$ | | $\times$ | $\times$ |