TOPOLOGY

ADAM GULLY AND DYLAN ZWICK

Abstract

These notes are based on a set of lectures given by Dan Margalit, Lars Lauder, and Mladen Bestvina on, respectively, March 2nd, 9th, and 16th of 2007 at the University of Utah. The notes were transcribed and prepared by Adam Gully and Dylan Zwick.

The subject of the talks was, broadly speaking, topology. As with any area of mathematics, it is essentially impossible to give a concise definition that is exact and yet covers the entire field, but it is sufficient to say that topology is the branch of mathematics concerned with space and shape. Topology tends to view large classes of seemingly disparate objects as related or even the same, in hopes of determining the essential characteristics that fundamentally define these shapes and differentiate them from others.

As an example, a sphere is homeomorphic to a cube, so in many branches of topology these shapes would be considered the same. Both would be distinct from, say, a torus, to which neither the sphere nor cube are homeomorphic. There is even an old joke that topologists treat all homeomorphic objects as the same, and so a topologist can't tell a coffee cup from a doughnut during breakfast.

These talks all address an area of topology related to the presenter's area of research. While not necessarily what the presenter is researching per se, each lecture gives a taste of the ideas within the field.

These lectures all address topics in topology with a highly algebraic flavor with problems that can be tackled utilizing ideas from algebra, especially group theory. This particular area of topology research is especially active at the University of Utah.

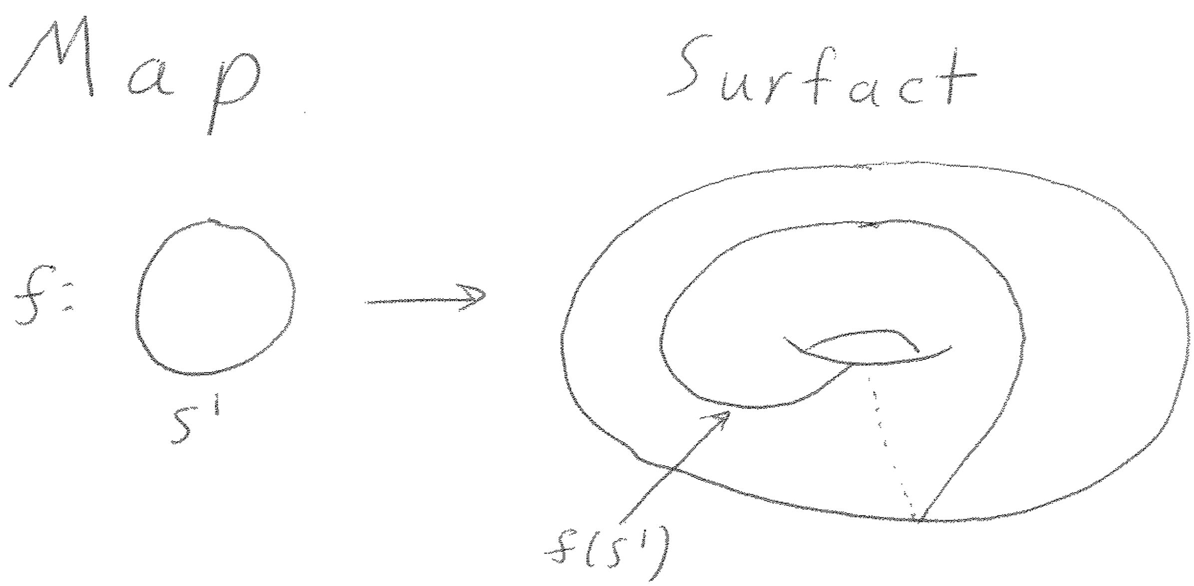
The Mapping Class Group

Notes based on a lecture by Prof. Dan Margalit given on March 2nd, 2007

The mapping class group is an interesting, complex, and powerful structure that is built by studying the types of curves you can produce on a surface, the relations between these curves, and the structures formed by these relations.

Surfaces and Curves

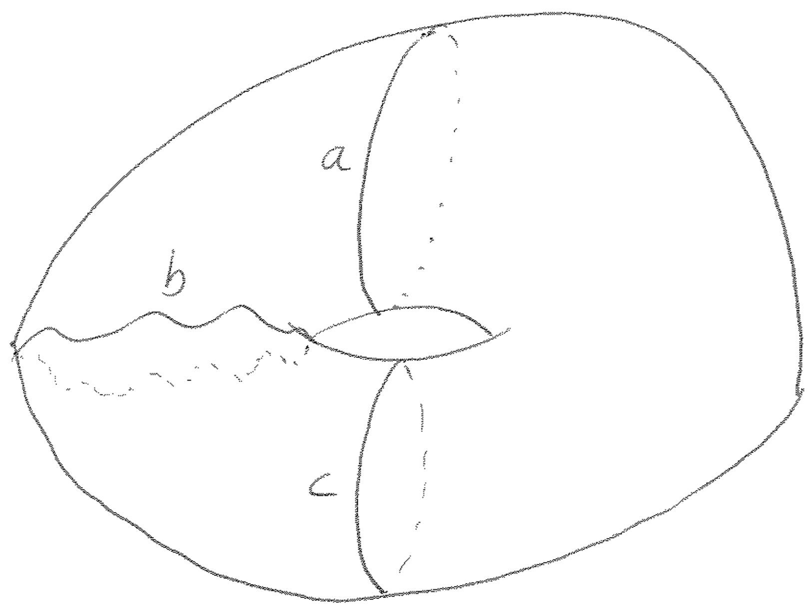
A surface is a 2-dimensional manifold, and a curve on this surface is an injective continuous map from S^1 to the surface.



We refer to a curve as being essential if it does not bound a disk or an annulus. In other words, a curve is essential if we can't shrink it down to a point or slide it to the surface's boundary.

Note finally that we consider two curves to be the same if we can homotope one curve to the other. That is, if we can stretch, wiggle, and slide one curve without breaking it and turn it into the other.

For example, the curves a, b, and c below are all homotopic, and so are treated as the same curve.



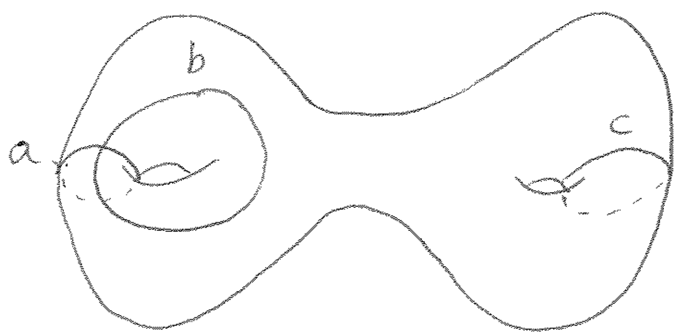
The Curve Complex

The first step in the creation of the mapping class group for a surface is the construction of the curve complex for that surface. The curve complex for a surface is a graph with the identifications:

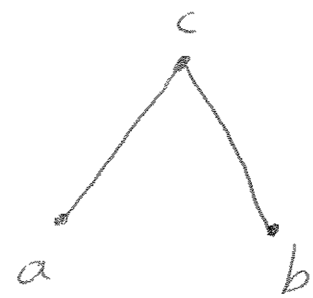
- **Vertices** - Essential curves on the surface. (Isotopy classes of simple closed curves that are not homotopic to a point or a section of the boundary.)
- **Edges** - An edge is drawn between two vertices if the two curves represented by those vertices do not intersect.

For example in the surface below the curves a and b intersect, so there is no edge between them in the curve complex, while the curves a and c do not intersect, and so there is an edge between them in the curve complex.

Surfaces and Curves

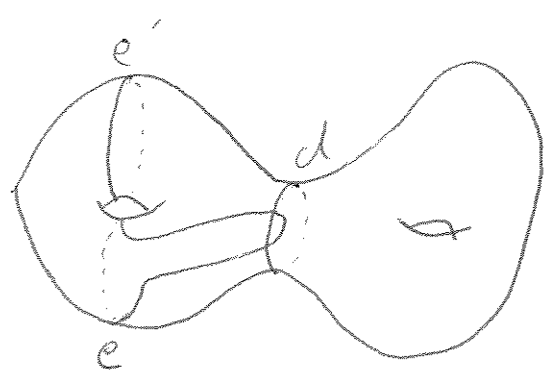


Curve Complex

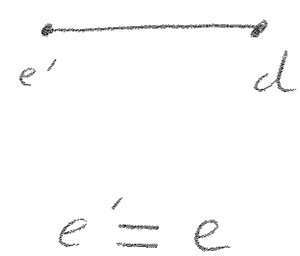


Note the importance here of the concept that two curves are the same if you can homotope one to the other. For example, the two curves on the left below are disjoint because curve e is homotopic to curve e' and curve e' is disjoint from curve d. Note in the earlier example above there is no way to homotope curve a so that it does not intersect curve b, so it is correct to claim they intersect.

Surfaces and Curves



Curve Complex

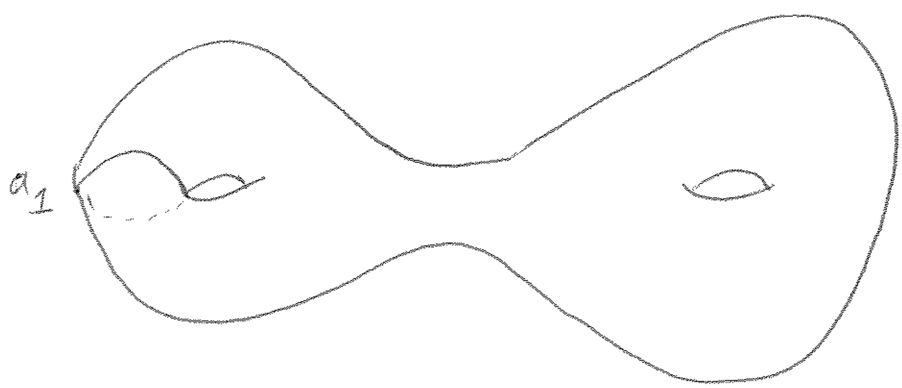


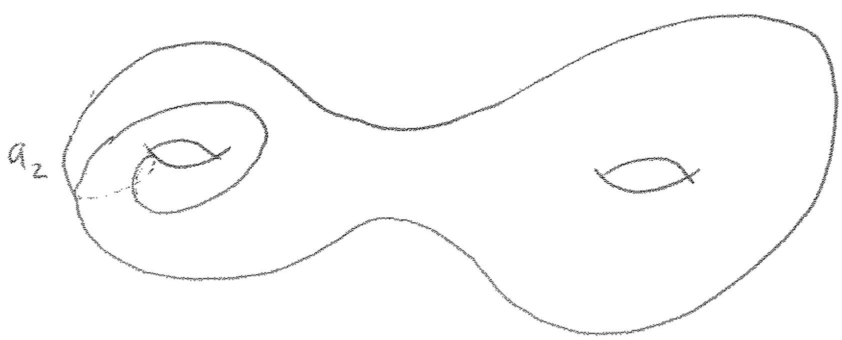
Some Properties of the Curve Complex

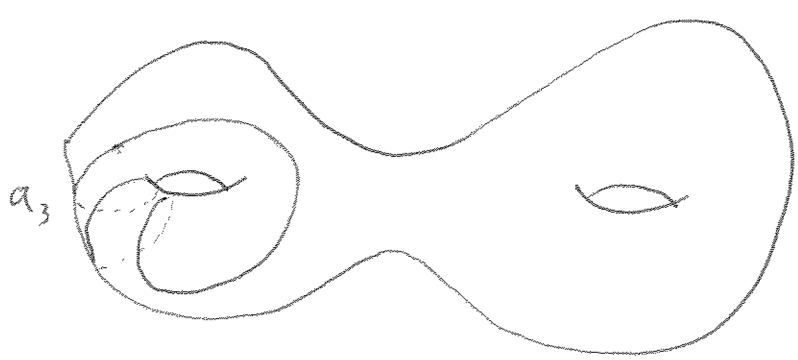
Note - When discussing these properties we assume we are dealing with a surface that has a non-trivial curve complex. The curve complex of the sphere, for example, is empty. On the sphere any closed curve is homotopic to a point, and so there are no non-essential curves. The properties enumerated below all apply to, for example, the curve complex of the genus-2 torus, and also to "more complicated" surfaces.

(1) **There are an infinite number of vertices in the curve complex.**

We can get a good idea for why this is by looking at the essential curves on the genus-2 torus. We can start with an essential curve such as curve a_1 below, and create from it another essential curve, a_2 below, by wrapping it around the torus twice. We can create another by wrapping it around the torus three times (curve a_3 below) and continuing this process we can create an infinite number of essential curves. Each of these curves is a vertex on the curve complex. Note also that as they all intersect these vertices are mutually disjoint.

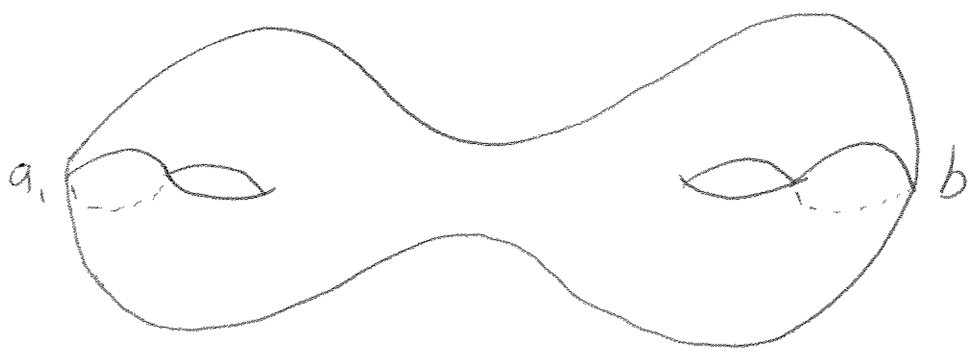






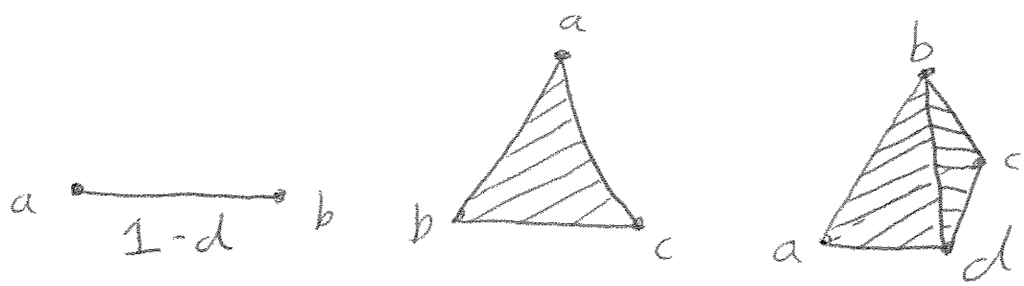
(2) **The curve complex is not locally finite.**

What this means is that every vertex in the curve complex has an infinite number of associated edges. This can be seen by examining the curve b below and noting that it is disjoint not just from curve a_1 (also given below for clarity) but also from all curves a_n discussed above. So, there are an infinite number of edges connected to the vertex for curve b, and a similar fact applies to every vertex in the complex.

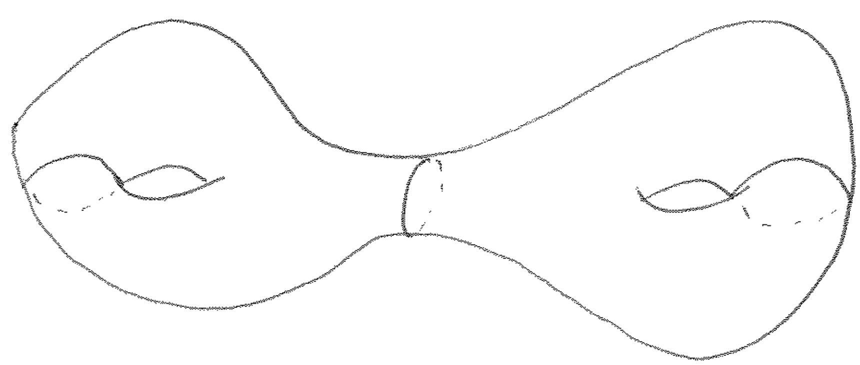


(3) **The dimension of the curve complex on a closed, orientable boundaryless surface is *3g-4***

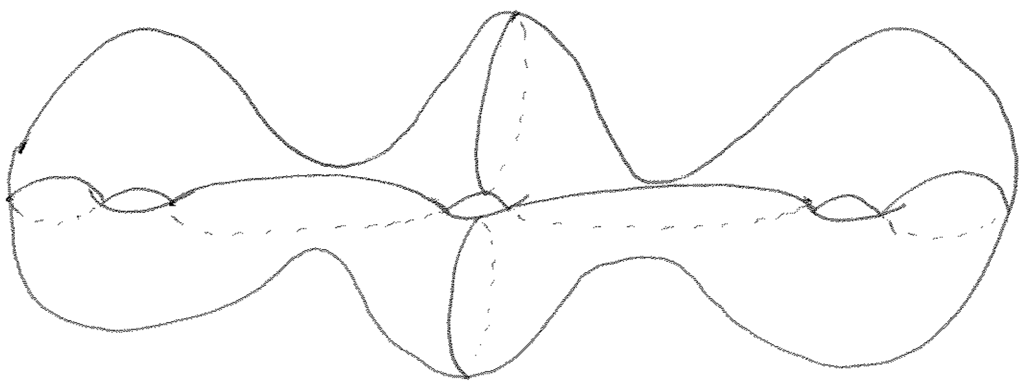
If the curve complex has two connected vertices, then we can view that as forming a 1-dimensional structure (the line connecting the two vertices). If we have three mutually connected vertices (corresponding to three mutually disjoint curves) then we can view this as forming a 2-dimensional structure (the area formed by the triangle whose corners are the three vertices). If we have 4 mutually connected vertices we can view these as the edges of a 3-dimensional tetrahedron, and so on. So, the dimension of the curve complex is one less than the maximum number of disjoint curves you can draw on the surface at the same time.



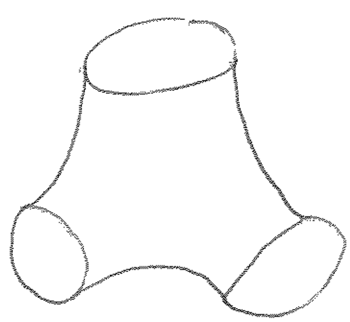
For the genus-2 torus we see that we can draw 3 mutually disjoint curves, and so the dimension of its curve complex is 2:



For the genus-3 torus we can draw 6 mutually disjoint curves, and so the dimension of its curve complex is 5.



Note that if we cut along the disjoint curves of the genus-3 torus above, we get four surfaces that look like this:



Topologists call this surface a "pair of pants", and the set of disjoint curves for the genus-3 torus above is called its "pants decomposition." Note that, like with the sphere, we cannot draw any essential curves on a pair of pants. All curves on a pair of pants can either be shrunk to a point or moved to the boundary.

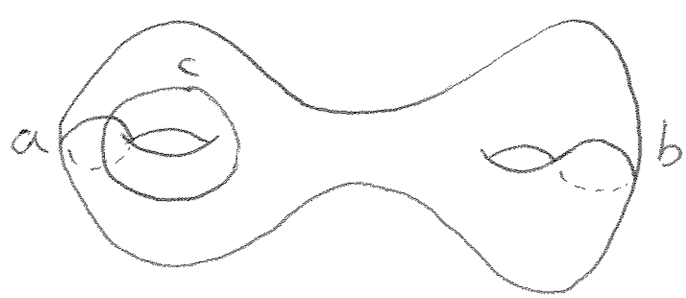
(4) **The Curve Complex is Connected**

This means that for any vertex in the curve complex there is a path connecting it to any other vertex.

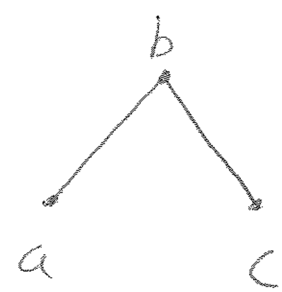
(5) **The Curve Complex has Infinite Diameter**

The distance between any two vertices is the minimum length path connecting them. So, for example, two connected vertices are distance 1 apart. The curves a and b below are distance 1 apart. The intersecting curves a and c below are distance 2 apart, as they can be connected through the curve b, from which both a and c are disjoint.

Surface and Curves



Curve Complex



Having an infinite diamater means that for any positive integer N there exist vertices in the curve complex that are a distance N apart.

(6) **The Distance Between Any Two Vertices is Computable**

It is, in theory, possible to calculate the distance between any two vertices, although in practice using current techniques this is essentially impossible for distances that are even moderately large.

The basic idea is that for any positive inter N there are only a finite number of vertices a distance N apart, and we can calculate an upper bound for the distance between any two vertices. So, in theory, we can figure out this upper bound, and then just check all the paths with distance less than this upper bound. There are a finite number of such paths, so we will eventually figure out the smallest distance between the two vertices.

Definition of the Mapping Class Group

The mapping class group for a surface S is defined as:

$$MCG(S) = \frac{Homeo(S)}{isotopy}$$

In words, this means that the mapping class group of a surface is the group created by looking at homeomorphisms of the surface, and treating two surfaces as being the same if they are isotopic.

Properties of the Mapping Class Group

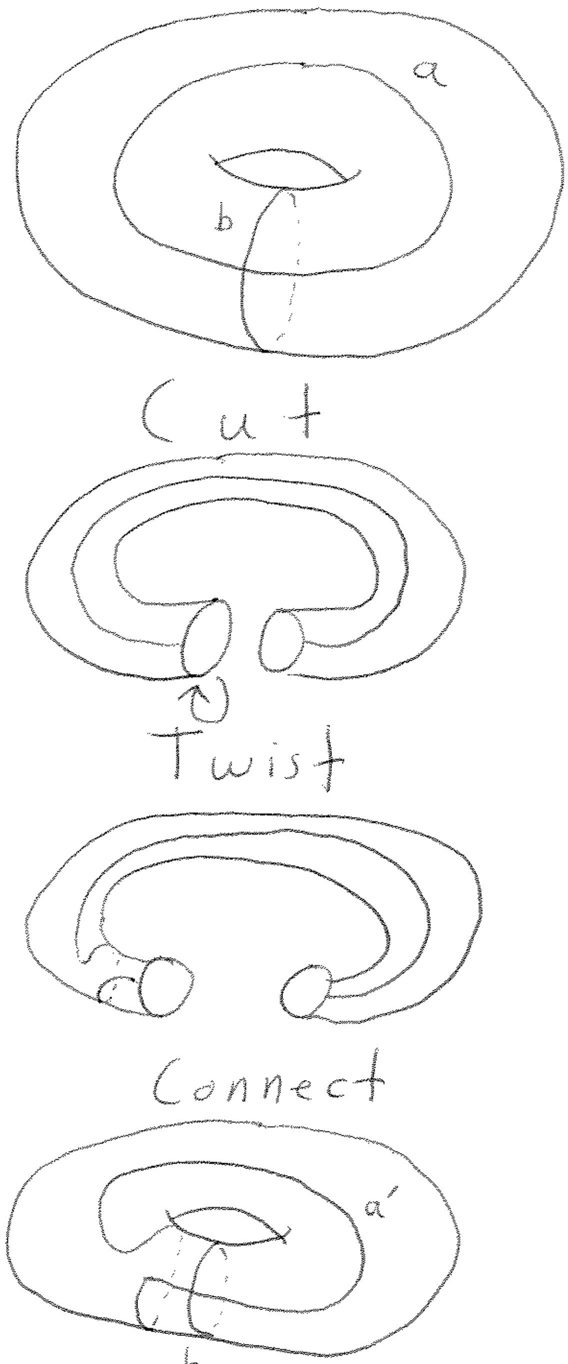
(1) **The Mapping Class Group Corresponds to Automorphisms of the Curve Complex**

It is an amazing fact that is not obvious at all, but there is an exact correspondence between the elements in the mapping class group and the automorphisms of the curve complex. Any element in the mapping class group represents an automorphism of the curve complex, and any automorphism of the curve complex has a corresponding element in the mapping class group.

(2) **The Mapping Class Group is Generated by Dehn Twists**

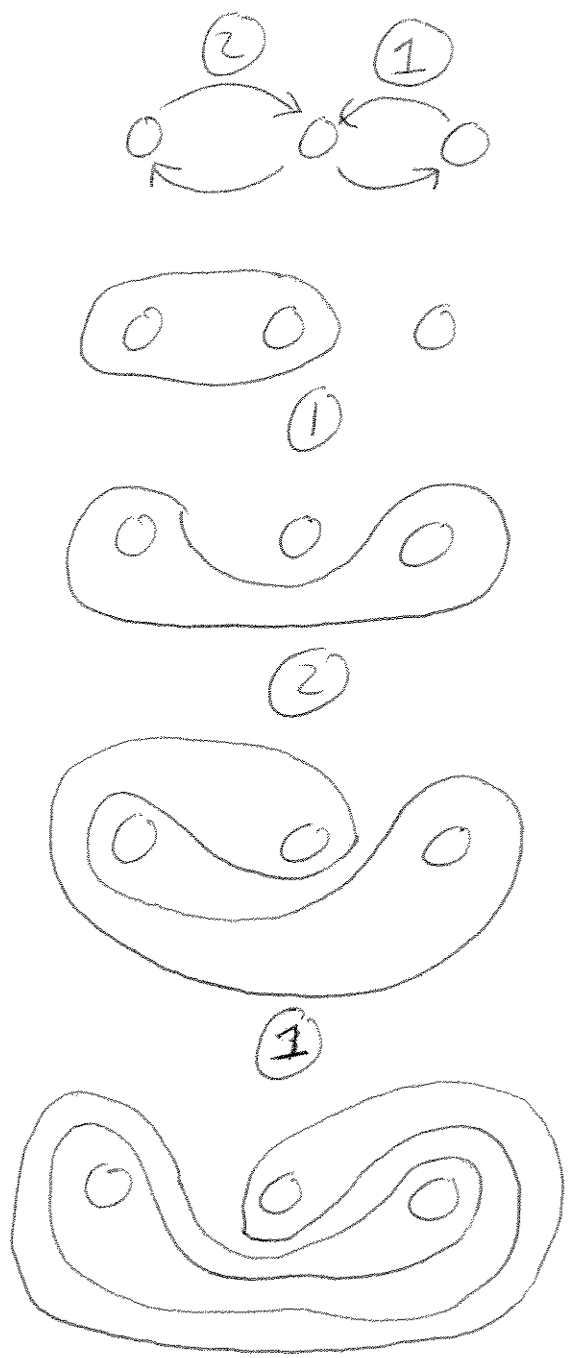
A Dehn twist corresponds to cutting the surface along a circle, and then rotating one of the resulting pieces one full turn, while leaving the other piece the same, and finally reconnecting the two pieces.

For example, on the genus-1 torus below if we perform a Dehn twist along the circle b the steps are as follows, and while the resulting surface is still a genus-1 torus, the curve a has been changed into a'.



Thurston's Game

As a final, somewhat whimsical, example of a mapping class group we can take a look at a "game." In this game we start with a plane with three holes, and draw a circle around two adjacent holes. The game is played by exchanging adjacent holes through rotations. As we perform these rotations, our initially simple curve gets more and more complicated.



The amazing thing is that the plane with three holes removed is a surface, and each of these rotations represent a different element in the surface's mapping class group.

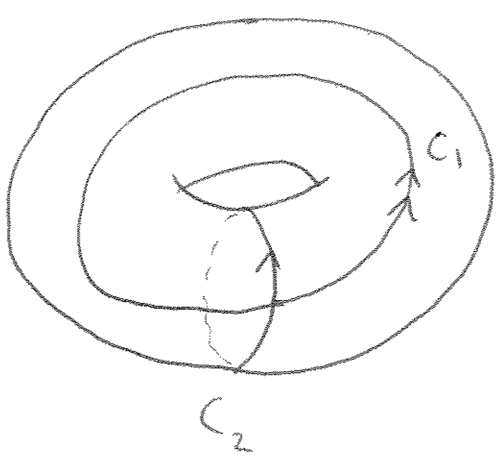
The Algebra of the Torus and the Farey Graph

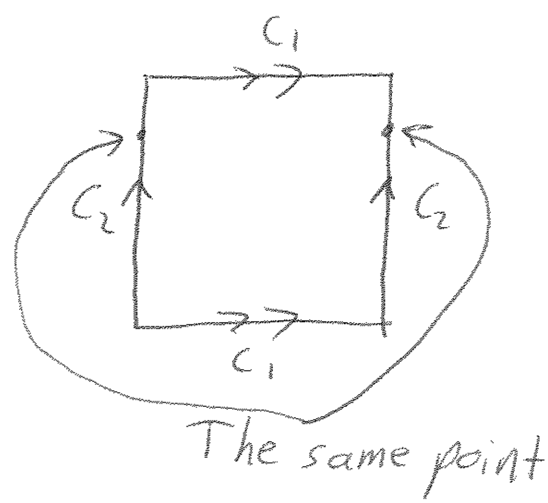
Notes based on a lecture by Graduate Student Lars Lauder given on March 9th, 2007

Here we explore in a little more depth the algebraic properties of homeomorphisms, particularly homeomorphisms of the genus 2 torus (the one holed doughnut, referred to as the torus for the rest of the notes on this lecture), and how these algebraic properties can be used in the study of curves on surfaces. We then use these ideas to construct a very interesting metric space using curves on a torus and properties about their intersections.

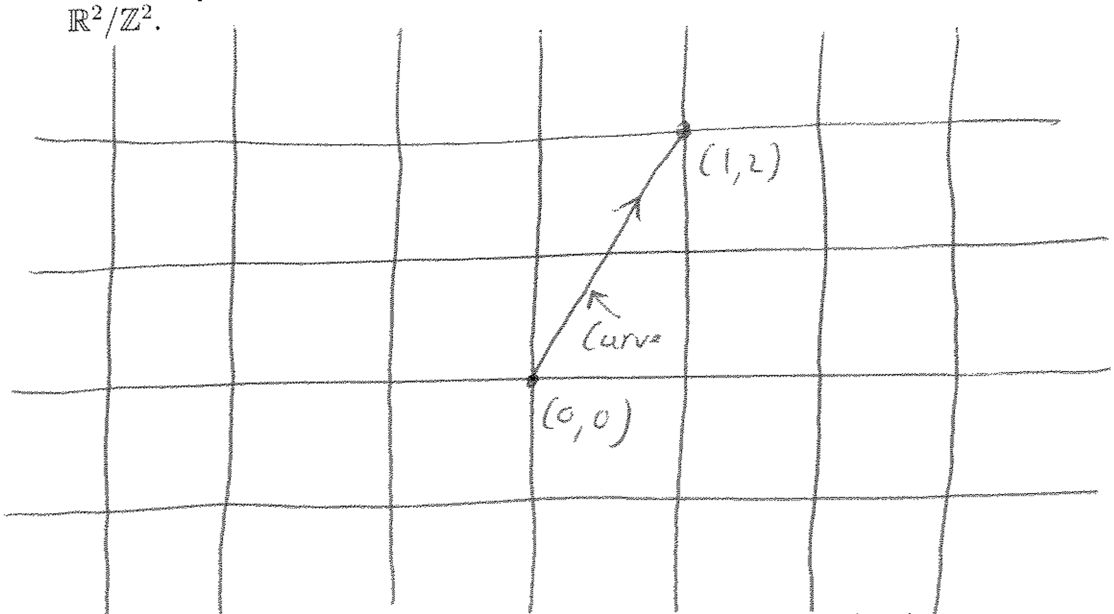
The Torus and its Homeomorphisms

There are two main ways in which mathematicians view and draw the torus. The first is the standard drawing of a doughnut, and the second is the drawing of a square with sides identified as equivalent. We can imagine constructing this square by cutting along the torus along the two closed curves, C_1 and C_2 below.

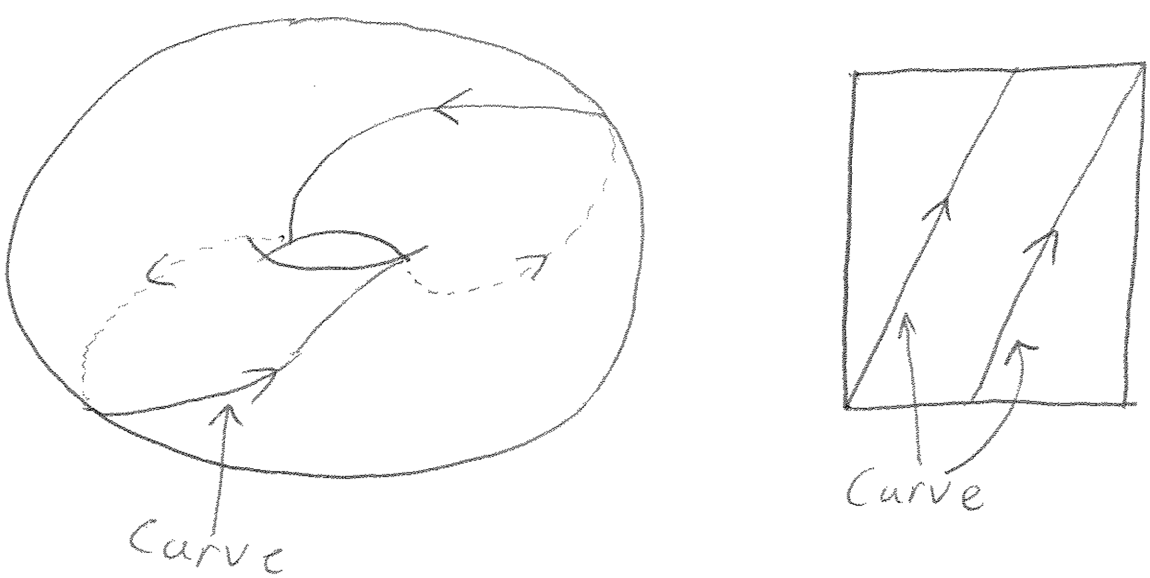




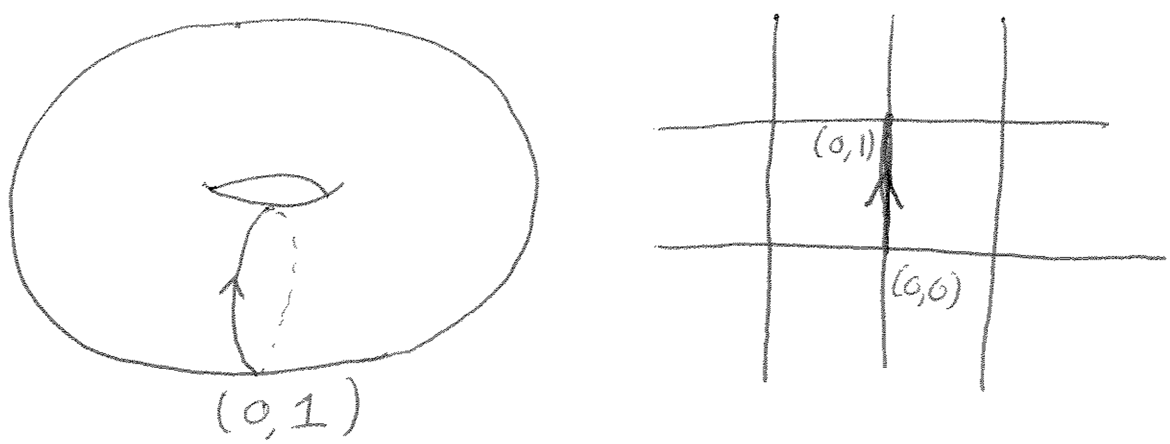
Now, we can tile $\mathbb{R}^2$ with these square representations of the torus and view each of these squares as representing the same torus. Mathematically, this would be taking the quotient of $\mathbb{R}^2$ by $\mathbb{Z}^2$, written as $\mathbb{R}^2/\mathbb{Z}^2$.

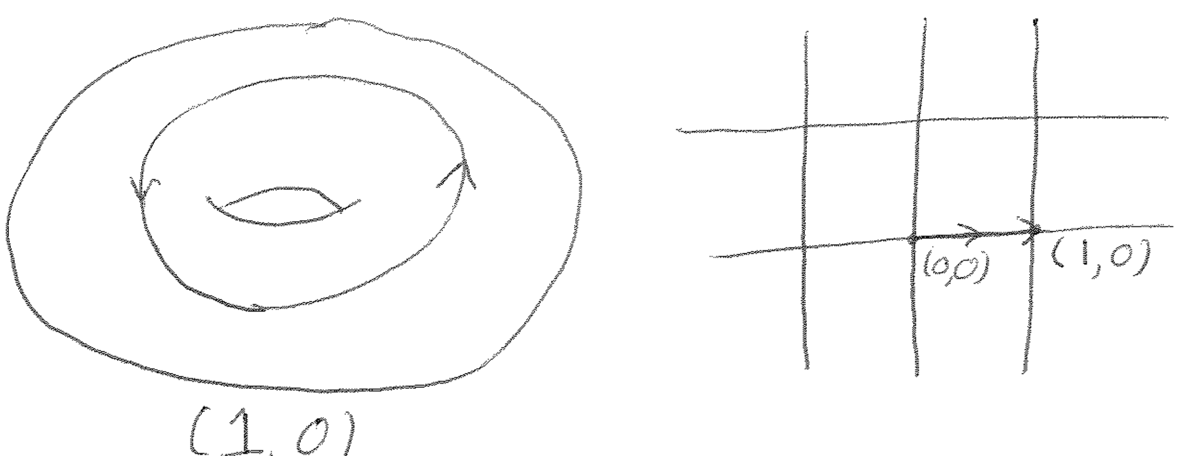


Let us take a look at curves on the torus. For example, the curve on the torus below can be represented on the "square" torus on the right, but the curve can also be represented is a more elegant and intuitive (as well as useful) manner by using the tiled plane.



Note as with the first lecture we treat simple, closed curves as the same if one can be homotoped to the other. Each simple, closed curve on the torus can be homotoped so that it passes through a given point, which we will denote $(0, 0)$. Each simple, closed curve in the plane can be represented by a pair of integers (p, q) where $lcd(p, q) = 1$. Here p represents the number of times the curve wraps around the doughtnut before it gets back to $(0, 0)$, while q represents the number of times the curve wraps around the doughnut hole before it gets back to $(0, 0)$. Some examples are given below:





Now, we will look as some algebraic concepts, and how these concepts relate to our study of simple, closed curves on the torus.

The matrix group $GL_2(\mathbb{Z})$ is the group of 2×2 matrices with determinant equal to ± 1:

$$GL_2(\mathbb{Z}) = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{matrix} a, b, c, d \in \mathbb{Z} \\ ad - bc = \pm 1. \end{matrix}$$

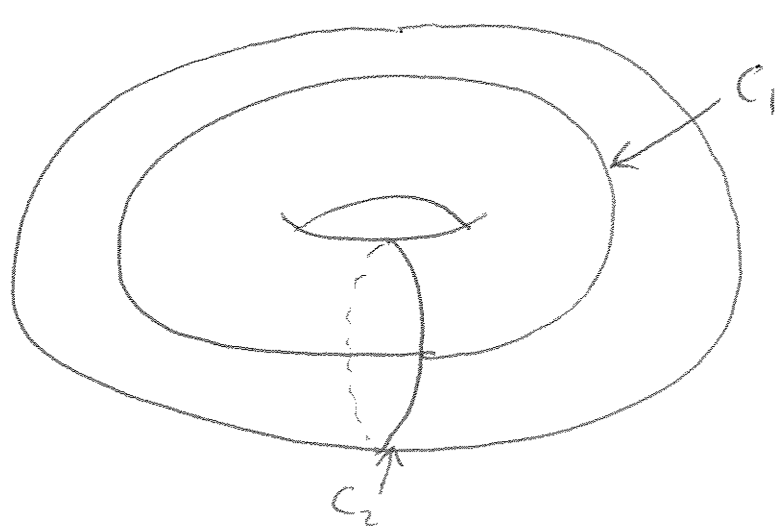
Note that this matrix maps points in $\mathbb{Z}^2$ to points in $\mathbb{Z}^2$, and we can view its action on the tiled plane as a homeomorphism of the torus. (Note the fact that it maps $\mathbb{Z}^2$ to itself is critical because in our tiled plane each point on the integer lattice represents the same point on the torus, and our homeomorphism is a function, so it better map one point to only one point!) Of particular interest is what this group does to simple closed curves on the torus. In fact, this map takes simple closed curves to simple closed curves!

We can restrict this group to the subgroup $SL_2(\mathbb{Z})$, which is the group of 2×2 matrices with determinant equal to 1:

$$SL_2(\mathbb{Z}) = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{matrix} a, b, c, d \in \mathbb{Z} \\ ad - bc = 1. \end{matrix}$$

This group's action on the torus also maps simple, closed curves to simple, closed curves, but it does so in a way that preserves orientation. (A curve starting out at $(0, 0)$ with a positive slope maps to another curve with a positive slope, and vice versa.)

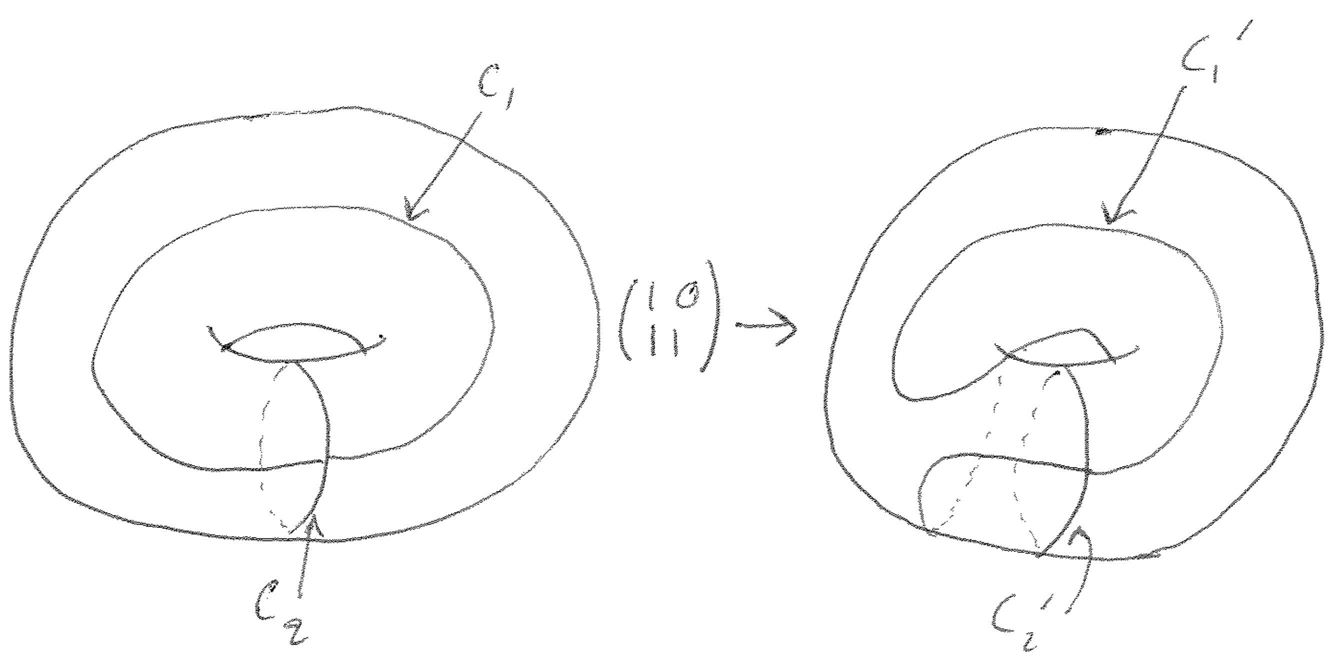
A particular example of a matrix in $SL_2(\mathbb{Z})$ is $\begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}$. Now, we can view the curve C_1 below as the vector $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and the vector C_2 below as the vector $\begin{pmatrix} 0 \\ 1 \end{pmatrix}$.



The action of the matrix on these vectors is:

$$\begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$
$$\begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

When viewed graphically we note that this matrix represents the Dehn twist about C_1:



Which is the same result we got with the Dehn twist described in the earlier lecture.

The Farey Graph

As a further exploration into these simple, closed curves on the torus we will construct a metric space whose points are these curves.

Note first of all that two disjoint simple, closed curves on the torus must be homotopic. (The only disjoint simple, closed curves we can draw on the torus are either two homotopies of C_1 mentioned above, or two homotopies of C_2. Any other simple, closed curves must intersect.)

However, some simple, closed curves on the torus intersect each other once, and some intersect more than once. As mentioned earlier, any simple, closed curve on the surface can be homotoped so that it crosses through the point $(0, 0)$. The simple, closed curves that intersect only once are those curves that only intersect at the point $(0, 0)$.

If we represent two simple, closed curves with the relatively prime pair of integers (p, q) and (p', q'). We can form a 2×2 matrix from this pair of curves like:

$$\begin{pmatrix} p & p' \\ q & q' \end{pmatrix}$$

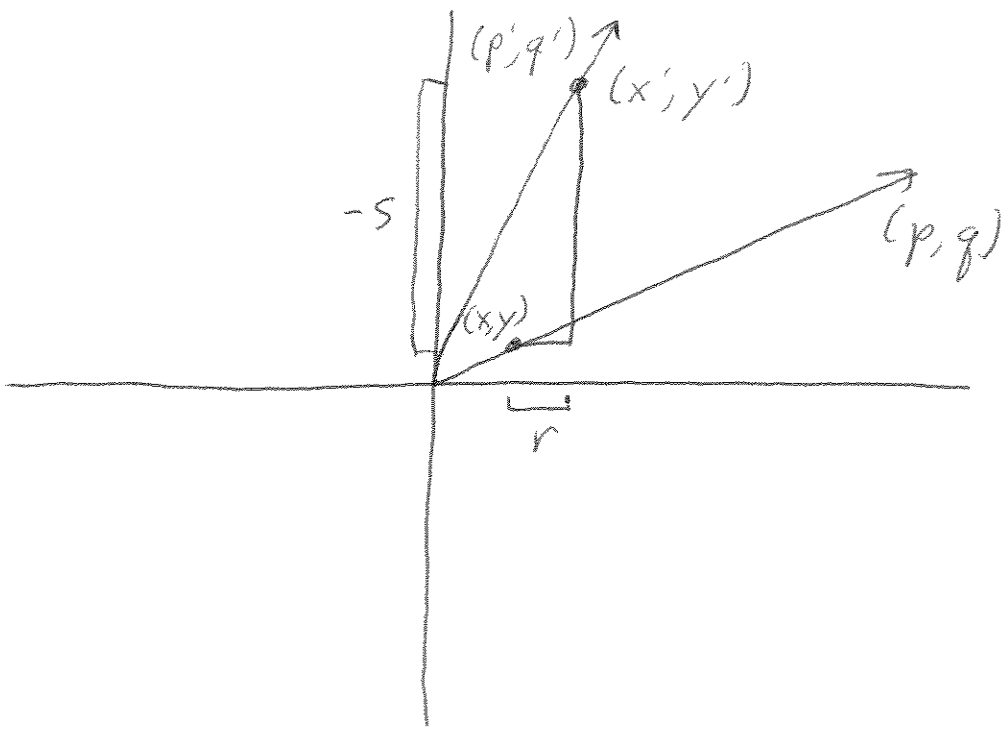
The claim is that the curves (p, q) and (p', q') intersect only once if and only if the corresponding matrix is in $GL_2(\mathbb{Z})$.

Proof

Necessity:

Suppose the curves (p, q) and (p', q') are such that the corresponding matrix is in $GL_2(\mathbb{Z})$. Specifically, this means that $(pq' - p'q) = \pm 1$. Now suppose we have a point (x, y) on the line through the origin with slope $\frac{p}{q}$, and another point (x', y') on the line through the origin with slope $\frac{p'}{q'}$, and that the two points differ by an integer in each term. In other words $x' = x + r$ and $y = y' + s$ where $r, s \in \mathbb{Z}$.

Graphically we can view (x, y) and (x', y') below, where the plane here is interpreted as the tiled plane with copies of the torus:



The point $(0, 0)$ is the common point through which both lines (p, q) and (p', q') cross. We can write y and y' as functions of x and x', respectively, just using the standard slope-intercept line equation $y = (\frac{p}{q})x$ and $y' = (\frac{p'}{q'})x'$. Using these relations and the earlier relations, $x' = x + r$ and $y = y' + s$, after some algebra we get:

$$x = \frac{p'qr + sqq'}{pq' - p'q}.$$

Given $(pq' - p'q) = \pm 1$ we have that x, and therefore also x' are integers. Noting that x is divisible by q we have that y is also an integer and therefore so is y'. So, these points must occur on the integer lattice in $\mathbb{R}^2$, and each of these points maps to the same point on the torus. So, the curves only intersect at one point.

Sufficiency:

Suppose the curves (p, q) and (p', q') only intersect once. Note that if we define y, x', y' in terms of x as above and pick r and s such that

$rp' + sq' = 1$ (which the Euclidean algorithm says we can do given $lcd(p', q') = 1$) then if we choose x using our earlier equation:

$$x = \frac{p'qr + sqq'}{pq' - p'q},$$

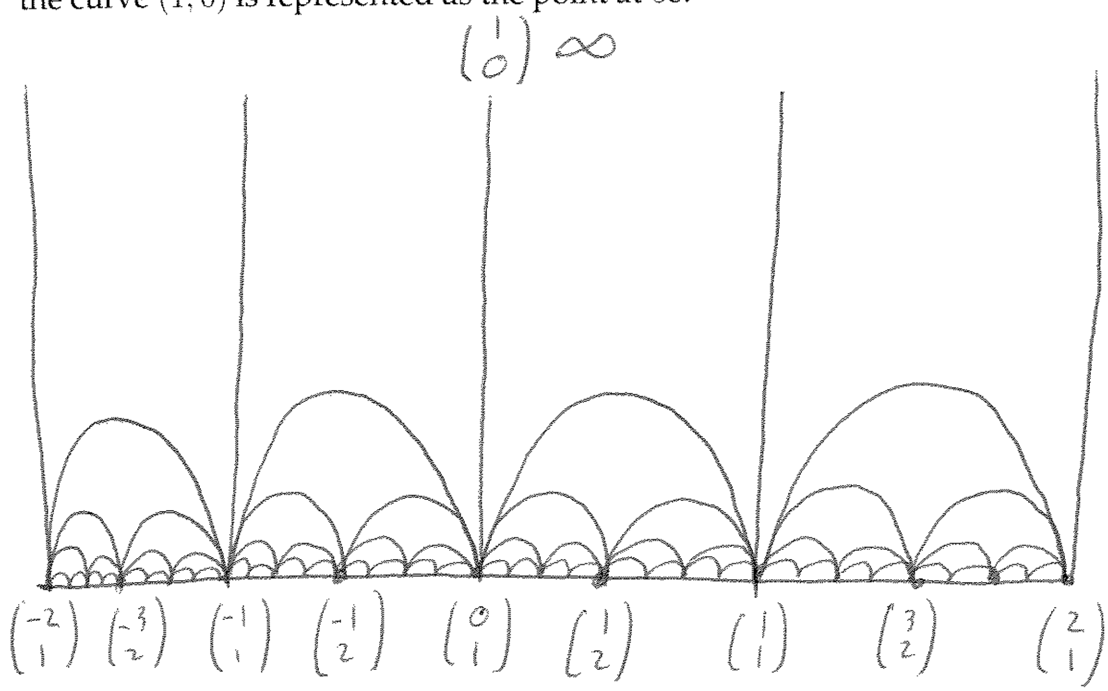
we recover a pair of points (x, y) on the first curve and (x', y') on the second curve that differ by a pair of integers. Given the curves only intersect once at $(0, 0)$, the points (x, y) (and therefore (x', y')) must be integers. Now, as $y = (\frac{p}{q})x$, given p and q are relatively prime, it must be that $q|x$. Given this fact and our choosing of r and s such that $rp' + sq' = 1$ we get that:

$$\frac{x}{q} = \frac{1}{pq' - p'q}.$$

And as $q|x$ we know that $\frac{x}{q}$ is an integer, and therefore $(pq' - p'q) = \pm 1$. Q.E.D.

Now, this fact allows us to construct the Farey graph. We define the vertices of the Farey graph as consisting of all simple, closed curves on the torus, and we connect two vertices if the corresponding curves only intersect once. In other words, the curves (p, q) and (p', q') are connected if and only if $\begin{pmatrix} p & p' \\ q & q' \end{pmatrix} \in GL_2(\mathbb{Z})$.

A subset of the vertices and edges of this graph that provides a clear and telling way to view the graph is drawn below. Note that the curve $(1, 0)$ is represented as the point at ∞.



There is much non-trivial and interesting mathematics that can be done on the Farey graph. A very elementary property of the graph that can be more or less deduced immediately is that the graph is not locally finite. Each vertex has an infinite number of connecting edges. Another elementary property is that it is connected, so each point can be reached by another through a series of edges. Now, we can turn this graph into a metric space by defining the distance between two curves C_1 and C_2 as the minimum number of edges on the graph that must be traversed to get from the vertex representing C_1 to the vertex representing C_2.

Mathematics related to and arising from the Farey graph, of which this is just the tip of the iceberg, is one area of mathematical research that is pursued in the study of geometric group theory here at the University of Utah.

Quasi-Homomorphisms and Words

Notes based on a lecture by Prof. Mladen Bestvina given on March 16th, 2007

In topology, a homeomorphism is a special transformation between topological spaces that is a structure preserving map with respect to topological properties. These properties on the most basic of levels essentially allow for the stretching and shrinking of space as long as you do not tear a whole in the space or close a whole. An example of this would be transforming a coffee cup into a doughnut. They are homeomorphic.

Definition

Let G be a group. Then a quasi-homeomorphism on G is a function $f : G \to \mathbb{R}$ such that

$$\sup_{x,y \in G} |f(xy) - f(x) - f(y)| \leq \delta < \infty$$

Quasi means "having a resemblance to something." So a quasi-homeomorphism can be thought of as a topological map that is similar to a homeomorphism. For instance, as explained in detail in the example below, any function within a bounded distance of a homeomorphism is a quasi-homeomophism. Below are some more detailed examples to demonstrate this definition.

Example

Any quasi-homeomorphism on $\mathbb{Z}$ can be obtained by some function within a bounded distance of a homeomorphism.

Proof:

First, we note that a quasi-homeomorphism on $\mathbb{Z}$ is a function mapping the integers to real numbers:

$$f : \mathbb{Z} \to \mathbb{R}$$

Using this mapping we can define a sequence $\{a_n\}$ as follows:

$$a_n = \frac{f(2^2)}{2^n}$$

Now, we can use the definition of a quasi-homeomorphism to show that this sequence is Cauchy:

$$|a_{n+1} - a_n| = |\frac{f(2^{n+1}) - 2f(2^n)}{2^{n+1}}| \leq \frac{\delta}{2^{n+1}}$$

Which given δ is constant this difference goes to 0 as $n \to \infty$. This means $\{a_n\}$ is Cauchy.
Now, as $\mathbb{R}$ is complete, this implies $a_n \to a \in \mathbb{R}$.

By subtracting ax from $f(x)$ we may assume $a = 0$. Now,

$$|a_n| \leq |a_n - a_{n+1}| + |a_{n+1} - a_{n+2}| + \ldots \leq \frac{\delta}{2^{n+1}} + \frac{\delta}{2^{n+2}} + \ldots = \frac{delta}{2^n}$$

Therefore, $|f(2^n)| \leq \delta$, and simiilarly $|f(k2^n)| \leq \delta$ for all k, n. Q.E.D.

Thus, it is observed that any quasi-homeomorphism on $\mathbb{Z}$ is indeed obtained by a function within a bounded distance of a homeomorphism. However, it is not true that any function within a bounded distance of a homeomorphism is the only way to obtain a quasi-homeomophism. Quasi-homeomorphisms can be obtained other ways. Below is an example of this phenomenon.

Example

When $G = F_2 =< a, b >$, then there are quasi-homeomorphisms that are not within bounded distance of a homeomorphism. Choose $w = ab$.

If we make the definition $f_w : F_2 \to \mathbb{R}$, then we might have something like $f_w(aabab^{-1}a^{-1}baba) = 1$. Further, a neat way to reduce a word in topology is like this:

$$f_w(reducedword) =$$
$$number(occurance\ of\ ab) - number(occurance\ of\ b^{-1}a^{-1}).$$

Therefore, f_w is a quasi-homeomorphism because we can reduce the word and find that $|f_w(xy) - f_w(x) - f_w(y)| \leq 3$.

Similarly, $f_w(a^n) = f_w(a \dots a) = 0$, $f_w(b^n) = 0$, and $f_w((ab)^n) = n$.

Open Question

Using just these relatively simple definitions we already run across an open question in geometric group theory. This open question involves the dimensionality of the set of quasi-homeomorphisms of the group when we take its quotient by a subgroup.

Define:

X - The set of all quasi-homeomorphisms of G.

Y - The set of all quasi-homeomorphisms of G within a bounded distance of a homeomorphism.

For a given group G we can define a space $QH(G) = X/Y$, which is the set of all quasi-homeomorphisms of G, where we consider two such homeomorphisms the same if they differ by a quasi-homeomorphism within bounded distance of a homeomorphism.

For example, we proved earlier that for the group $\mathbb{Z}$ all quasi-homeomorphisms are within a bounded distance of a homeomorphism, and so this quotient group is just the trivial map to the identity, which has dimension 0. On the other hand for our word group F_2 we investigated, the quotient is infinite dimensional.

So, in our two examples the quotient is either trivial or infinite dimensional. Are there any intermediate examples between these two extremes? This is actually still an open question, and is related to questions that are being researched here at the University of Utah.