Intersection Multiplicity, Chow Groups, and the Canonical Element Conjecture

Abstract

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1 Serre's Conjecture

All local rings are assumed to be Noetherian, M, N are finitely generated R-modules. If $\operatorname{proj}\dim(M)$ finite or $\operatorname{proj}\dim(N)$ finite and we have $\ell(M \otimes_R N) < \infty$, then we may define

$$\chi(M,N) := \sum_{i=0}^{d} (-1)^i \ell(\operatorname{Tor}_i^R(M,N)),$$

where d is $\operatorname{proj}\dim(M)$ or $\operatorname{proj}\dim(N)$ respectively.

1.1 Regular Rings

Conjecture 1 (Nonnegativity) *If R is a regular local ring, then $\chi(M,N) \geqslant 0$.*

This is was proved by Gabber.

Theorem 2 (Serre) *If R is a regular local ring and $\ell(M \otimes_R N) < \infty$, then $\dim(M) + \dim(N) \leqslant \dim(R)$*

Conjecture 3 (Peskine-Szpiro) *If R is any local ring, M an R-module with $\operatorname{proj}\dim(M) < \infty$, and $\ell(M \otimes_R N) < \infty$, then $\dim(M) + \dim(N) \leqslant \dim(R)$.*

This is wide open except for hypersurface case.

Conjecture 4 (Vanishing) *If R is a regular local ring and*

$$\dim(M) + \dim(N) < \dim(R),$$

then $\chi(M,N) = 0$.

This was proved independently by Roberts and Gillet-Saulé.

Conjecture 5 (Positivity) *If R is a regular local ring and*

$$\dim(M) + \dim(N) = \dim(R),$$

then $\chi(M, N) > 0$.

This conjecture is still open.

1.2 The General Case

Theorem 6 (Serre) *If R is a regular local ring, then*

$$\max\{j : \mathrm{Tor}_j^R(M, N) \neq 0\} = \dim(R) - \mathrm{depth}(M) - \mathrm{depth}(N).$$

Lemma 1 (Hochster) *Let R be Cohen-Macaulay and M and R-module with* $\mathrm{proj\,dim}(M) < \infty$. *Vanishing holds if and only if it holds for every pair of Cohen-Macaulay R-modules M, N such that,*

$$\dim(M) + \dim(N) = \dim(R) - 1.$$

Sketch of Proof Write $\dim(M) + \dim(N) < \dim(R)$. So

$$\dim(R) - \mathrm{ht}(\mathrm{Ann}(M)) + \dim(R) - \mathrm{ht}(\mathrm{Ann}(N)) < \dim(R),$$

and so

$$\mathrm{ht}(\mathrm{Ann}(M)) + \mathrm{ht}(\mathrm{Ann}(N)) > \dim(R)$$

or

$$\mathrm{ht}(\mathrm{Ann}(N)) > \dim(M).$$

If $r = \dim(M)$ and $s = \dim(N)$, we may choose $x_1, \ldots, x_{r+1} \in \mathrm{Ann}(N)$ such that $\ell(M/\mathbf{x}M) < \infty$ and any r elements of $x_1, \ldots, x_{r+1}$ is a system of parameters for M with $\mathbf{x}$ being R-regular.

Now we can construct T, a Cohen-Macaulay module, by taking a resolution of N over $R/\mathbf{x}R$

$$0 \to T \to \cdots \to (R/\mathbf{x}R)^{t_1} \to (R/\mathbf{x}R)^{t_0} \to N \to 0.$$

such that $\chi(M, T) = \chi(M, N)$. Note that $\dim(R/\mathbf{x}R) = n - r - 1$ where $n = \dim(R)$. So $\dim(T) = n-r-1$, but $\chi(M, R/\mathbf{x}R) = 0$ as $\#(\mathbf{x}) = r+1 > \dim(M)$. Hence, $\chi(M, T) = 0$ if and only if $\chi(M, N) = 0$. In a similar manner we can show M is a Cohen-Macaulay module. ■

Proposition 7 *Let R be Gorenstein, M, N are Cohen-Macaulay, where M has finite projective dimension and $\ell(M \otimes_R N) < \infty$, $i = \dim(R) - \dim(M) - \dim(N)$, $r = \dim(M)$, $s = \dim(N)$, $\check{M} = \mathrm{Ext}_R^{n-r}(M, R)$, and $\check{N} = \mathrm{Ext}_R^{n-s}(N, R)$. Now*

$$\chi(M, N) = (-1)^i \chi(\check{M}, \check{N}).$$

Sketch of Proof The crucial step is a simple spectral sequence argument. First note

$$\ell(\mathrm{Tor}_j^R(M,N)) = \ell(\mathrm{Tor}_j^R(M,N)\check{\ }).$$

Now write

$$\begin{aligned}
\mathrm{Tor}_j^R(M,N) &= \mathrm{Ext}_R^n(\mathrm{Tor}_j^R(M,N),R),\\
&= \mathrm{Ext}_R^{n+j-(n-s)}(M,\mathrm{Ext}_R^{n-s}(N,R)),\\
&= \mathrm{Ext}_R^{n-r+(i-j)}(M,\mathrm{Ext}_R^{n-s}(N,R)),\\
&= \mathrm{Tor}_{i-j}^R(\check{M},\check{N}).
\end{aligned}$$

■

Corollary 7.1 *If* $\dim(M)+\dim(N)=\dim(R)-1$, *then* $\chi(M,N) = -\chi(\check{M},\check{N})$.

Corollary 7.2 *If* $M \simeq \check{M}$, $N \simeq \check{N}$, *and* $\dim(R)-\dim(M)-\dim(N)$ *is odd, then* $\chi(M,N)=0$.

Corollary 7.3 *If* R, $R/\mathfrak{p}$, *and* $R/\mathfrak{q}$ *are all Gorenstein, where* $\mathfrak{p},\mathfrak{q} \in \mathrm{Spec}(R)$, *and* $\dim(R)-\dim(R/\mathfrak{p})-\dim(R/\mathfrak{q})$ *is odd, then* $\chi(R/\mathfrak{p},R/\mathfrak{q})=0$.

Corollary 7.4 *If* $i=0$, *then* $\ell(M\otimes_R N) = \ell(\check{M}\otimes_R \check{N})$.

Theorem 8 *If* R *is Gorenstein, then vanishing holds if and only if for every pair of Cohen-Macaulay modules* M, N *where* $\mathrm{proj}\dim(M)<\infty$ *and* $\dim(M)+\dim(N)=\dim(R)$, *we have* $\ell(M\otimes_R N) = \ell(M\otimes_R \check{N})$.

Sketch of Proof ($\Rightarrow$) Given M and N as above, we have by a result due to Serre that $\mathrm{Tor}_i^R(M,N)=0$ for $i>0$. So,

$$\chi(M,N) = \ell(M\otimes_R N).$$

Hence we have $\chi(M,\check{N}) = \ell(M\otimes_R \check{N})$. Now taking a prime filtration on N and using the additivity of χ we have

$$\chi(M,N) = \sum_{\dim(R/\mathfrak{p})=\dim(N)} \ell(N_\mathfrak{p})\chi(M,R/\mathfrak{p}) + \sum_{\dim(Q_i)<\dim(N)} \chi(M,Q_i)$$

Similarly we have

$$\chi(M,\check{N}) = \sum_{\dim(R/\mathfrak{p})=\dim(N)} \ell(\check{N}_\mathfrak{p})\chi(M,R/\mathfrak{p}) + \sum_{\dim(Q_i)<\dim(N)} \chi(M,Q_i)$$

But by vanishing we have $\sum \chi(M,Q_i)=0$. Since R is Gorenstein, we have $\ell(N_\mathfrak{p}) = \ell(\check{N}_\mathfrak{p})$. Thus $\chi(M,N)=\chi(M,\check{N})$.

Warning: One cannot use the same idea of additivity to prove an analogous statement when *both* M and N have finite projective dimension as $R/\mathfrak{p}$ may no longer have finite projective dimension.

($\Leftarrow$) Recall if M and N are Cohen-Macaulay, then $\dim(M) + \dim(N) = \dim(R) - 1$. Write

$$0 \to T \to \left(\frac{R}{(x_1, \ldots, x_r)}\right)^t \to N \to 0$$

where the $x_i's \in \operatorname{Ann}(N)$ as earlier. So

$$\chi(M, N) = t\ell(M/\mathbf{x}M) - \ell(M \otimes_R T),$$

which leads us to:

$$0 \to \left(\frac{R}{(x_1, \ldots, x_r)}\right)^t \to \check{T} \to \check{N} \to 0$$

This shows that

$$\chi(M, \check{N}) = \ell(M \otimes_R \check{T}) - t\ell(M/\mathbf{x}M),$$

So

$$\chi(M, N) = -\chi(M, \check{N}).$$

Applying the above technique once more we see $\chi(\check{M}, \check{N})$. From a previous proposition we see that $\chi(M, N) = -\chi(\check{M}, \check{N})$. Hence, $\chi(M, N) = 0$. Note that the argument for this part of the proof would work if the projective dimension of both M and N are finite. ■

Remark When $\dim(M) + \dim(N) = \dim(R)$ (as in the above theorem) we say we have a "proper intersection."

Sketch of Proof This is implied by the fact that for every pair of Cohen-Macaulay modules T and Q with finite projective dimension such that $\ell(T \otimes_R Q) < \infty$ and $\dim(T) + \dim(Q) = \dim(R)$, we have $\ell(T \otimes_R Q) = \ell(T \otimes_R \check{Q})$. ■

Remark If R is regular and is a complete intersection ring, then $\ell(T \otimes_R Q) = \ell(T \otimes_R \check{Q})$ can be shown by local Chern characters.

Theorem 9 *If R is Gorenstein and $\dim(R) \leqslant 5$, then vanishing holds for R-modules M, N when both M and N have finite projective dimension.*

Open Problem 10 *If R is Gorenstein and $\dim(R) > 5$, does vanishing holds for pairs of R-modules M, N when both M and N have finite projective dimension?*

Theorem 11 *If R is Gorenstein, then positivity (or nonnegativity) implies vanishing.*

Proof We can assume M to be Cohen-Macaulay with the projective dimension of M finite. We know that if $\dim(M) < \dim(R)$ and $\ell(N) < \infty$, then $\chi(M, N) = 0$.

Suppose that $R/\mathfrak{p}$ has the least dimension such that we do not know about vanishing. Then

1. We have

$$\chi(M, R/\mathfrak{p}^t) = \ell(R_\mathfrak{p})/\mathfrak{p}^t R_\mathfrak{p})\chi(M, R/\mathfrak{p}) + \sum_{\dim(Q_i)<\dim(R/\mathfrak{p})} \chi(M, Q_i).$$

   However, the last term in this sum goes to zero by our choice of $R/\mathfrak{p}$.

2. If $\dim(M) = r$ chose $x_1, \ldots, x_r \in \mathfrak{p}$ such that $\ell(M/\mathbf{x}M) < \infty$. Set $\overline{R} = R/\mathbf{x}$ and $\overline{M} = M/\mathbf{x}M$, then $\chi^R(M, R/\mathfrak{p}) = \chi^{\overline{R}}(\overline{M}, R/\mathfrak{p})$ as $\mathbf{x}$ is also an M-sequence and an R-sequence.

Thus we can assume that the projective dimension of M is finite, $\ell(M) < \infty$, and $\dim(R/\mathfrak{p}) = \dim(R) - 1$. So

$$\chi(M, R/\mathfrak{p}) = \lim_{t\to\infty} \frac{\chi(M, R/\mathfrak{p}^t)}{\ell(R_\mathfrak{p}/\mathfrak{p}^t R_\mathfrak{p})}$$

Now look at

$$0 \to \mathfrak{p}^t \to R \to R/\mathfrak{p}^t \to 0$$

So, $\chi(M, R/\mathfrak{p}^t) = \ell(M) - \chi(M, \mathfrak{p}^t)$. Now we have

$$\chi(M, R/\mathfrak{p}) = -\lim_{t\to\infty} \frac{\chi(M, \mathfrak{p}^t)}{\ell(R_\mathfrak{p}/\mathfrak{p}^t R_\mathfrak{p})}$$

as the $\ell(M)/\ell(R_\mathfrak{p}/\mathfrak{p}^t R_\mathfrak{p})$ term goes to zero in the limit, note $\dim(\mathfrak{p}^t) = \dim(R)$. If positivity or nonnegativity holds, then $\chi(M, \mathfrak{p}^t) \geqslant 0$ and thus $\chi(M, R/\mathfrak{p}) \leqslant 0$.

So take $y_1, \ldots, y_n$ a maximal R-sequence. Since $\ell(M) < \infty$ we may assume that $y_i \in \operatorname{Ann}(M)$. Write

$$0 \to N \to (R/\mathbf{y})^t \to M \to 0$$

Then $\chi((R/\mathbf{y})^t, R/\mathfrak{p}) = \chi(M, R/\mathfrak{p}) + \chi(N, R/\mathfrak{p})$. But the left-hand side is zero by a result due to Serre and so each term of the right-hand side is less than or equal to zero. Thus both $\chi(M, R/\mathfrak{p}) = 0$ and $\chi(N, R/\mathfrak{p}) = 0$. ■

2 χ_i-Conjecture

In this section we will assume that R is local, M, N are R-modules, the projective dimension of M is finite, $\ell(M \otimes_R N) < \infty$, and we define

$$\chi_i(M, N) := \sum_{j=0}^{\operatorname{proj}\dim(M)-i} (-1)^j \ell(\operatorname{Tor}^R_{i+j}(M, N)).$$

Conjecture 12 (Serre) *If R is a regular local ring, then $\chi_i(M, N) \geqslant 0$, or $\chi_i(M, N) = 0$ if and only if $\operatorname{Tor}^R_j(M, N) = 0$ for $j \geqslant i$.*

Remark in the above conjecture, the conclusion $\operatorname{Tor}^R_j(M, N) = 0$ for $j \geqslant i$ implies rigidity.

Theorem 13 (Serre-Auslander) *The above conjecture is true when R is of equal characteristic.*

Theorem 14 (Lichtenbaum) *The above conjecture is true when R is unramified for all χ_i except possibly $i = 1$.*

Theorem 15 (Hochster) *The above conjecture is true when R is unramified for χ_1.*

Remark Gabber also claims to have independently proven the above conjecture when R is unramified for χ_1.

Open Problem 16 *The above conjecture is open if R is ramified. To clarify, it is still open when*

$$R = \frac{V[\![x_1, \dots, x_n]\!]}{f}$$

where

$$\begin{aligned} f &= x_n^t + a_1 x^{t-1} + \cdots + a_n, \\ a_i &\in (\mathfrak{p}, x_1, \dots, x_{n-1}), \\ a_t &\in (\mathfrak{p}, x_1, \dots, x_{n-1}) - (\mathfrak{p}, x_1, \dots, x_{n-1})^2. \end{aligned}$$

In this case, $S = R \widehat{\otimes}_V R$ is no longer regular.

Theorem 17 *If R is a regular local ring where the χ_2-conjecture is valid, then $\chi(M, N) > 0$ when M is Cohen-Macaulay and $\dim(M) + \dim(N) = \dim(R)$.*

Remark The above conjectures make sense when R is not regular and the projective dimension of M or the projective dimension of N is finite.

2.1 Counterexamples

Example (Dutta-Hochster-Mclaughlin) Let

$$R = \left(\frac{k[X, Y, U, V]}{(XY - UV)} \right)_{(X,Y,U,V)}.$$

Now there exists an R-module M such that $\ell(M) < \infty$, $\operatorname{proj}\dim(M) < \infty$, $\chi(M, R/\mathfrak{p}) = -1 \neq 0$, $\dim(M) = 0$, $\dim(R/\mathfrak{p}) = 1$ where $\mathfrak{p} = (X, U)$, and hence positivity is false, which implies χ_i is false.

Example (Levine) Let

$$R = \left(\frac{k[X_1, \dots, X_n, Y_1, \dots, Y_n]}{\sum X_i Y_i} \right)_{(X_1, \dots, X_n, Y_1, \dots, Y_n)}.$$

This was done using non-constructive K-theoretic techniques.

Example (Roberts-Srinivas)

1. $R = k[X, Y, Z, W]/f$, where f has degree three and k is separable and algebraically closed - the coordinate ring of a cubic surface in $\mathbb{P}^3$.

2. R the coordinate ring $\mathbb{P}^n \times \mathbb{P}^n$.

Theorem 18 (Roberts, Gillet-Soulé) *Vanishing holds over complete intersection rings when both M and N have finite projective dimension.*

Theorem 19 (Dutta) *There exist complete intersection rings R along with R-modules M and N both with finite projective dimension such that $\chi(M, N) = 0$ but $\chi_2(M, N) < 0$. In fact, one can produce examples where all the χ_i's are negative for $i \geqslant 2$!*

In light of the above theorem, we are not sure whether we should believe the positivity conjecture when both M and N have finite projective dimension over complete intersection rings.

To prove the above theorem, we need the following special case of a theorem by Auslander and Bridger.

Theorem 20 (Auslander-Bridger) *Let R be Gorenstein and N any finitely generated R-module, then there exists an exact sequence*

$$0 \to T \to N \oplus R^t \to Q \to 0$$

where $\operatorname{proj}\dim(Q) < \infty$ *and T is a maximal Cohen-Macaulay module.*

Theorem 21 (Auslander-Buchweitz) *Let R be Gorenstein and N any finitely generated R-module, then there exists an exact sequence*

$$0 \to N \to Q \to T \to 0$$

where $\operatorname{proj}\dim(Q) < \infty$ *and T is a maximal Cohen-Macaulay module.*

Definition Given a pair M, N such that $\operatorname{proj}\dim(M) < \infty$, $\ell(M \otimes_R N) < \infty$, and $\dim(M) + \dim(N) = \dim(R)$, we say a finitely generated R-module N' is a **companion module** of N with respect to M if the following hold:

1. $\dim(N') = \dim(N)$.

2. $\operatorname{depth}(N') = \dim(N') - 1$.

3. $\ell(M \otimes_R N') < \infty$ and $\chi(M, N') = \chi(M, N)$.

Proposition 22 *With the above setup, if R is Gorenstein, N has a companion module.*

Proof If $\dim(M) = r$ we can find $x_1, \ldots, x_n \in \operatorname{Ann}(N)$ a system of parameters that is an R-sequence. Set $\overline{R} = R/\mathbf{x}R$, so M is an $\overline{R}$-module. Applying Auslander-Bridger over $\overline{R}$,

$$0 \to T \to N \oplus \overline{R}^t \to Q \to 0,$$

where Q and T are $\overline{R}$-modules and $\operatorname{proj}\dim(Q) < \infty$ and T is a maximal Cohen-Macaulay module. Now we have two cases. Case a: $\dim(Q) = \dim(\overline{R})$; and case b: $\dim(Q) < \dim(\overline{R})$. We want to reduce case a to case b. By the lectures of Paul Roberts in this mini-course, we have that $\dim(Q) = \dim(\overline{R})$ and $\operatorname{proj}\dim(Q) < \infty$ implies that $\operatorname{Supp}(Q) = \operatorname{Supp}(\overline{R})$. If S is the set of non-zero-divisors of $\overline{R}$, then $S^{-1}Q$ is $S^{-1}\overline{R}$-free of rank s. Therefore we have the exact sequence

$$0 \to \overline{R}^s \to Q \to Q' \to 0,$$

where $\dim(Q') < \dim(\overline{R})$ and the $\operatorname{proj}\dim(Q') < \infty$. So we have a diagram that looks like:

$$\begin{array}{ccccccccc}
 & & & & & & 0 & & \\
 & & & & & & \downarrow & & \\
 & & & & & & \overline{R}^s & & \\
 & & & & & & \downarrow & & \\
0 & \longrightarrow & T & \longrightarrow & N \oplus \overline{R}^t & \longrightarrow & Q & \longrightarrow & 0 \\
 & & & & & & \downarrow & & \\
 & & & & & & Q' & & \\
 & & & & & & \downarrow & & \\
 & & & & & & 0 & &
\end{array}$$

From this we obtain the exact sequence:

$$0 \to T \oplus \overline{R}^s \to N \oplus \overline{R}^t \to Q' \to 0$$

So $\dim(Q') < \dim(\overline{R})$.

Now we may assume case b. Write

$$0 \to T \to N \oplus \overline{R}^t \to Q \to 0$$

with $\dim(Q) < \dim(\overline{R})$. So we have

$$\chi^R(M, N) + t\chi^R(M, \overline{R}) = \chi^R(M, Y) + \chi^R(M, Q)$$

but

$$\chi^R(M, Q) = \sum (-1)^i \chi(\operatorname{Tor}_i^R(M, R/\mathbf{x}), Q).$$

Since $\operatorname{Tor}_i^R(M, R/\mathbf{x})$ has finite length, we are left with

$$\chi(M, N) = \chi(M, T) - t\chi(M, \overline{R}).$$

Since $\dim(Q) < \dim(\overline{R})$,

$$0 \to \overline{R}^t \to T \to N' \to 0$$

is an exact sequence. So

$$\chi(M, N') = \chi(M, T) - t\chi(M, \overline{R}) = \chi(M, N).$$

So $\operatorname{depth}(N') = \dim(N') - 1$ by depth counting, $\dim(N') = \dim(N)$. ∎

2.1.1 Discussion of Proof

Step 1 R is Gorenstein, so suppose vanishing does not hold. So we can find M Cohen-Macaulay with finite projective dimension, $\mathfrak{p}$ a prime ideal such that $\chi(M, R/\mathfrak{p}) > 0$, $\dim(M) + \dim(R/\mathfrak{p}) < \dim(R)$, and $\chi(M, R/\mathfrak{q}) = 0$ if $\mathfrak{q} \supset \mathfrak{p}$.

Step 2 From the previous section, we may assume that $\ell(M) < \infty$ and so we have

$$\chi(M, R/\mathfrak{p}) = \frac{\chi(M, R/\mathfrak{p}^t)}{\ell(R_\mathfrak{p}/\mathfrak{p}^t R_\mathfrak{p})} = \frac{-\chi(M, \mathfrak{p}^t)}{\ell(R_\mathfrak{p}/\mathfrak{p}^t R_\mathfrak{p})}.$$

So $\chi(M, R/\mathfrak{p}) > 0$ which implies $\chi(M, \mathfrak{p}^t) < 0$, note that $\dim(\mathfrak{p}^t) = \dim(R)$.

Step 3 By an easy spectral sequence argument (which reduces to a long exact sequence) we find

$$\chi(M, N) > \ell(\mathrm{Tor}_1^R(\check{M}, \mathrm{Ext}_R^1(N, R)) - \ell(\check{M} \otimes_R \mathrm{Ext}_R^1(N, R))$$

$\dim(\mathrm{Ext}_R^1(N, R)) < \dim(R)$ since R is Gorenstein. Suppose that

$$\chi(\check{M}, \mathrm{Ext}_R^1(N, R)) = 0.$$

Then $0 > \chi(M, N) > \chi_2(\check{M}, \mathrm{Ext}_R^1(N, R))$. Letting $x \in \mathrm{Ann}(\mathrm{Ext}_R^1(N, R))$ a non-zero-divisor on R, apply Auslander-Buchweitz to $\mathrm{Ext}_{\overline{R}}^1(N, R)$. Write

$$0 \to \mathrm{Ext}_{\overline{R}}^1(N, R) \to Q' \to T \to 0,$$

with the projective dimension of Q' finite and T a maximal Cohen-Macaulay module over $\overline{R}$. We have

$$\chi(\check{M}, T) = \ell(\mathrm{Tor}_0^R(\check{M}, T)) - \ell(\mathrm{Tor}_1^R(\check{M}, T)).$$

So $\chi_2(\check{M}, \mathrm{Ext}^1(N, R)) = \chi_2(\check{M}, Q') < 0$, but $\chi(\check{M}, Q') = 0$.

The condition $\chi(\check{M}, \mathrm{Ext}_R^1(N, R)) = 0$ happens:

1. For all counterexamples to vanishing listed above,
2. When R is Gorenstein of dimension 3.

3 Some on Positivity

In this section we will assume that R is local and Noetherian, $\dim(R) = d$, $\mathrm{char}(R) = p$ where p is a prime, and that $R/\mathfrak{m}$ is perfect (Cohen-Macaulay with finite projective dimension) for convenience. M and N will be R-modules with $\ell(M \otimes_R N) < \infty$ and $\dim(M) + \dim(N) = \dim(R)$. Finally, f will denote the Frobenius endomorphism, specifically:

$$f : R \to R \qquad\qquad f^n : R \to R$$
$$r \mapsto r^p \qquad\qquad r \mapsto r^{p^n}$$

The notation ${}^{f^n}R$ represents the R-algebra structure defined by

$$r \cdot x := r^{p^n} x \qquad \text{and} \qquad x \cdot r := xr,$$

where $x \in {}^{f^n}R$. The notation ${}^{f^n}N$ represents the left R-module structure defined by

$$r \cdot x := r^{p^n} x$$

where $x \in {}^{f^n}N$. We define the Frobenius functor $\mathbf{F}$ via

$$\mathbf{F^n}(-) := - \otimes_R {}^{f^n}R,$$

where the R-module structure is the normal one on the right.

Theorem 23 (Peskine-Szpiro) *If* $\operatorname{proj\,dim}(M) < \infty$, *then* $\operatorname{proj\,dim}(\mathbf{F^n}(M)) < \infty$. *Also* $\operatorname{Supp}(\mathbf{F^n}(M)) = \operatorname{Supp}(M)$, *so* $\ell(M \otimes_R N) = \ell(\mathbf{F^n}(M) \otimes_R N) < \infty$.

3.1 Some Facts

Supposing $\operatorname{proj\,dim}(M) < \infty$ and we have $M_R \xrightarrow{f^n} {}_R N$, we have

$$\operatorname{Tor}_i^R(M, {}^{f^n}N) = \operatorname{Tor}_i^R(\mathbf{F^n}(M), N).$$

This is because given a resolution $F_\bullet$ of M,

$$F_\bullet \otimes_R {}^{f^n}N \simeq F_\bullet \otimes_R {}^{f^n}R \otimes_R N \simeq \mathbf{F^n}(F_\bullet) \otimes_R N.$$

Now supposing R is a complete local domain, where $k = R/\mathfrak{m}$, we have the following diagram:

$$\begin{array}{ccc} R & \xhookrightarrow{\quad f^n \quad} & R \\ {\scriptstyle\text{module finite}}\uparrow & & \uparrow \\ k[[X_1, \ldots, X_d]] & \xhookrightarrow{\ f^n\ } & k[[X_1, \ldots, X_d]] \end{array}$$

Note that because k is perfect, the image of the bottom map is $k[[X_1^{p^n}, \ldots, X_d^{p^n}]]$. So the torsion-free rank of ${}^{f^n}R$ is p^{dn}.

Now we have a question: When $\operatorname{proj\,dim}(M) < \infty$, how are $\chi(\mathbf{F^n}(M), N)$ and $\chi(M, N)$ related?

When attacking this question we may assume that $N = R/\mathfrak{p} = \overline{R}$ as χ is additive. So we have

$$0 \to \bigoplus_{i=1}^{p} \overline{R} \to {}^{f}\overline{R} \to Q \to 0$$

where $\dim(Q) < \dim(\overline{R})$. So

$$0 \to \bigoplus_{i=1}^{p^r} {}^{f^{n-1}}\overline{R} \to {}^{f^n}\overline{R} \to {}^{f^{n-1}}Q \to 0,$$

and so

$$\chi(M, {}^{f^n}\overline{R}) = \underbrace{p^r \chi(M, {}^{f^{n-1}}\overline{R})}_{\text{repeat for this term, etc.}} + \chi(M, {}^{f^{n-1}}Q).$$

We obtain:

$$\chi(\mathbf{F^n}(M), \overline{R}) = p^{nr}\chi(M, \overline{R}) + c_n \chi(M, R/\mathfrak{p}_i) + \cdots .$$

By recalling: $\chi(\mathbf{F^n}(M), N) = \chi(M, {}^{f^n}N)$.

Definition Define:

$$\chi_\infty := \lim_{n\to\infty} \frac{\chi(\mathbf{F^n}(M), N)}{p^{n\cdot \operatorname{codim}(M)}}$$

and

$$\alpha_\infty := \lim_{n\to\infty} \frac{\chi(\mathbf{F^n}(M), N)}{p^{n\cdot \dim(M)}}$$

Note that since $\dim(M) + \dim(N) \leqslant \dim(R)$, we have $\dim(M) \leqslant \operatorname{codim}(N)$ and that we have equality in the positivity case.

Theorem 24 *We have that*

$$\alpha_\infty(M, R/\mathfrak{p}) = \chi(M, R/\mathfrak{p}) + \sum_{\dim(R/\mathfrak{p}_i) < \dim(R/\mathfrak{p})} c_i \chi(M, R/\mathfrak{p}_i)$$

where each $c_i \in \mathbb{Q}$.

So when $\dim(M) + \dim(N) < \dim(R)$, $\chi_\infty(M, N) = 0$ and when $\dim(M) + \dim(N) = \dim(R)$, $\chi_\infty(M, N) = \alpha_\infty(M, N)$.

Theorem 25 *If R is local,* $\operatorname{proj}\dim(M) < \infty$, *$M$ is Cohen-Macaulay, and* $\dim(M) + \dim(N) = \dim(R)$, *then* $\chi_\infty(M, N) > 0$.

Remark If M is not assumed to be Cohen-Macaulay, then the theorem is still open!

Proof of the above statement can be made much simpler by the fact:

$$\lim_{n\to\infty} \frac{\ell(\operatorname{Tor}_i^R(\mathbf{F^n}(M), N)}{p^{n\cdot \operatorname{codim}(M)}} = \begin{cases} 0 & \text{for } i > 0. \\ \neq 0 & \text{for } i = 0. \end{cases}$$

The first proof of this fact needed R to be Gorenstein. Now we know it for all R. Also note that this is really a special case of the New Intersection Theorem.

Theorem 26 (Seibert)

1. *If $F_\bullet$ is a finite complex of finitely generated free R-modules, N a finitely generated R-module of dimension r such that for each $i \geqslant 0$,*

$$\ell(H_i(F_\bullet \otimes_R N)) < \infty,$$

define

$$\chi(F_\bullet, N) = \sum (-1)^i \ell(H_i(F_\bullet \otimes_R N)).$$

Then $\chi(\mathbf{F^n}(F_\bullet), N) = c_r p^{nr} + c_{r-1}p^{n(r-1)} + \cdots + c_0$, *where* $c_i \in \mathbb{Q}$.

2. *Given an exact sequence*

$$0 \to N' \to N \to N'' \to 0,$$

we have for some constant K

$$\ell(H_i(F_\bullet \otimes_R N)) - \ell(H_i(F_\bullet \otimes_R N')) - \ell(H_i(F_\bullet \otimes_R N'')) \leqslant K p^{n(r-1)}.$$

Applications

Theorem 27 *If R is a regular local ring, p a non-zero-divisor on M, where M is a Cohen-Macaulay module, and $p^t N = 0$ for some $t > 0$, then $\chi(M, N) > 0$.*

Proof Write

$$N \supset pN \supset \cdots \supset p^{t-1}N \supset 0$$

$\chi(M, N) = \sum \chi(M, p^i N / p^{i+1} N)$. So we can assume that $pN = 0$. Since p is a non-zero-divisor on R and on M we have

$$\chi^R(M, N) = \chi^{R/pR}(M/pM, N)$$

but $\overline{M} = M/pM$ is Cohen-Macaulay. So by vanishing,

$$\underbrace{\chi_\infty^{\overline{R}}(\overline{M}, N)}_{>0} \underbrace{=}_{\text{by vanishing}} \chi^{\overline{R}}(\overline{M}, N) = \chi^R(M, N)$$

So we see that $\chi^R(M, N) > 0$. ■

Remark This theorem was extended by Kurano and Roberts.

Theorem 28 (Foxby) *If R is local and M is an R-module with finite projective dimension and the dimension of N is one, then $\chi(M, N) > 0$.*

Theorem 29 (Tennison) *If R is regular, M and N are R-modules, and suppose that*

$$\ell(G_{\mathfrak{m}}(M) \otimes G_{\mathfrak{m}}(N)) < \infty.$$

Then $\chi(M, N) = e_{\mathfrak{m}}(M) e_{\mathfrak{m}}(N)$.

More generally, if $M = R/\mathfrak{p}$, $N = R/\mathfrak{q}$, $Y = \operatorname{Spec}(M)$, $Z = \operatorname{Spec}(N)$, and $\widetilde{Y}$, $\widetilde{Z}$ are the blow-ups of Y and Z, then

$$\ell(G_{\mathfrak{m}}(M) \otimes G_{\mathfrak{m}}(N)) < \infty \Leftrightarrow \widetilde{Y} \cap \widetilde{Z} = \varnothing.$$

Theorem 30 (Dutta) *If $\widetilde{Y} \cap \widetilde{Z}$ is a finite set of points, then $\chi(M, N) \geqslant e_{\mathfrak{m}}(M) e_{\mathfrak{m}}(N)$.*

The proof of this last theorem uses nonnegativity results by Gabber and Intersection Theory as introduced in Fulton's book.

3.2 Chow Groups

Let $\mathbb{A}_i(R)$ denote the ith Chow Group of R.

Theorem 31 (Claborn-Fossum)

1. *For a field k, if $R = k[X_1, \ldots, X_n]$, then $\mathbb{A}_i(R) = 0$ for $i < n$ and $\mathbb{A}_n \simeq \mathbb{Z}$. For a DVR V, if $R = V[X_1, \ldots, X_n]$, then $\mathbb{A}_i(R) = 0$ for $i < n + 1$.*
2. *For a field k, if $R = k[[X_1, \ldots, X_n]]$, then $\mathbb{A}_i(R) = 0$ for $i < n$ and $\mathbb{A}_n \simeq \mathbb{Z}$. For a DVR V, if $R = V[[X_1, \ldots, X_n]]$, then $\mathbb{A}_i(R) = 0$ for $i < n + 1$.*

Conjecture 32 (Gersten) *If R is any regular local ring, of dimension n, then $\mathbb{A}_i(R) = 0$ for $i < n$.*

Theorem 33 (Quillen) *If R is a regular local ring smooth over k, then $\mathbb{A}_i(R) = 0$ for $i < n$,*

His proof was geometric, looking at the tangent cone and tangent space.

Theorem 34 (Gillet-Levine) *If R is regular local and smooth over an excellent DVR V, then $\mathbb{A}_i(R) = 0$ for $i < n$.*

This proof is an extension of Quillen's arguments.

Remark Cannot assume R is complete for the Chow group problem.

Question For $R \to \widehat{R}$, can we say

$$\mathbb{A}_i(R) \hookrightarrow \mathbb{A}_i(\widehat{R})$$

While this is not true in general, (Hochster gave a counterexample in the non-normal case) we do have this:

Theorem 35 (Kamoi-Kurano) *If R is an excellent regular local ring, then*

$$\mathbb{A}_i(R) \hookrightarrow \mathbb{A}_i(\widehat{R}).$$

Gersten's Conjecture is still open when R is ramified regular local. We have the following result:

Theorem 36 (Dutta) *If R is a ramified regular local ring, then $\mathbb{A}_1(R)$.*

For

$$R = \frac{V[[X_1, \ldots, X_n]]}{\mathfrak{p} - \sum x_i^2},$$

the result that $\mathbb{A}_i(R) = 0$ when $i < n$ was first proved by Levine using K-theoretic techniques. Dutta gives an algebraic proof which does not work for when the ring R is not so nice.

Conjecture 37 (Bass-Quillen) *If R is a regular local ring and P a finitely generated projective module over $R[X_1, \dots, X_n]$, then $P = P_0 \otimes_R R[\mathbf{X}]$ where P_0 is a finitely generated projective module over R.*

The case where R is a field, conjectured by Serre, was proved independently by Quillen and Suslin.

Theorem 38 (Lindel) *Proved the above conjecture when R is geometrically regular local ring. That is, when R is a local ring which is smooth over k.*

Lindel had a special proposition, which we will call a theorem:

Theorem 39 (Lindel) *If A is an affine domain over k of dimension d with maximal ideal $\mathfrak{m}$ such that $A_{\mathfrak{m}}$ is a regular local ring, and $A/\mathfrak{m}$ is a finite separable extension of k, then there exists $x_1, \dots, x_t \in A$ such that*

1. *$A = k[x_1, \dots, x_t]$ and $\mathfrak{m} = (f(x_1), x_2, \dots, x_t)$ where f is the monic irreducible polynomial of $\overline{x}_1 \in A/\mathfrak{m}(= k(\overline{x}_1))$ over k.*
2. *$B = k[x_1, \dots, x_d]$, $\mathfrak{n} = B \cap \mathfrak{m} = (f(x_1), x_2, \dots, x_d)$ and $B_{\mathfrak{n}} \to A_{\mathfrak{m}}$ is étale (flat with $\Omega_{A_{\mathfrak{m}}/B_{\mathfrak{n}}} = 0$).*

Using Zariski's Main Theorem we obtain an extension of this result:

Theorem 40 *If $(R, \mathfrak{m}, k)$ is a regular local ring which is smooth over k, or an excellent DVR V, and $R/\mathfrak{m}$ is separably generated over k or $V/\mathfrak{m}_V$, then there exists $(B, \mathfrak{n}, k)$ another regular local ring contained in R such that*

1. *$B = W[X_1, \dots, X_d]_{(f(X_1), X_2, \dots, X_d)}$ where W is a field or an excellent DVR contained in R and $f(X_1)$ is a monic irreducible polynomial in $W[X_1]$.*
2. *If we take any $a \in \mathfrak{m}^2$ ($a \neq 0$), then we can choose $(B, \mathfrak{n}, k)$ such that $B \to R$ is étale, $B \cap aR = (h)$ and $B/hb \simeq R/aR$.*

This theorem helps us to give an alternate proof of Serre's Theorem on Intersection-Multiplicities without using "complete-Tor." This also provides an alternate proof of Quillen's Theorem on Chow groups. Take $a \in \operatorname{Ann}(M) \cap \operatorname{Ann}(N) \cap \mathfrak{m}^2$ and apply the above theorem. This pulls back our problem to the polynomial case. Thus, it brings the Intersection-Multiplicities and the Chow group problems back to the polynomial case. Hence, only the ramified case is left.

4 Canonical Element Conjecture

Let $(A, \mathfrak{m}, k)$ be a local ring of dimension n and $\mathbf{x} = x_1, \dots, x_n$ a system of parameters for A. If we consider the Koszul complex $K(\mathbf{x}, A)$ we can find a chain-map from the Koszul complex to a minimal free resolution $F_\bullet$ of k:

$$
\begin{array}{ccccccccccccccc}
 & & 0 & \longrightarrow & A & \longrightarrow & A^n & \longrightarrow \cdots \longrightarrow & A^n & \longrightarrow & A & \longrightarrow & A/\mathbf{x} & \longrightarrow & 0 \\
 & & & & \downarrow{\scriptstyle \varphi_n} & & \downarrow{\scriptstyle \varphi_{n-1}} & & \downarrow{\scriptstyle \varphi_1} & & \Big| {\scriptstyle \mathrm{id}=\varphi_0} & & \downarrow & & \\
\cdots & \longrightarrow & A^{t_{n+1}} & \longrightarrow & A^{t_n} & \longrightarrow & A^{t_{n-1}} & \longrightarrow \cdots \longrightarrow & A^{t_1} & \longrightarrow & A & \longrightarrow & k & \longrightarrow & 0
\end{array}
$$

Conjecture 41 *In the situation above, $\varphi_n \neq 0$ for any system of parameters* $\mathbf{x}$.

4.1 Supposing $\varphi_n = 0$

Suppose $\varphi_n = 0$. Applying $\mathrm{Hom}_A(-, A)$, and denoting this with a $(-)^*$, to the diagram above, we obtain:

$$
\begin{array}{ccccccccccccc}
0 & \longrightarrow & A & \longrightarrow & (A^{t_1})^* & \longrightarrow & \cdots & \longrightarrow & (A^{t_{n-1}})^* & \longrightarrow & (A^{t_n})^* & \longrightarrow & \cdots \\
 & & \Big| \, \scriptstyle\mathrm{id} & & \Big\downarrow \scriptstyle\varphi_1^* & & & & \Big\downarrow \scriptstyle\varphi_{n-1}^* & & \Big\downarrow \scriptstyle\varphi_n^*=0 & & \\
0 & \longrightarrow & A & \longrightarrow & A^n & \longrightarrow & \cdots & \longrightarrow & A^n & \longrightarrow & A & \longrightarrow & 0
\end{array}
$$

Letting $G = \mathrm{Coker}(A^{\binom{n}{2}} \to A^n)$ and $\widetilde{G} = \mathrm{Coker}((A^{t_{n-2}})^* \to (A^{t_{n-1}})^*)$

$$
\begin{array}{ccccccccc}
0 & \longrightarrow & \mathrm{Ext}_A^{n-1}(k, A) & \longrightarrow & \widetilde{G} & \longrightarrow & \mathrm{Im}(\widetilde{G} \hookrightarrow (A^{t_n})^*) & \longrightarrow & 0 \\
 & & \Big\downarrow & \swarrow & \Big\downarrow \scriptstyle\eta & & \Big\downarrow \scriptstyle 0 & & \\
0 & \longrightarrow & H_1(\mathbf{x}, A) & \longrightarrow & G & \longrightarrow & \mathbf{x}A & \longrightarrow & 0
\end{array}
$$

So, we have the complexes:

$$
\begin{array}{ccccccccccccc}
0 & \longrightarrow & A & \longrightarrow & (A^{t_1})^* & \longrightarrow & \cdots & \longrightarrow & (A^{t_{n-1}})^* & \longrightarrow & \widetilde{G} & \longrightarrow & 0 \\
 & & \Big| \, \scriptstyle\mathrm{id} & & \Big\downarrow \scriptstyle\varphi_1^* & & & & \Big\downarrow \scriptstyle\varphi_{n-1}^* & & \Big\downarrow \scriptstyle\eta & & \\
0 & \longrightarrow & A & \longrightarrow & A^n & \longrightarrow & \cdots & \longrightarrow & A^n & \longrightarrow & G & \longrightarrow & 0
\end{array}
$$

and

$$
\begin{array}{ccccccccccccc}
 & & & & & & & & & & H_1(\mathbf{x}, A) & & \\
 & & & & & & & & & & \Big\downarrow & & \\
0 & \longrightarrow & A & \longrightarrow & A^n & \longrightarrow & \cdots & \longrightarrow & A^n & \longrightarrow & G & \longrightarrow & 0
\end{array}
$$

Though $K_\bullet(\mathbf{x}, A)$ is not necessarily exact, we still can prove the following:

Proposition 42 *There exists a free complex $L_\bullet$ of finitely generated free modules and maps $\psi_\bullet : L_\bullet \to K_\bullet(\mathbf{x}, A)_{+1}$ such that*

1. *$L_\bullet$ is minimal and*
2. *$\psi_\bullet$ induces an isomorphism $H_i(L_\bullet) \simeq H_i(K_\bullet(\mathbf{x}, A))_{+1}$ for $i > 0$.*

Then the mapping cone of $\psi_\bullet$ gives a free resolution of $\mathbf{x}A$.

This forces $\psi_{n-1} : A^{r_{n-1}} \to A$ to be onto. Actually, $\varphi_n \neq 0$ if and only if ψ_{n-1} is not onto, which is the case if and only if $K_\bullet(\mathbf{x}, A)$ embeds into the free minimal resolution of $A/\mathbf{x}A$. This seems to be Robert's way of looking at the Canonical Element Conjecture.

Consider the diagram

$$
\begin{array}{ccccccccccccc}
\cdots \longrightarrow A^{r_n} & \xrightarrow{\alpha_n} & A^{r_{n-1}} & \xrightarrow{\alpha_{n-1}} & A^{r_{n-2}} & \xrightarrow{\alpha_{n-2}} & \cdots & \longrightarrow & A^{r_0} & \longrightarrow & H_1(\mathbf{x}, A) & & \\
 & & \downarrow{\scriptstyle \psi_{n-1}} & & \downarrow{\scriptstyle \psi_{n-2}} & & & & \downarrow{\scriptstyle \psi_0} & & \hookdownarrow & & \\
0 & \longrightarrow & A & \longrightarrow & A^n & \longrightarrow & \cdots & \longrightarrow & A^n & \longrightarrow & G & \longrightarrow & 0
\end{array}
$$

and suppose that ψ_{n-1} is onto. Then we can break it up into:

1. $A^{r_{n-1}} = Ae_1 \oplus (\bigoplus_{i=2}^{r_{n-1}-1} Ae_i)$.
2. $\alpha_n(A) \subset \bigoplus_{i=2}^{r_{n-1}} Ae_i$

$\mathrm{Coker}(\alpha_n) = A \oplus S'_{n-1}$ so the cokernel is a free summand. So if the Canonical Element Conjecture is true, this cannot happen.

From this with some work we get the following theorem:

Theorem 43 *If $(A, \mathfrak{m}, k)$ is local, take a minimal resolution of k and let $S_i = \mathrm{Syz}^i(k)$. Then A is regular if and only if S_i has a free summand for some $i > 0$.*

Applying $\mathrm{Hom}_A(-, A)$ to the diagram above, we obtain

$$
\begin{array}{ccccccccccccc}
 & 0 \longrightarrow A \longrightarrow & A^n & \longrightarrow \cdots \longrightarrow & A^n & \longrightarrow & A & \longrightarrow & A/\mathbf{x} & \longrightarrow 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \downarrow{\scriptstyle \theta} & \\
P_\bullet : & 0 \longrightarrow & (A^{r_0})^* & \longrightarrow \cdots \longrightarrow & (A^{r_{n-2}})^* & \xrightarrow[\alpha^*_{n-1}]{} & (A^{r_{n-1}})^* & \longrightarrow & M & \longrightarrow 0
\end{array}
$$

where $M = \mathrm{Coker}(\alpha^*_{n-1})$ and $\theta(1) = \nu$, a minimal generator of M. such that $\mathbf{x}\nu = 0$. So we have that $P_\bullet$ is a complex of finitely generated free A-modules such that $\ell(H_i(P_\bullet)) < \infty$ for $i > 0$ and $H_0(P_\bullet)$ has a minimal generator killed by $\mathbf{x}$, and hence is killed by a power of $\mathfrak{m}$. Thus the Canonical Element Conjecture is true if and only if the Improved New Intersection Theorem is true. It is enough to prove the Improved New Intersection Conjecture when M is locally free on $\mathrm{Spec}(A) - \{\mathfrak{m}\}$.

Suppose that $\mathrm{depth}(A) = \dim(A) - 1$ and A is the homomorphic image of a Gorenstein ring R such that $\dim(R) = \dim(A)$. Then the Canonical Element Conjecture holds in the following cases:

1. $\mathrm{Ext}^1_R(A, R)$ is decomposable.
2. $\mathrm{Ext}^1_R(A, R)$ is cyclic.

Now if $\theta : \mathrm{Ext}^n_A(k, \Omega) \to H^n_{\mathfrak{m}}(\Omega)$ where $\Omega = \mathrm{Hom}_R(A, R)$, the Canonical Element Conjecture says that $\theta \neq 0$. Write

$$I^\bullet : \quad 0 \to \Omega \to I^0 \to I^1 \to \cdots \to I^{n-1} \to E \to 0$$

where E is the injective hull of $A/\mathfrak{m}A$. By the same kind of argument as used before, but now using injective complexes we get a complex of injective modules $J^\bullet$ with $\varphi^\bullet : I^\bullet \to J^\bullet$ such that $\varphi^\bullet$ induces an isomorphism on cohomology,

$$\begin{array}{ccccccccccccc}
0 \longrightarrow \Omega \longrightarrow I^0 \longrightarrow & I^1 & \longrightarrow \cdots \longrightarrow & I^{n-1} & \longrightarrow & E & \longrightarrow 0 & \\
& \Big\downarrow \varphi^0 & & \Big\downarrow \varphi^{n-2} & & \Big\downarrow \varphi^{n-1} & & \\
0 \longrightarrow & J^0 & \longrightarrow \cdots \longrightarrow & J^{n-2} & \longrightarrow & J^{n-1} & \longrightarrow J^n \longrightarrow 0 &
\end{array}$$

thus the mapping cone of $\varphi^\bullet$ gives an injective resolution of Ω.

Following the same line of arguments, we can show that $\theta \neq 0$ if and only if φ_{n-1} is not injective. Not injective means that the socle must get killed! See Shamash's article.

Using these ideas we get that

1. If $x \in \mathfrak{m}\,\mathrm{Ann}(\mathrm{Ext}^1_R(A,R))$, then A/xA satisfies the Canonical Element Conjecture.
2. If $\mathrm{Ext}^1_R(A,R) = 0$, then A satisfies the Canonical Element Conjecture. In particular
   (a) If Ω is S_3, A satisfies the Canonical Element Conjecture.
   (b) $0 \to \Omega \to R \to R/\Omega \to 0$, R/Ω satisfies the Canonical Element Conjecture.
   (c) If A is an almost complete intersection ring and p is a non-zero-divisor on A, then A satisfies the Canonical Element Conjecture.
   (d) If A is almost a complete intersection ring, with $A = R/\lambda R$. Take $x_1, \ldots, x_n$ a system of parameters of R. Is $\ell(A/\mathbf{x}) > \ell(\mathrm{Tor}^R_1(\mathbf{x}, R/\lambda R))$?

Remark For Canonical Element Conjecture, we may assume A is almost a complete intersection ring and that p is a parameter on A.

4.2 The Intersection Theorem in Characteristic p

Let us consider the Intersection Theorem in characteristic p which is due independently to both Roberts and Peskine-Szpiro.

The statement is as follows: Consider a complex of finitely generated free modules of length s

$$F_\bullet : \quad 0 \to F_s \to \cdots \to F_1 \to F_0 \to 0$$

where $\ell(H_i(F_\bullet)) < \infty$ and not all are zero for every i, then $s \geqslant d = \dim(A)$.

Theorem 44 *Let A be local with dimension d and of non-zero characteristic p. And consider the complex of free A-modules $F_\bullet$ with $\ell(H_i(F_\bullet)) < \infty$ for $i > 0$ and $H^0_\mathfrak{m}(H_0(F_\bullet)) \neq 0$. Assume $M = H_0(F_\bullet)$ is locally free on* $\mathrm{Spec}(A) - \mathfrak{m}$ *and take any finitely generated A-module N. Define*

$$\chi(F_\bullet, N) := \ell(H^0_\mathfrak{m}(M \otimes_A N)) + \sum_{i>0} (-1)^i (H_i(F_\bullet \otimes_A N)).$$

Similarly define

$$\chi_\infty(F_\bullet, N) := \lim_{n\to\infty} \frac{\chi(\mathbf{F^n}(F_\bullet), N)}{p^{na}}.$$

Then we have the following:

1. *If* $\dim(N) < d$, *then* $\chi_\infty(F_\bullet, N) = 0$.

2. (a) *If* $\dim(N) = d$ *and* $s < d$, *then* $\chi_\infty(F_\bullet, N) = 0$.
   (b) *If* $\dim(N) = d$ *and* $s = d$, *then* $\chi_\infty(F_\bullet, N) > 0$.

Corollary 44.1 *The Improved New Intersection Theorem is true is characteristic* p.

Proof M has a minimal generator which is killed by $\mathfrak{m}^t$. So,

$$M \to A/I$$

where the minimal generator maps onto $\overline{1}$ in A/I. Hence we get an onto map $\mathbf{F^n}(M) \to A/I^{[p^n]}$. This implies that

$$\lim_{n\to\infty} \frac{\ell(A/I^{[p^n]})}{p^{nd}} > 0.$$

But higher homologies go to zero in the limit, hence by the previous theorem, $s \geqslant d$. ∎

References and Further Reading

[1] M. Auslander and M. Bridger, Stable module theory, *Mem. Amer. Math. Soc.*, **94** (1969), 129.

[2] M. Auslander and R.-O. Buchweitz, The homological theory of Cohen-Macaulay approximations, *Mem. Soc. Math. de France*, **38** (1989), 5–37.

[3] H. Bass, Some problems in "classical" algebraic K-theory, *Algebraic K-theory II*, Lecture Notes in Math., **342** Springer-Verlag, (1972), 1–70.

[4] L. Claborn and R. Fossum, Generalizations of the notion of class group, *Illinois J. Math.*, **12** (1968), 228–253.

[5] S.P. Dutta, Frobenius and Multiplicities, *J. of Algebra*, **85** (1983), 424–448.

[6] ———, A Note on Chow Groups and Intersection Multiplicity of Modules, *J. of Algebra*, **161** (1993), 186–198.

[7] ———, A Note on Chow Groups and Intersection Multiplicity of Modules II, *J. of Algebra*, **171** (1995), 370–382.

[8] ______, A Theorem on Smoothness-Bass-Quillen, chow groups and Intersection Multiplicity of Serre, *Trans. Amer. Math. Soc.*, **352** (2000), 1635–1645.

[9] S.P. Dutta, M. Hochster, J. E. McLaughlin. Modules of finite projective dimension with negative intersection multiplicities, *Invent. Math.*, **79** (1985), 253–291.

[10] H.-B. Foxby, The MacRae Invariant, *Commutative Algebra*, London Math. Soc. Lecture Note Series **72** (1982), 121–128.

[11] W. Fulton, *Intersection Theory*, Springer-Verlag, (1984).

[12] H. Gillet and M. Levine, The relative form of Gersten's conjecture over a discrete valuation ring: the smooth case, *J. Pure Appl. Algebra*, **46** (1987), 59–71.

[13] H. Gillet and C. Soulé, K-théorie et nullité des multiplicités d'intersection, *C. R. Acad. Sc. Paris Ser. I Math.*, **300** (1985), 71–74.

[14] M. Hochster, Cohen-Macaulay modules, *Conference on Commutative Algebra*, Lecture Notes in Math., **311** Springer-Verlag, (1973), 120–152.

[15] ______, Nonnegativity of Intersection Multiplicities in Ramified Regular Local Rings following Gabber/De Jong/Berthelot, preprint.

[16] Y. Kamoi and K. Kurano, On Chow groups of G-graded rings, *Comm. Alg.*, **31** (2003), 2141–2160.

[17] K. Kurano and P. Roberts, Adams operations, localized Chern characters, and the positivity of Dutta multiplicity in characteristic 0, *Trans. Amer. Math. Soc.*, **352** (2000), 3103–3116.

[18] M. Levine, A K-theoretic approach to multiplicities. *Math. Ann.*, **271** (1985), 451–458.

[19] S. Lichtenbaum, On the vanishing of Tor in regular local rings, *Illinois J. of Math.*, **10** (1966), 220–226.

[20] H. Lindel, On the Bass-Quillen conjecture concerning projective modules over polynomial rings. *Invent. Math.*, **65** (1981/82), 319–323.

[21] C. Peskine and L. Szpiro, Dimension projective finie et cohomologie locale. *Publ. Math. I.H.E.S.*, **42** (1972), 47–119.

[22] ______, Syzygies et multiplicités. *C. R. Acad. Sci. Paris Sr. A*, **278** (1974), 1421–1424.

[23] D. Quillen, Projective modules over polynomial rings. *Invent. Math.*, **36** (1976), 167–171.

[24] ———, *Higher algebraic K-theory: I; Algebraic K-theory I—Higher K theories*, Lecture Notes in Math., **341** Springer-Verlag.

[25] P. Roberts, Two applications of dualizing complexes over local rings. *Ann. Sci. École Norm. Sup.*, **9** (1976), 103–106.

[26] ———, Intersection multiplicities in commutative algebra. *Algebra colloquium (Rennes, 1985)*, 25–39, *Univ. Rennes I, Rennes*, (1985).

[27] ———, Recent developments on Serre's multiplicity conjectures: Gabber's proof of the nonnegativity conjecture, *Enseign. Math.*, **44** (1998), 305–324.

[28] P. Roberts and V. Srinivas, Modules of finite length and finite projective dimension. *Invent. Math.*, **151** (2003), 1–27.

[29] G. Seibert, Complexes with homology of finite length and Frobenius functors, *J. Algebra*, **125** (1989), 278–287.

[30] J.-P. Serre, *Local Algebra.* Springer Monographs in Math., Springer, (2000).

[31] J. Shamash, The Poincaré series of a local ring. *J. Algebra*, **12** (1969), 453–470.

[32] A.A. Suslin, Projective modules over polynomial rings are free. *Dokl. Akad. Nauk SSSR*, **229** (1976), 1063–1066.

[33] B.R. Tennison, Intersection multiplicities and tangent cones, *Math. Proc. Cambridge Philos. Soc.*, **85** (1979), 33–42.