LOCAL COMOLOGY, LOCAL DUALITY AND BASIC NOTIONS OF TIGHT CLOSURE THEORY

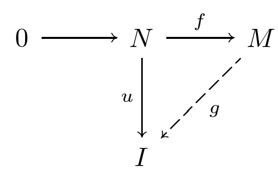
Abstract. This lectures were given by Florian Enescu at the mini-course on classical problems in commutative algebra held at University of Utah, June 2004. The references listed were used extensively in preparing these notes and the author makes no claim of originality. Moreover, he encourages the reader to consult these references for more details and many more results that had to be omitted due to time constraints.

This version has been typed and prepared by Bahman Engheta.

1. Injective modules, essential extensions, and local cohomology

Throughout, let R be a commutative Noetherian ring.

Definition. An R-module I is called *injective* if given any R-module monomorphism $f : N \to M$, every homomorphism $u : N \to I$ can be extended to a homomorphism $g : M \to I$, i.e. $g\,f = u$.



Or equivalently, if the functor $\mathrm{Hom}(\_, I)$ is exact.

Example. $\mathbb{Q}$ and $\mathbb{Q}/\mathbb{Z}$ are injective $\mathbb{Z}$-modules.

Definition. An R-module M is called *divisible* if for every $m \in M$ and M-regular element $r \in R$, there is an $m' \in M$ such that $m = rm'$.

Exercise. An injective R-module is divisible. The converse holds if R is a principal ideal domain.

A note on existence: If $R \to S$ is a ring homomorphism and I is an injective R-module, then $\mathrm{Hom}_R(S, I)$ is an injective S-module. [The S-module structure is given by $s \cdot \varphi(\_) := \varphi(s\_)$.] In particular, $\mathrm{Hom}_{\mathbb{Z}}(R, \mathbb{Q})$ is an injective R-module and any R-module can be embedded in an injective R-module.

Definition. An injective map $M \xrightarrow{h} N$ is called an *essential extension* if one of the following equivalent conditions holds:

a) $h(M) \cap N' \neq 0 \quad \forall\ 0 \neq N' \subseteq N$.
b) $\forall\ 0 \neq n \in N \quad \exists\ r \in R \quad$ such that $\quad 0 \neq rn \in h(M)$.
c) $\forall\ N \xrightarrow{\varphi} Q$, if $\varphi \circ h$ is injective, then φ is injective.

Example.
1) If R is a domain and $Q(R)$ its fraction field, then $R \subseteq Q(R)$ is an essential extension.
2) Let M be a submodule of N. By Zorn's lemma there is a maximal submodule $N' \subseteq N$ containing M such that $M \subset N'$ is an essential extension.
3) Let $(R, \mathfrak{m}, k)$ be a local ring and N an R-module such that every element of N is annihilated by some power of the maximal ideal $\mathfrak{m}$. Let $\mathrm{Soc}(N) := \mathrm{Ann}_N(\mathfrak{m})$ denote the socle of N. Then $\mathrm{Soc}(N) \subseteq N$ is an essential extension. (See exercise below.)

Note: The socle of a module N over a local ring $(R, \mathfrak{m}, k)$ is a k-vector space.

Exercise. An R-module N is Artinian if and only if $\mathrm{Soc}(N)$ is finite dimensional (as a k-vector space) and $\mathrm{Soc}(N) \subseteq N$ is essential.

Definition. If $M \subseteq N$ is an essential extension such that N has no proper essential extension, then $M \subseteq N$ is called a *maximal essential extension*.

Proposition.
a) *An R-module I is injective if and only if it has no proper essential extension.*
b) *Let M be an R-module and I and injective R-module containing M. Then any maximal essential extension of M contained in I is a maximal essential extension. In particular, it is injective and thus a direct summand of I.*
c) *If $M \subseteq I$ and $M \subseteq I'$ are two maximal essential extensions, then there is an isomorphism $I \cong I'$ that fixes M.*

Definition. A maximal essential extension of an R-module M is called an *injective hull* of M, denoted $E_R(M)$.

Definition. Let M be an R-module. Set $I^{-1} := M$ and $I^0 := E_R(M)$. Inductively define $I^n := E_R({}^{I^{n-1}}/_{\mathrm{im}(I^{n-2})})$. Then the acyclic complex

$$\mathbb{I}: \quad 0 \to I^0 \to I^1 \to \cdots \to I^n \to \cdots$$

is called an *injective resolution* of M, where the maps are given by the composition

$$I^{n-1} \;\to\; {}^{I^{n-1}}/_{\mathrm{im}(I^{n-2})} \;\hookrightarrow\; E_R({}^{I^{n-1}}/_{\mathrm{im}(I^{n-2})}).$$

Conversely, an acyclic complex $\mathbb{I}$ of injective R-modules is a minimal injective resolution of M if

- $M = \ker(I^0 \to I^1)$,
- $I^0 = E_R(M)$,
- $I^n = E_R(\mathrm{im}(I^{n-1} \to I^n))$.

<u>I-torsion:</u> Given an ideal $I \subseteq R$ and an R-module M, set $\Gamma_I(M) := \bigcup_n (0 :_M I^n)$. Then $\Gamma_I(\_)$ defines a covariant functor and for a homomorphism $f : M \to N$, $\Gamma_I(f)$ is given by the restriction $f|_{\Gamma_I(M)}$.

Proposition. $\Gamma_I(\_)$ *is a left exact functor.*

Proof. Let

$$0 \to L \xrightarrow{f} M \xrightarrow{g} N \to 0$$

be a short exact sequence. We want to show that

$$0 \to \Gamma_I(L) \xrightarrow{\Gamma_I(f)} \Gamma_I(M) \xrightarrow{\Gamma_I(g)} \Gamma_I(N)$$

is an exact sequence.

Exactness at $\Gamma_I(L)$: $\Gamma_I(f)$ is injective as it is the restriction of the injective map f. Exactness at $\Gamma_I(M)$: It is clear that $\operatorname{im}(\Gamma_I(f)) \subseteq \ker(\Gamma_I(g))$. Conversely, let $m \in \ker(\Gamma_I(g))$. Then $m \in \ker(g)$ and therefore $m = f(l)$ for some $l \in L$. It remains to show that $l \in \Gamma_I(L)$. As $m \in \Gamma_I(M)$, we have $I^k m = 0$ for some integer k. Then $f(I^k l) = I^k f(l) = I^k m = 0$. As f is injective, $I^k l = 0$ and $l \in \Gamma_I(L)$. $\square$

Exercise. $\Gamma_I = \Gamma_J$ if and only if $\sqrt{I} = \sqrt{J}$.

Definition. The i-th local cohomology functor $H^i_I(\_)$ is defined as the right derived functor of $\Gamma_I(\_)$.

More precisely, given an R-module M, let $\mathbb{I}$ be an injective resolution of M:

$$\mathbb{I}: \quad 0 \to I^0 \xrightarrow{d^0} I^1 \xrightarrow{d^1} \cdots \to I^n \xrightarrow{d^n} \cdots$$

Apply $\Gamma_I(\_)$ to $\mathbb{I}$ and obtain the complex:

$$\Gamma_I(\mathbb{I}): \quad 0 \to \Gamma_I(I^0) \to \Gamma_I(I^1) \to \cdots \to \Gamma_I(I^n) \to \cdots$$

Then set $H^0_I(\_) := \Gamma_I(\_)$ and $H^i_I(M) := \ker(\Gamma_I(d^i))/\operatorname{im}(\Gamma_I(d^{i-1}))$ for $i > 0$. Note that $H^i_I(\_)$ is a covariant functor.

Proposition. 1) *If* $0 \to L \to M \to N \to 0$ *is a short exact sequence, then we have an induced long exact sequence*

$$\begin{aligned} 0 \to H^0_I(L) \to H^0_I(M) \to H^0_I(N) \to \\ H^1_I(L) \to H^1_I(M) \to H^1_I(N) \to \cdots \end{aligned}$$

2) *Given a commutative diagram with exact rows:*

$$\begin{array}{ccccccccc} 0 & \longrightarrow & L & \longrightarrow & M & \longrightarrow & N & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & L' & \longrightarrow & M' & \longrightarrow & N' & \longrightarrow & 0 \end{array}$$

then we have the following commutative diagram with exact rows:

$$\begin{array}{ccccccccc} \cdots & \longrightarrow & H^i_I(M) & \longrightarrow & H^i_I(N) & \longrightarrow & H^{i+1}_I(L) & \longrightarrow & \cdots \\ & & \downarrow & & \downarrow & & \downarrow & & \\ \cdots & \longrightarrow & H^i_I(M') & \longrightarrow & H^i_I(N') & \longrightarrow & H^{i+1}_I(L') & \longrightarrow & \cdots \end{array}$$

3) $\sqrt{I} = \sqrt{J}$ *if and only if* $H^i_I(\_) = H^i_J(\_)$.

A note on localization: Let $S \subseteq R$ be a multiplicatively closed set. Then $S^{-1}\Gamma_I(M) = \Gamma_{S^{-1}I}(S^{-1}M)$ and the same holds for the higher local cohomology modules.

Clearly, if E is an injective R-module, then $H^i_I(E) = 0$ for $i > 0$.

2. Local cohomology

An alternate way of constructing the local cohomology modules: Consider the module $\operatorname{Hom}_R(R/I^n, M) \cong (0 :_M I^n)$. Now, if $n \geqslant m$, then one has a natural map $R/I^n \to R/I^m$, forming an inverse system. Applying $\operatorname{Hom}_R(\_, M)$, we get a direct system of maps:

$$\varinjlim_n \operatorname{Hom}_R(R/I^n, M) \cong \bigcup_n (0 :_M I^n) = \Gamma_I(M).$$

As one might guess (or hope), it is also the case that

$$\varinjlim_n \operatorname{Ext}^i_R(R/I^n, M) \cong H^i_I(M).$$

This follows from the theory of negative strongly connected functors – see [R].

Definition. Let R, R' be commutative rings. A sequence of covariant functors $\{T^i\}_{i \geqslant 0}$: R-modules $\to$ R'-modules is said to be negative (strongly) connected if

(i) Any short exact sequence $0 \to L \to M \to N \to 0$ induces a long exact sequence

$$0 \to T^0(L) \to T^0(M) \to T^0(N) \to$$
$$T^1(L) \to T^1(M) \to T^1(N) \to \cdots$$

(ii) For any commutative diagram with exact rows

$$\begin{array}{ccccccccc} 0 & \longrightarrow & L & \longrightarrow & M & \longrightarrow & N & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & L' & \longrightarrow & M' & \longrightarrow & N' & \longrightarrow & 0 \end{array}$$

there is a chain map between the long exact sequences given in (i).

Theorem. *Let $\psi^0 : T^0 \to U^0$ be a natural equivalence, where $\{T^i\}_{i \geqslant 0}, \{U^i\}_{i \geqslant 0}$ are strongly connected. If $T^i(I) = U^i(I) = 0$ for all $i > 0$ and injective modules I, then there is a natural equivalence of functors $\psi = \{\psi^i\}_{i \geqslant 0} : \{T^i\}_i \to \{U^i\}_i$.*

The above theorem implies that $\varinjlim_n \operatorname{Ext}^i_R(R/I^n, M) \cong H^i_I(M)$.

Remark. i) One can replace the sequence of $\{I^n\}$ by any decreasing sequence of ideals $\{J_t\}$ which are cofinal with $\{I^n\}$, i.e. $\forall t\ \exists n$ such that $I^n \subseteq J_t$ and $\forall n\ \exists t$ such that $J_t \subseteq I^n$.

ii) Every element of $H^i_I(M)$ is killed by some power of I, as every $m \in H^i_I(M)$ is the image of some $\operatorname{Ext}^i_R(R/I^n, M)$ which is killed by I^n.

iii) For any $x \in R$, the homomorphism $M \xrightarrow{\cdot x} M$ induces a homomorphism $H^i_I(M) \xrightarrow{\cdot x} H^i_I(M)$.

Proposition. *Let M be a finitely generated R-module and $I \subseteq R$ an ideal. Then $IM = M \iff H^i_I(M) = 0\ \forall i$. If $IM \neq M$, then* $\min\{i \mid H^i_I(M) \neq 0\} = \operatorname{depth}_I(M)$.

Proof. $IM = M \iff I^tM = M\ \forall t$. So $I^t + \operatorname{Ann}(M) = R$, as otherwise $I^t + \operatorname{Ann}(M) \subseteq \mathfrak{m}$ for some $\mathfrak{m} \in$ max-Spec(R). Since $I^t + \operatorname{Ann}(M)$ annihilates $\operatorname{Ext}^i_R(R/I^t, M)$, we have $\operatorname{Ext}^i_R(R/I^t, M) = 0$ and therefore $H^i_I(M) = 0$.

It suffices to assume now that $IM \neq M$. Set $d := \operatorname{depth}_I(M)$ and let $x_1, \dots, x_d$ be a maximal M-regular sequence in I. We show by induction on d that $H^i_I(M) = 0$ for $i < d$ and $H^d_I(M) \neq 0$.

If $d = 0$, that is, if $\operatorname{depth}_I(M) = 0$, then there is an $0 \neq m \in M$ killed by I. So $m \in H^0_I(M) \neq 0$. Now let $d \geqslant 1$ and set $x := x_1$. Consider the short exact sequence

$$0 \to M \xrightarrow{\cdot x} M \to M/xM \to 0$$

and the induced long exact sequence

$$\cdots \to H^{i-1}_I(M/xM) \to H^i_I(M) \xrightarrow{\cdot x} H^i_I(M) \to \cdots$$

If $i < d$, then $H^{i-1}_I(M/xM) = 0$ by induction hypothesis and x is a nonzerodivisor on $H^i_I(M)$. As all elements of $H^i_I(M)$ are killed by some power of I, we conclude that $H^i_I(M) = 0$.

It remains to show that $H^d_I(M) \neq 0$. This follows from the induction hypothesis and the long exact sequence

$$\cdots \xrightarrow{\cdot x} H^{d-1}_I(M) \to H^{d-1}_I(M/xM) \to H^d_I(M) \xrightarrow{\cdot x} \cdots$$

which yield $0 \neq H^{d-1}_I(M/xM) \hookrightarrow H^d_I(M)$. □

The Koszul interpretation

Let $\mathbb{K}, \mathbb{L}$ be two complexes of R-modules with differentials d', d'', respectively. Define the complex $\mathbb{M} := \mathbb{K} \otimes \mathbb{L}$ via $\mathbb{M}_k := \bigoplus_{i+j=k} \mathbb{K}_i \otimes \mathbb{L}_j$ with differential $d(a_i \otimes b_j) := d'a_i \otimes b_j + (-1)^i a_i \otimes d''b_j$ where $a_i \in K_i,\ b_j \in L_j$. If $\mathbb{K}^{(1)}, \dots, \mathbb{K}^{(n)}$ are n complexes, then $\mathbb{K}^{(1)} \otimes \cdots \otimes \mathbb{K}^{(n)}$ is defined inductively as $(\mathbb{K}^{(1)} \otimes \cdots \otimes \mathbb{K}^{(n-1)}) \otimes \mathbb{K}^{(n)}$.

To any $x \in R$ one can associate a complex $K_\bullet(x; R):\ 0 \to R \xrightarrow{\cdot x} R \to 0$. Given a sequence $\underline{x} = x_1, \dots, x_n$ of elements in R, we define the Koszul complex $K_\bullet(\underline{x}; R) := \bigotimes_{i=1}^n K_\bullet(x_i; R)$. For an R-module M we define $K_\bullet(\underline{x}; M) := K_\bullet(\underline{x}; R) \otimes M$. In cohomological notation we write $K^\bullet(\underline{x}; M) = K^\bullet(\underline{x}; R) \otimes_R M \cong \operatorname{Hom}_R(K_\bullet(\underline{x}; R), M)$.

Discussion: Let $x \in R$ and M an R-module. Consider the complex

$$M \xrightarrow{\cdot x} M \xrightarrow{\cdot x} M \to \cdots$$

and set $N := \ker(M \to M_x)$ and $M' := M/N$. Note that $N = H^0_{xR}(M)$. Then

$$\varprojlim(M \xrightarrow{\cdot x} M \xrightarrow{\cdot x} M \to \cdots) \cong$$
$$\varprojlim(M' \xrightarrow{\cdot x} M' \xrightarrow{\cdot x} M' \to \cdots) =$$
$$\varprojlim(M' \subseteq x^{-1}M' \subseteq \cdots \subseteq x^{-t}M' \subseteq \cdots) \cong M'_x \cong M_x.$$

Notation: Whenever $\underline{x} = x_1, \ldots, x_n$ denotes a sequence of elements in R, we will denote by $\underline{x}^t$ the sequence of the individual powers $x_1^t, \ldots, x_n^t$

For $x \in R$ we have a chain map of complexes $K^\bullet(x^t; R) \to K^\bullet(x^{t+1}; R)$ via the commutative diagram

$$\begin{array}{lccccccc} K^\bullet(x^t;R): & 0 & \longrightarrow & R & \xrightarrow{\cdot x^t} & R & \longrightarrow & 0 \\ & & & \big\downarrow \text{id} & & \big\downarrow \cdot x & & \\ K^\bullet(x^{t+1};R): & 0 & \longrightarrow & R & \xrightarrow{\cdot x^{t+1}} & R & \longrightarrow & 0 \end{array} \tag{1}$$

By tensoring we get $K^\bullet(x^t; M) \to K^\bullet(x^{t+1}; M)$. Take the direct limit and denote the resulting complex by $K^\bullet(\underline{x}^\infty; M)$. We can look at $H^i(K^\bullet(\underline{x}^\infty; M)) = \varinjlim_t H^i(K^\bullet(\underline{x}^t; M))$ for which we simply write $H^i(\underline{x}^\infty; M)$.

Theorem. *If $I = (x_1, \ldots, x_n)$ and M is an R-module, then there is a canonical isomorphism $H^i(\underline{x}^\infty; M) \cong H^i_I(M)$.*

Observation: Say $\underline{x} = x_1, \ldots, x_n$ is an R-regular sequence. Then it is known that $K_\bullet(\underline{x}^t; R)$ is a projective resolution of $R/\underline{x}^t R$. Apply $\mathrm{Hom}_R(\_, M)$ and note that on the one hand $K^\bullet(\underline{x}^t; M)$ gives $H^\bullet(\underline{x}^t; M)$ while on the other hand we get $\mathrm{Ext}^\bullet_R(R/\underline{x}^t; M)$. Now take the direct limit: $H^i(\underline{x}^\infty; M) \cong H^i_I(M)$.

The Cěch complex and a detailed look at $K^\bullet(\underline{x}^\infty; M)$

Let $x \in R$. Then $K^0(x^\infty; R) = R$ and $K^1(x^\infty; R) = R_x$. (Recall diagram (1).) So $K^\bullet(\underline{x}^\infty; R) = \bigotimes_{i=1}^n (0 \to R \to R_{x_i} \to 0)$, that is,

$$K^j(\underline{x}^\infty; R) = \bigoplus_{|S|=j} R_{x(S)}$$

where $S \subseteq \{1, \ldots, n\}$ and $x(S) = \prod_{i \in S} x_i$. Similarly, $K^j(\underline{x}^\infty; M) = \bigoplus_{|S|=j} M_{x(S)}$. The map $R_{x(S)} \to R_{x(T)}$, where $|T| = |S| + 1$, is the zero map unless $S \subset T$, in which case it is the localization map times $(-1)^a$ where a is the number of elements in S preceding the element in $T \setminus S$.

Exercise. Let $x, y, z \in R$. Write down the maps in

$$0 \to R \to R_x \oplus R_y \oplus R_z \to R_{xy} \oplus R_{yz} \oplus R_{zx} \to R_{xyz} \to 0.$$

Corollary. *If $I \subseteq R$ is an ideal which can be generated by n elements up to radical, then $H^i_I(R) = 0$ for $i > n$.*

Remark. The modules occurring in $K^\bullet(\underline{x}^\infty; R)$ are flat.

3. Properties of local cohomology

Proposition. 1) *Let $R \to S$ be a ring homomorphism of Noetherian rings, I an ideal of R, and M an S-module. Then $H^i_I(M) \cong H^i_{IS}(M)$ as S-modules.*
2) *Let Λ be a directed set and $\{M_\lambda\}_{\lambda\in\Lambda}$ a direct system of R-modules. Then $\varinjlim_\lambda H^i_I(M_\lambda) \cong H^i_I(\varinjlim_\lambda M_\lambda)$.*
3) *Is S is flat over R, then $H^i_I(M) \otimes_R S = H^i_{IS}(M \otimes_R S)$.*
4) *If $\mathfrak{m} \subseteq R$ is a maximal ideal, then $H^i_\mathfrak{m}(M) \cong H^i_{\mathfrak{m}R_\mathfrak{m}}(M_\mathfrak{m})$.*
5) *If $(R, \mathfrak{m})$ is local, then $H^i_\mathfrak{m}(M) \cong H^i_{\mathfrak{m}\hat{R}}(\hat{R} \otimes_R M)$ which is isomorphic to $H^i_{\mathfrak{m}\hat{R}}(\hat{M})$ if M is finitely generated.*

Proposition. *Let $I \subseteq R$ be an ideal which, up to radical, is generated by a regular sequence of length n. Then $H^i_I(M) \cong \operatorname{Tor}_{n-i}(M, H^n_I(R))$ for $i \leqslant n$.*

Proof. If $i < \operatorname{depth}_I(R) = n$, then $H^i_I(R) = 0$. Therefore $K^\bullet(\underline{x}^\infty; R)$ gives a flat resolution of $H^n_I(R)$, numbered backwards:

$$\cdots \to K^{n-1} \to K^n \to H^n_I(R) = K^n/\operatorname{im}(d^{n-1}) \to 0.$$

On the one hand $\operatorname{Tor}^R_{n-i}(M, H^n_I(R)) = H^i(K^\bullet(\underline{x}^\infty; R) \otimes_R M)$ by definition of Tor. On the other hand, by the preceding theorem, $H^i(K^\bullet(\underline{x}^\infty; R) \otimes_R M) \cong H^i_I(M)$. □

Corollary. *Let $(R, \mathfrak{m}, k)$ be a Cohen-Macaulay local ring of dimension n. Then $H^i_\mathfrak{m}(M) \cong \operatorname{Tor}^R_{n-i}(M, H^n_\mathfrak{m}(R))$.*

Proof. The maximal ideal $\mathfrak{m}$ is the radical of an ideal generated by a(ny) regular sequence of length n. □

<u>Grothendieck's Theorems</u>

1. Vanishing Theorem: Let $I \subseteq R$ be an ideal and M an R-module. Then $H^i_I(M) = 0$ for $i > \dim(R)$.

2. Non-Vanishing Theorem: Let $(R, \mathfrak{m}, k)$ be a local ring and M a finitely generated R-module. Then $H^n_\mathfrak{m}(M) \neq 0$ for $n = \dim(M)$.

Proof of 1. We may assume that R is local with maximal ideal $\mathfrak{m}$. Further, as M is the direct limit of its finitely generated submodules, we may also assume that M is finitely generated. Set $S := R/\operatorname{Ann}(M)$ so that $n := \dim(M) = \dim(S)$. The maximal ideal of S is generated by n elements up to radical, so $H^i_{\mathfrak{m}S}(M) = 0$ for $i > n$.

We want to show that $H^i_I(M) = 0$ for $i > \dim(M)$. By induction, we assume the theorem is true for all finitely generated modules of dimension less than n. We leave the case $n = 0$ as an exercise and assume $n > 0$.

Note that if a module is I-torsion, then all its higher local cohomology modules vanish. So, as $\Gamma_I(M)$ is I-torsion, without loss of generality $\Gamma_I(M) \neq M$. Also, the long exact sequence induced by

$$0 \to \Gamma_I(M) \to M \to {}^{M}/_{\Gamma_I(M)} \to 0$$

yields that $H^i_I(M) \cong H^i_I(M/\Gamma_I(M))$ for all $i > 0$. Hence, by passing to $M/\Gamma_I(M)$, we may assume that $M \neq 0$ is I-torsionfree. It follows that I contains an M-regular element r. (Otherwise I is contained in the union of the associated primes of M, and by prime avoidance I is contained in one of those primes which is of the form $(0 :_R m)$ for some $0 \neq m \in M$. That is $Im = 0$ — a contradiction.)

Let $i > n$ and let t be an integer. Consider the short exact sequence

$$0 \to M \xrightarrow{\cdot r^t} M \to {}^{M}/_{r^t M} \to 0$$

and the induced long exact sequence

$$\cdots \to H^{i-1}_I(M/r^t M) \to H^i_I(M) \xrightarrow{\cdot r^t} H^i_I(M) \to \cdots$$

Since $\dim(M/r^t M) < \dim(M)$, $H^{i-1}_I(M/r^t M) = 0$ by induction hypothesis and r^t is a nonzerodivisor on $H^i_I(M)$. But any element of $H^i_I(M)$ is killed by a power of I. As $r \in I$, that implies $H^i_I(M) = 0$. □

Remark. If $\dim(R) = n$ and M is an R-module, then $H^n_I(M) \cong M \otimes_R H^n_I(R)$. This follows from the fact that $H^n_{\mathfrak{m}}(-)$ is a right exact functor, a consequence of Grothedieck's Vanishing Theorem

4. Local duality

Let $(R, \mathfrak{m}, k)$ be a local ring and $E = E_R(k)$ the injective hull of k and let $\Delta_R(\_) := \mathrm{Hom}_R(\_, E)$ denote the Matlis duality functor. Assume R is the homomorphic image of a Gorenstein local ring R' of dimension n'. Then for all i and for all finitely generated R-modules M one has

$$\begin{aligned} H^i_{\mathfrak{m}}(M) &\cong \mathrm{Hom}_R(\mathrm{Ext}^{n'-i}_{R'}(M, R'), E) \\ &= \Delta_R(\mathrm{Ext}^{n'-i}_{R'}(M, R')). \end{aligned}$$

In particular, if R is Gorenstein, setting $R' = M = R$ and $i = n$ yields $H^n_{\mathfrak{m}}(R) \cong E$.

The Cohen-Macaulay case

Let $(R, \mathfrak{m}, k)$ be a Cohen-Macaulay local ring of dimension n. Recall that the type of M is defined as $\dim_k \mathrm{Soc}(M/\underline{x}M) = \dim_k \mathrm{Hom}_R(k, M/\underline{x}M) = \dim_k \mathrm{Ext}^n_R(k, M)$ where $\underline{x}$ is a system of parameters for M. The type of M is independent of the choice of the system of parameters chosen for M.

Definition. A canonical module for R, denoted here by ω_R, is a maximal Cohen-Macaulay module of type 1.

Proposition.
- *A ring admits a canonical module if and only if it is the homomorphic image of a Gorenstein local ring.*
- *Any two canonical modules are (non-canonically) isomorphic.*
- $\mathrm{End}_R(\omega_R) \cong R$.

Theorem. *Let R be a Cohen-Macaulay local ring of dimension n which is the homomorphic image of a Gorenstein local ring R' of dimension n'. Then $\omega_R \cong \mathrm{Ext}_{R'}^{n'-n}(R, R')$. In particular, if R is Gorenstein, then $\omega_R \cong R$.*

In general, if R is a domain, then $\mathrm{rank}(\omega_R) = 1$ and ω_R can be embedded in R as an ideal J such that $\mathrm{ht}(J) = 1$ or $J = R$.

Theorem. *Let R be a Cohen-Macaulay local ring which admits a canonical module ω_R. Then for all i and for all finitely generated R-modules M one has*

$$H_{\mathfrak{m}}^{n-i}(M) \cong \Delta_R(\mathrm{Ext}_R^i(M, \omega_R)).$$

Proof. First case is $i = 0$. In this case we use the local duality theorem stated for images of Gorenstein rings:

Let R' Gorenstein surjects on R. Using the notations introduced above

$$H_{\mathfrak{m}}^{n-i}(M) \cong \mathrm{Hom}_R(\mathrm{Ext}_R^{n'-n+i}(M, R'), E).$$

Now take $M = R$, $i = 0$ and see that we get

$$H_{\mathfrak{m}}^{n}(M) \cong \mathrm{Hom}_R(\mathrm{Ext}_R^{n'-n}(R, R'), E) = \mathrm{Hom}_R(\omega_R, E).$$

But, $H_{\mathfrak{m}}^n(-) \cong - \otimes_R H_{\mathfrak{m}}^n(R) \cong - \otimes_R \mathrm{Hom}_R(\omega_R, E)$.

We can then apply the following canonical isomorphism:

$$M \otimes \mathrm{Hom}_R(I, J) \cong \mathrm{Hom}_R(\mathrm{Hom}_R(M, I), J),$$

where M is finitely generated and J is injective over R. Use this for M, $I = \omega_R$ and $J = E$ and therefore obtain the case $i = 0$

The case $i > 0$ follows through from homological algebra considerations by interpreting both sides as positively strongly connected functors on the category of finitely generated R-modules.

□

Lemma. *Let $(R, \mathfrak{m}, k)$ be a local ring of dimension n and I an ideal of R. Then I is generated by n elements up to radical.*

Proof. Set $A_0 := \{P \in \mathrm{Min}(R) \mid P \not\supseteq I\}$. By prime avoidance, $I \not\subseteq \bigcup_{P \in A_0} P$. Let $x_1 \in I$ such that $x_1 \notin P$ for any $P \in A_0$. Assume we have a sequence of elements $x_1, \ldots, x_k \in I$ such that all primes of height $\leqslant k-1$ that contain $(x_1, \ldots, x_k)$ also contain I.

If P is a prime of height at most k containing $(x_1, \ldots, x_k)$, but not containing I, then P is minimal over $(x_1, \ldots, x_k)$. If not, then there is a prime $Q \subsetneq P$ minimal over $(x_1, \ldots, x_k)$. As $\mathrm{ht}(P) \leqslant k$, we must have $\mathrm{ht}(Q) \leqslant k-1$. Then, by the above assumption, Q contains I and therefore so does P. That is,

$$A_k := \{P \in \mathrm{Spec}(R) \mid \mathrm{ht}(P) \leqslant k,\ P \supseteq (x_1, \ldots, x_k),\ P \not\supseteq I\} \subseteq \mathrm{Min}(x_1, \ldots, x_k)$$

and therefore A_k is a finite set of primes. So we can choose $x_{k+1} \in I$ such that $x_{k+1} \notin P$ for any $P \in A_k$. This process has to terminate at $k = n$ with $A_n = \varnothing$ and $\sqrt{(x_1, \ldots, x_n)} = \sqrt{I}$. □

Remark. This lemma allows us to give an alternate proof of the Vanishing Theorem of Grothendieck:

Up to radical I is generated by n elements. So, we can assume that we are in the case where I is generated by exactly n elements. Using the Koszul interpretation of the local cohomology we can see now that if $i > n$, the Koszul cocomplex is trivial and hence all local cohomology modules $H^i_I(M)$ must vanish.

5. Basic notions of tight closure theory

Let R be a Noetherian ring of characteristic p and let q denote a power of p, i.e. $q = p^e$ for some integer e. We use the notation R° for the complement of the minimal primes of R, i.e. $R^\circ := R \setminus \bigcup_{P \in \mathrm{Min}(R)} P$. For an ideal $I \subseteq R$, we denote by $I^{[q]}$ the ideal $(r^q \mid r \in I)$. It is easily seen that if $I = (r_1, \ldots, r_n)$, then $I^{[q]} = (r_1^q, \ldots, r_n^q)$.

Let $F : R \to R$ denote the Frobenius homomorphism sending $r \mapsto r^p$ and denote by $F^e : R \to R$ its e-th iteration sending $r \mapsto r^q$. This gives R a new R-algebra structure which we denote by $R^{(e)}$. One can think of $R^{(e)}$ as a right R-algebra.

Definition. The *tight closure* of an ideal $I \subseteq R$, written I^*, is the set of all elements $x \in R$ for which $\exists\, c \in R^\circ$ such that $cx^q \in I^{[q]}$ for $q \gg 0$.

Some basic properties: I^* is an ideal which itself is tightly closed, i.e. $(I^*)^* = I^*$. Furthermore, the following inclusions hold: $I \subseteq I^* \subseteq \bar{I} \subseteq \sqrt{I}$. ($\bar{I}$ denotes the integral closure of I which will be defined later in the text.)

Some motivation for the notion of tight closure:

• It captures the essence of some arguments that appear in the proofs of some homological conjectures and delivers it in a unified way.

• It naturally leads to new classes of rings in characteristic $p > 0$ (or in characteristic 0) that are important, for instance, in view of birational geometry: F-regular, F-rational, F-pure, F-injective.

Definition. A Noetherian ring R is said to be *weakly F-regular* if every ideal of R is tightly closed. If for every multiplicatively closed set W, the ring $W^{-1}R$ is weakly F-regular, then R is said to be *F-regular*.

Proposition. 1. *If $I \subseteq J$, then $I^* \subseteq J^*$.*

2. *The intersection of tightly closed ideals is again tightly closed:* $I_\lambda = I_\lambda^* \implies (\bigcap_\lambda I_\lambda)^* = \bigcap_\lambda I_\lambda$.
3. $(I \cap J)^* \subseteq I^* \cap J^*$.
4. $(I + J)^* = (I^* + J^*)^*$

Furthermore, $0^* = \sqrt{0}$ and $I^* = \pi^{-1}\left[(IR_{\text{red}})^*\right]$ where π is the natural projection $\pi : R \to R_{\text{red}} = R/\sqrt{0}$. Also, if I is tightly closed, then the quotient $I : J$ is as well, for all ideals J.

Proposition. 1. *If R is reduced, or if* $\text{ht}(I) > 0$*, then* $x \in I^*$ *iff* $\exists\, c \in R^\circ$ *such that* $cx^q \in I^{[q]}\ \forall\, q$.
2. $x \in I^*$ *iff* $\bar{x} \in \left({}^{I+P}/_{P}\right)^*\ \ \forall\, P \in \text{Min}(R)$.

Proof of 1. Let $\text{ht}(I) > 0$. We have $I^* \subseteq (x \mid x \in I^* \cap R^\circ) \cup \bigcup_{P \in \text{Min}} P$. As $I \not\subseteq P$ for any minimal prime P, prime avoidance yields that $I^* = (x \mid x \in I^* \cap R^\circ)$.

We can assume that $x \in I^* \cap R^\circ$. So $\exists\, c \in R^\circ$ such that $cx^q \in I^{[q]}\ \forall q \geqslant q_0$. Set $c' := cx^{q_0}$, that is, $c'x^q = cx^{q_0+q}$. If $q \geqslant q_0$, then $c'x^q \in I^{[q]}$. And if $q < q_0$, then $cx^{q_0+q} \in I^{[q_0]} \subseteq I^{[q]}$.

If R is reduced, then set $S := (R^\circ)^{-1}R = \prod_{i=1}^n K_i$ where K_i are fields. So IS is generated by an idempotent and $IS = I^{[q]}S\ \forall\, q$. We have $x \in I^*$, so $\exists\, c \in R^\circ$ such that $cx^q \in I^{[q]}\ \forall q \geqslant q_0$. Mapping this element to S we get $x^q \in I^{[q]}S\ \forall\, q \geqslant q_0$. As S is a product of fields, F^e is flat. So $x \in IS = I^{[q]}S$ and for every q there exists a $c_q \in R^\circ$ such that $c_q x \in I^{[q]}$. Setting $c' := c \cdot \prod_{q' < q_0} c_{q'}$ we get $c'x^q = c \cdot \prod_{q' < q_0} c_{q'}\ x^q \in I^{[q]}\ \forall\, q$. □

Theorem. *If R is regular, then $I = I^*$ for all ideals $I \subseteq R$. In particular, R is F-regular.*

Proof. Note that since R is regular, F is flat. Assume $I \neq I^*$ for some I and let $x \in I^* \setminus I$. Note that $(I : x) \neq R$, so $(I : x) \subseteq \mathfrak{m} \in \text{max-Spec}(R)$. After localizing at $\mathfrak{m}$ we may assume that R is local. There exists a $c \in R^\circ$ such that $cx^q \in I^{[q]}\ \forall\, q$. But then $c \in \bigcap_q (I^{[q]} : x^q) \overset{\text{flatness}}{=} \bigcap_q (I : x)^{[q]} \subseteq \bigcap_q \mathfrak{m}^q = 0$ — a contradiction. □

If $R \to S$ is flat, $I \subseteq R$ is an ideal, and $x \in R$, then $(I :_R x)S = (IS :_S x)$. To see this, consider the following short exact sequence

$$0 \to {}^{R}/_{I:x} \xrightarrow{\cdot x} {}^{R}/_{I} \to {}^{R}/_{(I,x)R} \to 0,$$

and tensor it with S:

$$0 \to {}^{S}/_{(I:x)S} \xrightarrow{\cdot x} {}^{S}/_{IS} \to {}^{S}/_{(I,x)S} \to 0.$$

Apply the above to $R \xrightarrow{F^e} R$, i.e. $S = R$ via F^e. Then $(I :_R x)S = (I :_R x)^{[q]}R$ and

$$\begin{aligned}(IS :_S x) &= \{c \in S \mid x \cdot c \in IS\} \\ &= \{c \in R \mid cx^q \in I^{[q]}\} = I^{[q]} :_R x^q.\end{aligned}$$

Some additional properties of tight closure and its connections to integral closure:

Definition. An element $x \in R$ is said to be in the *integral closure* of an ideal $I \subseteq R$, written $x \in \bar{I}$, if $\exists\, a_i \in I^{n-i}$ such that $x^n + a_{n-1}x^{n-1} + \cdots + a_0 = 0$.

In terms of valuations, an element x is in the integral closure of an ideal I if and only if $h(x) \in IV$ for all homomorphisms $h : R \to V$ where V is a DVR and $\ker(h)$ is contained in a minimal prime of R. Yet another characterization: $x \in \bar{I}$ if and only if $\exists\, c \in R^\circ$ such that $cx^n \in I^n$ for $n \gg 0$. In particular, $I^* \subseteq \bar{I}$.

Theorem (Briançon-Skoda)**.** *Let R be a Noetherian ring of characteristic p and $I \subseteq R$ an n-generated ideal. Then $\overline{I^n} \subseteq I^*$. In particular, of R is regular, then $\overline{I^n} \subseteq I$.*

Proof. If $x \in \overline{I^n}$, then there exists a $c \in R^\circ$ such that $cx^m \in I^{mn}$ for all $m \gg 0$. Say $I = (x_1, \ldots, x_n)$. Then an element in I^{mn} is a sum of terms $x_1^{a_1} \cdots x_n^{a_n}$ where $\sum_{i=1}^n a_i \geqslant mn$ and so at least one $a_i \geqslant m$. Now if $m = q$, then $cx^q \in I^{[q]}$ and $x \in I^*$. □

Example. Let $R = {}^{k[\![x,y,z]\!]}/_{(x^3+y^3+z^3)}$ where k is a field of characteristic $p \neq 3$ and let I be the ideal generated by the system of parameters (y, z). Claim: $x^2 \in I^*$.

If $I \neq I^*$, then $I^*/I \cap \operatorname{Soc}(R/I) \neq 0$. Note that $\operatorname{Soc}(R/I) = \operatorname{Soc}(k[\![x]\!]/(x^3))$ is generated by x^2. We want to find an element $c \in R^\circ$ such that $cx^{2q} \in (y,z)^{[q]}$ for all $q \gg 0$. In the following we show that c can be taken to be x^2. As $p \neq 3$, the remainder of $2q$ modulo 3 is either 1 or 2. Write $2q = 3k+i$ with $i = 1$ or 2. Adding 2 to both sides we can write $2q + 2 = 3k + 3 + \epsilon$ with $\epsilon = 0$ or 1. So we have

$$\begin{aligned} x^2 x^{2q} &= x^\epsilon x^{3k+3} \\ &= x^\epsilon (x^3)^{k+1} = x^\epsilon(-y^3 - z^3)^{k+1}. \end{aligned}$$

After expansion of the last term, a general monomial will contain the factor $y^{3i}z^{3j}$ with $i + j = k + 1$. If $3i \leqslant q - 1$ and $3j \leqslant q - 1$, then $3(k+1) \leqslant 2q - 2$ — a contradiction to the fact that $2q - 2 \leqslant 3k$. So $x^\epsilon(-y^3 - z^3)^{k+1} \subseteq (y^q, z^q)R$. ✓

We observe that in the above example $x \in \bar{I}$. Namely, $y^3 + z^3 \in I^3$ and x satisfies the integral dependency relation $x^3 + (y^3 + z^3) = 0$ (in R). By the same token, it is easily seen that $x^2 \in \overline{I^2}$. Indeed, $(x^2)^3 - (y^3 + z^3)^2 = 0$ and $(y^3 + z^3)^2 \in (I^3)^2 = (I^2)^3$. So, as I is 2-generated, the Briançon-Skoda Theorem delivers $x^2 \in \overline{I^2} \subseteq I^*$.

<u>Contractions:</u> If $R \subseteq S$ is a module finite extension of domains, then for an ideal $I \subseteq R$, $(IS)^* \cap R \subseteq I^*$. In particular, $IS \cap R \subseteq I^*$.

<u>Persistence:</u> Let $R \xrightarrow{\phi} S$ be a homomorphism of Noetherian rings, $I \subseteq R$ an ideal, and $x \in I^*$. If R is a localization of a finitely generated algebra over an excellent ring, or $R_{\text{red}} = R/\sqrt{R}$ is F-finite (i.e. $F : R_{\text{red}} \to R_{\text{red}}$ is finite), then $\phi(x) \in (IS)^*$.

Recall that elements $x_1, \ldots, x_t$ of a Noetherian ring R are called parameters if they are part of a system of parameters in R_P for any prime $P \supseteq (x_1, \ldots, x_t)$.

Exercise. $x_1, \dots, x_t$ are parameters if and only if $\mathrm{ht}(x_1, \dots, x_t) = t$ or ∞.

Colon-Capturing:

Theorem. *Let $(R, \mathfrak{m})$ be a local equidimensional ring which is the homomorphic image of a Cohen-Macaulay local ring. Then*

(a) $(x_1, \dots, x_{t-1}) : x_t \subseteq (x_1, \dots, x_{t-1})^*$
(b) $(x_1^n, \dots, x_t^n) : (x_1 \cdots x_t)^{n-1} \subseteq (x_1, \dots, x_t)^*$

Theorem (Monomial Conjecture)**.** *Let $(R, \mathfrak{m})$ be a local ring containing a field. Let $x_1, \dots, x_d$ be a system of parameters in R. Then $(x_1 \cdots x_d)^{n-1} \notin (x_1^n, \dots, x_d^n)$ for all n.*

Sketch of proof. First reduce to the case where R is complete. Then, by part (b) of the previous theorem, $1 \in (x_1, \dots, x_d)^*$. Write this out explicitly to obtain a contradiction. □

Let us look at the Frobenius action the highest local cohomology of R. Let $x_1, \dots, x_d$ be a system of parameters for R. We can compute the $H^n\mathfrak{m}(R)$ as the direct limit of $R/(x_1^q, \dots, x_d^q 0$ where the transition maps are given by multplication by $(x_1 \cdots x_d)^{q-1}$.

Hence, each element of $H^n_\mathfrak{m}(R)$ can be seen as the class $\eta = [a + (x_1^q, \dots, x_d^q)]$. The Frobenius action is then simply $F(\eta) = [a^p + x_1^{qp}, \dots, x_d^{qp})]$.

Using local cohomology, one can show that the monomial conjecture holds in general if and only if the image of 1 under the composition

$$R \to R/(x_1, \dots, x_d) \to H^d_\mathfrak{m}(R)$$

is non-zero. However, in characteristic $p > 0$, any element of the local cohomology module is a multiple of a Frobenius iteration of this element:

indeed, $\eta = [a + (x_1^q, \dots, x_d^q)] = a[1 + (x_1^q, \dots, x_d^q)] = aF^e([1 + (x_1^q, \dots, x_d^q)])$.

So, in characteristic p, the monomial conjecture is equivalent to $H^d_\mathfrak{m}(R) \neq 0$.

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