RAY HEITMANN'S LECTURES

1. The Rigidity conjecture

Let R be a Noetherian ring and $\mathbf{x} = x_1, ..., x_k$ elements of R. Let $K(\mathbf{x})$ be the Koszul complex of $\mathbf{x}$ and N be a finitely generated module. Then $K(\mathbf{x}) \otimes N$ is the Koszul complex for N and it is well known that $H_i(\mathbf{x}, N)$ =0 if and only if $i > k - \text{grade}_I N$ where $I = \mathbf{x}R$. A consequence of this fact is that once the homology becomes 0, it stays 0. Auslander described this property by saying K(x) is *rigid*.

Auslander focused on acyclic complexes, which are projective resolutions of finitely generated modules. We may consider an acyclic complex X, which is a projective resolution of a module M, and the rigid property with $H_i(X \otimes N) \cong \text{Tor}_i(M, N)$. Obviously X is rigid iff every projective resolution of M is rigid. So it makes sense to speak of *rigid modules* as modules with rigid resolutions.

Theorem 1.1. (Auslander,Lichtenbaum) Let R be a regular local ring. Then any finitely generated module M is rigid.

Auslander(1960) proved this result for any equicharacteristic (regular, local) R and also for R of mixed characteristic p provided that R is unramified and p is M-regular. Lichtenbaum(1966) proved the general case. Below we will give Lichtenbaum's proof in the unramified case.

Proof. Suppose there is a module M that is not rigid. Then there is a f.g module N and integer i such that $\text{Tor}_i(M, N) = 0$ but $\text{Tor}_{i+1}(M, N) \neq 0$. Replacing M by its $i - 1$st syzygy, we may assume $i \leq 1$. On the other hand, $\text{Tor}_i(M, N) = 0$ implies $M = 0$ or $N = 0$, so we may assume $i = 1$.

Next, we can also assume R is complete, since $\text{Tor}_i^{\hat{R}}(\hat{M}, \hat{N}) \cong \text{Tor}_i^R(M, N)$. By Cohen's Structure Theorem, $R \cong A[[T_1, ..., T_n]]$ where A is a field or a DVR with maximal ideal pA.

Let $S = R\hat{\otimes}_A R = A[[T_1, ...T_n, T'_1, ..., T'_n]]$. Then we may view R as an S module via $R \cong S/(T_1 - T'_1, ...T_n - T'_n)S$. Note that R is S modulo an S-sequence, so its resolution over S is a Koszul complex. By previous discussion, R is rigid.

The key ingredient of the proof is a spectral sequence introduced by Serre:

$$\mathrm{Tor}_i^S(R, \hat{\mathrm{Tor}}_j^A(M, N)) \Longrightarrow \mathrm{Tor}_{i+j}^R(M, N)$$

(Here $\hat{\mathrm{Tor}}$ means complete Tor). Since $\mathrm{Tor}_j^A(M, N) = 0$ for $i > 1$ ($\dim A \leq 1$!!), this spectral sequence degenerates to a long exact sequence, which we only need to look at the tail :

$\mathrm{Tor}_1^S(R, \hat{\mathrm{Tor}}_1^A(M, N)) \to \mathrm{Tor}_2^R(M, N) \to \mathrm{Tor}_2^S(R, M\hat{\otimes}_A N) \to R\otimes_S \hat{\mathrm{Tor}}_1^A(M, N) \to \mathrm{Tor}_1^R(M, N) \to \mathrm{Tor}_1^S(R, M\hat{\otimes}_A N) \to 0$

Since $\mathrm{Tor}_1^R(M, N) = 0$ by assumption, $\mathrm{Tor}_1^S(R, M\hat{\otimes}_A N) = 0$. But R is rigid as an S-module, so $\mathrm{Tor}_2^S(R, M\hat{\otimes}_A N) = 0$. This forces the next term $R \otimes_S \hat{\mathrm{Tor}}_1^A(M, N)$ to be 0. But $R \neq 0$, so $\hat{\mathrm{Tor}}_1^A(M, N) = 0$ which implies the leftmost term is 0. But that means $\mathrm{Tor}_2^R(M, N) = 0$, completing the proof.

$\square$

The idea is similar in the unramified case, but we need a more complicated spectral sequence.

When R is not regular, there are easy examples of non-rigid modules :

Example 1.2. Let $R = k[[x, y]]/(xy), M = R/(x), N = R/(y) \cong k[[x]]$. Then M has an infinite resolution :

$$\xrightarrow{x} R \xrightarrow{y} R \xrightarrow{x} R \xrightarrow{y} R \xrightarrow{x} M \longrightarrow 0$$

Tensor with N and note that the map given by multiplication with x is injective on N, while the one with y is 0. So $\mathrm{Tor}_1^R(M, N) = 0$ but $\mathrm{Tor}_2^R(M, N) \neq 0$.

Since regular local rings can be characterized by the property that any f.g module has finite projective dimension, it is natural to extend Theorem 1.1 to:

Conjecture 1.3. (Rigidity Conjecture) Let R be a local ring, M a f.g module of finite projective dimension. Then M is rigid.

Unfortunately, there is also a counterexample to this conjecture :

Example 1.4. Let k be any field. We will construct an affine k-algebra R and R-modules M of projective dimension 2 and N of length 3 such that $\mathrm{Tor}_1(M, N) = 0$ but $\mathrm{Tor}_2(M, N) \neq 0$. We are going to construct N directly, M indirectly, and R last of all !!.

The idea is to build N through exact sequences :

$$0 \to N_1 = k \to N_2 \to k \to 0$$

$$0 \to N_2 \to N_3 = N \to k \to 0$$

For a module M such that $\mathrm{pd}_R M = 2$, these give long exact sequences :

$0 \to \mathrm{Tor}_2(M, N_i) \to \mathrm{Tor}_2(M, N_{i+1}) \to \mathrm{Tor}_2(M, k) \to \mathrm{Tor}_1(M, N_i) \to \mathrm{Tor}_1(M, N_{i+1}) \to \mathrm{Tor}_1(M, k) \to M \otimes N_i$

When $i = 1$ this shows $\mathrm{Tor}_2(M, N_2) \neq 0$. That couples with the exact sequence for $i = 2$ to give $\mathrm{Tor}_2(M, N_3) \neq 0$. All we need is to make sure $\mathrm{Tor}_1(M, N_3) = 0$.

Let $S = k[s,t]/(s^2, st, t^2)$ and $N = S \oplus S/((t, 0), (0, s), (s, t))$. The homomorphism:

$$N \xrightarrow{(s\ t)} N^2$$

maps N onto the socle of N^2, and the sequence :

$$N \xrightarrow{(s\ t)} N^2 \xrightarrow{\alpha} N^4$$

is exact, where α is the map given by :

$$\begin{pmatrix} s & 0 & t & 0 \\ 0 & s & 0 & t \end{pmatrix}$$

We would be done if $\mathrm{Tor}(M, N)$ is the homology of :

$$0 \longrightarrow N \xrightarrow{(s\ t)} N^2 \xrightarrow{\alpha} N^4 \longrightarrow 0$$

For this idea (which doesn't quite work!) to succeed we need M to have the projective resolution:

$$0 \longrightarrow R \xrightarrow{X} R^2 \xrightarrow{Y} R^4 \longrightarrow M \longrightarrow 0$$

and N must have the module structure given by a homomorphism $\varphi : R \to S$ such that :
$\varphi(X) = (s\ t)$ and $\varphi(Y) = \begin{pmatrix} s & 0 & t & 0 \\ 0 & s & 0 & t \end{pmatrix}$

For that purpose, the maximal ideal m of R must contain all the entries of X and Y. It would be sufficient if the entries are linearly independent in R/m^2.

What actually works is the set of Betti numbers 2,4,8 instead of 1,2,4:

$$0 \longrightarrow R^2 \xrightarrow{X} R^4 \xrightarrow{Y} R^8 \longrightarrow M \longrightarrow 0$$

By Buchsbaum-Eisenbud criterion, this sequence is exact if :

1. $XY = 0$
2. $rank X = 2$
3. $rank Y = 2$
4. The grade of ideal of 2-minors of X is at least 2 (Or the ideal is R).
5. The grade of ideal of 2-minors of Y is at least 1 (Or the ideal is R).

To do this, we build X and Y as generic matrices (entries are indeterminates). Then we kill the ideal generated by the entries of XY and the 3-minors of Y. Finally we have to adjoin the multipliers needed for the grade condition on X. For more details, see [Heitmann] .

2. Closure operations

In this lecture we want to discuss the search for a "good" closure operation on ideals. Let ^{cl} denotes that operation. A dream closure would satisfy the following properties :

1. I^{cl} is an ideal.
2. $I \subseteq I^{cl} \subseteq (I^{cl})^{cl}$.
3. $I \subseteq J \Rightarrow I^{cl} \subseteq J^{cl}$
4. (Persistence) If $\varphi : R \to S$ is a ring homomorphism and $x \in I^{cl}$ then $\varphi(x) \in (\varphi(I)S)^{cl}$.
5. (Localization) $(IR_p)^{cl} = I^{cl}R_p$ for $p \in Spec(R)$.
6. $I^+ \subseteq I^{cl} \subseteq \bar{I}$.
7. (Colon-capturing) If $x_1, ..., x_n$ are part of a system of parameters, then: $((x_1, ..., x_{n-1}) : x_n) \subseteq ((x_1, ..., x_{n-1})R)^{cl}$.
8. (Tightness) If R is a regular local ring then $I^{cl} = I$
9. If $\dim R = d$, then $\overline{I^d} \subseteq I^{cl}$.
10. We can figure out what I^{cl} is !!

We would like such a closure to exist for all Noetherian (or at least excellent) local rings, but we get most of what we wish for if we have a closure for complete local domains. To some extent, we can live without persistence and localization. The others are more critical. For example, 8 and 9 combine to generalize Briançon-Skoda Theorem while 6 and 8 give the Direct Summand Conjecture.

In equicharacteristic p, tight closure satisfies 9 out of 10 properties (localization is still open). Tight closure in equicharacteristic 0 works nearly as well though it is probably fair to say it doesn't really meet 10. Alas, it cannot be readily adapted to mixed characteristic.

Property 6 suggest 2 obvious choices : *plus* and *integral* closure. It is quite easy to show that integral closure satisfy all properties except 8. However, it fails badly here, even in dimension 2, and 8 is essential. So let's take a look at the plus closure.

Definition 2.1. Let R be an integral domain with quotient field E. Let F be an algebraic closure of E and R^+ be the integral closure of R in F.For an ideal I of R, the plus closure of I is $I^+ = IR^+ \cap R$. Equivalently, $x \in I^+$ iff there is a finite integral extension S of R such that $x \in IS$.

Whether or not it works, plus closure is important. To see that, we want to discuss the following conjectures:

Conjecture 2.2. (Direct Summand) Let R be a regular local ring and S a module-finite R-algebra containing R. Then R is a direct summand of S as R-modules.

Conjecture 2.3. Let R be a regular local ring and let $x_1, ..., x_n$ be a s.o.p.If S is a module-finite R-algebra containing R, then $(x_1 x_2 ... x_n)^k \notin (x_1^{k+1}, ..., x_n^{k+1})S$

Conjecture 2.4. Plus closure satisfies 8 (the tightness property).

Hochster (1973) showed that these 3 conjectures are equivalent. The implication $2.4 \to 2.3$ is obvious. The hardest part is $2.3 \to 2.2$ and we will skip that. Here is a proof of the implication $(2.2 \to 2.4)$:

Let R be a regular local ring, I an ideal of R, and $x \in I^+$. Then $x = y_1 z_1 + ... + y_n z_n$ with $z_1, ..., z_n \in R^+$. Let $S = R[z_1, ..., z_n]$, a module- finite extension of R. Then $x \in IS$. By 2.2 we have an R-linear map $f : S \to R$ whose composition with the inclusion $R \hookrightarrow S$ is the identity map. So:

$$x = f(x) = f(z_1 y_1 + ... + z_n y_n) = f(z_1)y_1 + ... + f(z_n)y_n \in I$$

Hence, to prove the Direct Summand Conjecture, it suffices to show that plus closure, or a closure that contains it, has the tightness property.

Plus closure itself trivially satisfies properties 1-6. Moreover, we can use persistence to extend the definition to all Noetherian rings (For non-domains, we just let I^+ be the largest ideal allowed by the persistence property : $x \in I \Leftrightarrow \varphi(x) \in (\varphi(I))(R/p)^+$ for every surjection $\varphi : R \to R/p, p \in Spec(R)$).

However,in equicharacteristic 0, plus closure is too small to be of value:

Theorem 2.5. Let R be an integrally closed domain of equicharacteristic 0 and $I \subseteq R$ is an ideal. Then $I^+ = I$.

Proof. As in the preceeding proof, it is enough to show that $IS \cap R = I$ for any module-finite extension S of R. We have a diagram :

$$\begin{tikzcd} E \arrow[r, hook] & K \\ R \arrow[u, hook] \arrow[r, hook] & S \arrow[u, hook] \end{tikzcd}$$

with E, K the respective quotient fields. The trace map $Tr : S \to R$ which takes each element to its sum of images under Galois group action (allowing repetition) is a module homomorphism. Note that $Tr(r) = nr$ for $n = [K : E]$ and all $r \in R$. So $\frac{1}{n}Tr$ is a splitting map and, as in the earlier proof, $IS \cap R = I$.

□

In equicharacteristic p plus closure satisfies properties 1-9. (The problem is 10!. It is much easier to decide if $x \in I^*$ than if $x \in I^+$). We will discuss the proof of the very important property 7, colon-capturing :

Theorem 2.6. Let R be an excellent local domain of characteristic p. Then R^+ is a big Cohen-Macaulay algebra for R, i.e, every system of parameters for R is a regular sequence on R^+.

As a consequence :

$$((x_1, ..., x_{n-1})R :_R x_n) \subseteq ((x_1, ..., x_{n-1})R^+ :_{R^+} x_n) \cap R = (x_1, ...x_{n-1})R^+ \cap R = (x_1, ...x_{n-1})^+$$

So Theorem 2.6 implies colon-capturing (Actually, it can be shown that they are equivalent). The following lemma is very useful:

Lemma 2.7. Let R be a domain of characteristic p and $x_1, ..., x_n$ a system of parameters. Suppose that:

$$((x_1^k, ..., x_{n-1}^k) :_R x_n^k) = (x_1^k, ..., x_{n-1}^k)R + (x_1...x_{n-1})^{k-1}((x_1, ..., x_{n-1}) :_R x_n)$$

for every integer k. Then there exist a module finite extension S of R such that:

$$((x_1, ..., x_{n-1}) :_R x_n) \subseteq (x_1, ..., x_{n-1})S$$

In other words: $((x_1, ..., x_{n-1}) :_R x_n) \subseteq (x_1, ..., x_{n-1})^+$

Proof. Suppose $z \in ((x_1, ..., x_{n-1}) :_R x_n)$. So $x_n z = x_1 a_1 + ... + x_{n-1}a_{n-1}$ Then $x_n^p z^p = x_1^p a_1^p + ... + x_{n-1}^p a_{n-1}^p \in (x_1^p, ..., x_{n-1}^p)R$. Thus $z^p \in (x_1^p, ..., x_{n-1}^p)R + (x_1 ... x_{n-1})^{p-1}((x_1, ..., x_{n-1}) :_R x_n)$. Let $z_1, ..., z_t$ be generators of $(x_1, ..., x_{n-1}) :_R x_n$ and $x = x_1 ... x_{n-1}$. We get a system of equations:

$$z_i^p = \sum_{j=1}^{n-1} a_{ij} x_j^p + x^{p-1} \sum_{k=1}^{t} b_{ik} z_k \qquad (*)$$

Our plan is to first find integral elements $u_{ij} | 1 \leq i \leq t, 1 \leq j \leq n-1$ such that $\bar{z}_i = \sum_{j=1}^{n-1} u_{ij} x_j$ is a "solution" to (*). Since $\bar{z_i}^p = \sum_{j=1}^{n-1} u_{ij}^p x_j^p$, (*) becomes :

$$\sum_{j=1}^{n-1} u_{ij}^p x_j^p = \sum_{j=1}^{n-1} [a_{ij} + (\frac{x}{x_j})^{p-1} \sum_{k=1}^{t} b_{ik} u_{kj}] x_j^p$$

and we can solve this by equating the coefficients of x_j^p. That leads to a new system:

$$u_{ij}^p = a_{ij} + (\frac{x}{x_j})^{p-1} \sum_{k=1}^{t} b_{ik} u_{kj} \qquad (**)$$

This shows that $R[u_{ij}]$ can be generated as an R module by elements of the forms $\prod u_{ij}^{n_{ij}}$, $0 \leq n_{ij} < p$. Therefore, it is an integral extension of R. Now, in this extension, we get :

$$z_i^p - \bar{z_i}^p = (\frac{x}{x_j})^{p-1} \sum_{k=1}^{t} b_{ik}(z_k - \bar{z}_k)$$

If we let $w_i = z_i - \bar{z}_i$, this becomes:

$$w_i^p = x^{p-1} \sum_{k=1}^{t} b_{ik} w_k$$

or:

$$(\frac{w_i}{x})^p = \sum_{k=1}^{t} b_{ik} (\frac{w_k}{x})$$

Hence the $\frac{w_i}{x}$ are integral. It follows that $z_i = \sum_{j=1}^{n-1} u_{ij} x_j + (\frac{w_i}{x})x$ is in $(x_1, ..., x_{n-1})^+$

□

The lemma is a great tool but in order to use it to show colon-capturing, we need to satisfy the hypothesis :

$$((x_1^k, ..., x_{n-1}^k) :_R x_n^k) = (x_1^k, ..., x_{n-1}^k)R + (x_1...x_{n-1})^{k-1}((x_1, ..., x_{n-1}) :_R x_n)$$

First,let us make the harmless assumption that R is integrally closed and for simplicity, that $n = 3$ (if $n = 2$ R would be Cohen-Macaulay when we normalize).

$$((y_1, y_2) : y_3) \cong H^2(y_1, y_2, y_3, R)$$

the Koszul cohomology. If $w \in ((y_1^k, y_2^k) : y_3^k)$ and $m > k$, then $(y_1y_2)^{m-k}w \in ((y_1^m, y_2^m) : y_3^m)$ We get a system :

$$(y_1, y_2) : y_3 \hookrightarrow (y_1^2, y_2^2) : y_3^2 \hookrightarrow ...$$

whose limit is the local cohomology module $H_m^2(R)$. This is a finite module, due to the fact that if R is an excellent domain such that R_p is Cohen-Macaulay for all $p \neq m$, then $H_m^i(R)$ has finite length for $i < \dim R$. Here R_p is an normal domain of dimension at most 2, so R_p is Cohen-Macaulay.

Since the limit of the system is a finite module, it stabilizes at $(y_1^l, y_2^l) : y_3^l$ for some l. Letting $x_i = y_i^l$, it follows that the map $((x_1, x_2) : x_3 \to ((x_1^k, x_2^k) : x_3^k)$ is surjective But it is the same as : $((x_1^k, x_2^k) : x_3^k) = (x_1^k, x_2^k)R + (x_1x_2)^{k-1}((x_1, x_2) : x_3)$ So by the lemma, $((x_1, x_2) : x_3) \subseteq (x_1, x_2)^+$. Now, suppose $w \in ((y_1, y_2) : y_3)$. So $y_3w \in (y_1, y_2)R$, which implies $y_3^l(y_1y_2)^{l-1}w \in (y_1^l, y_2^l)R$, or $x_3(y_1y_2)^{l-1} \in (x_1, x_2)R$. So $(y_1y_2)^{l-1}w \in (x_1, x_2)^+$. Thus we can find integral element α, β such that :

$$(y_1y_2)^{l-1}w = y_1^l\alpha + y_2^l\beta$$

Which we can rewrite as $w = y_1\alpha^* + y_2\beta^*$, where $\alpha^* = \frac{\alpha}{y_1^{l-1}}$ and $\beta^* = \frac{\beta}{y_2^{l-1}}$. But α^*, β^* are both integral, and we are done.

For $n = 4$, this argument breaks down. The cohomology module $H^3_m(R)$ won't be finite. However, there has to be an overwhelming feeling that we can kill the homology if we can make the obstruction to killing that homology go away. The rest of the proof is clever but too intricate to present here.

3. Plus closure in mixed characteristic

Suppose that R is an integrally closed domain of mixed characteristic p.Then $R[p^{-1}]$ is equicharacteristic 0 and the plus closure there is trivial. It follows that for any ideal I of R and any $x \in I^+$, $p^n x \in I$ for some n. Hence neither colon-capturing nor property 9 is satisfied. On the other hand, plus closure is clearly not trivial:

Example 3.1. Let $R = Z[x, \sqrt{4-x^2}]$. Clearly $\sqrt{4-x^2} \notin (2, x)R$. Let $i^2 = -1$ and $\alpha = \frac{\sqrt{4-x^2}-ix}{2}$ Then i is integral over R, and so is α since $\alpha^2 + 2ix\alpha - 1 = 0$. So $\sqrt{4-x^2} = ix + 2\alpha \in (2, x)^+$.

The idea to deal with the mixed characteristic case is to create a closure operation with plus closure at its core. It is not clear what the right definition is but I have been working with 2 candidates. In the definitions below let (R, m) be a local ring of mixed characteristic and I an ideal of R

Definition 3.2. $x \in R$ is in the *full extended plus closure* of I if there exists $c \neq 0 \in R$ such that for every positive integer n, $c^{1/n}x \in (I, p^n)R+$. We write $x \in I^{epf}$.

Definition 3.3. $x \in R$ is in the *full rank one closure* of I if for every valuation v on R of rank at most 1, every positive integer n, and every $\epsilon > 0$, there exists $d \in R^+$ with $v(d) < \epsilon$ such that $dx \in (I, p^n)R^+$. We write $x \in I^{r1f}$.

Remark. We can ignore rank zero valuations unless R is a field. Occasionally we will use I^{cl} to indicate both closures. The word "full" will also be obmitted.

The definition of extended plus closure is reminiscent of tight closure: $cz^q = ax^q + by^q$ with $q = p^e$ means $c^{1/q}z = a^{1/q}x + b^{1/q}y$. In fact the definitions make sense in equicharacteristic p and for complete local mdomains, both coincide with tight closure.

It is easy to see that these closures satisfy 1,2,3,6. Properties 4,5 are hard and will be swept under rug!. Property 9 also holds (the definition of extended plus closure is motivated by that fact). In fact we can get a more general result:

Theorem 3.4. Supose $\dim R = d$ and I is an ideal of R generated by n elements. If $y \in \overline{I^{n+d}}$ then $y \in (I^{d+1})^{cl}$.

While persistence and localization remains elusive, the most critical properties we want are tightness and colon-capturing. Tightness seems almost impossible to prove directly, but we were delighted to find that it is implied by colon-capturing.

To prove this, we first note that if tightness fails for a regular local ring R, it fails for $\hat{R}$:

$$\begin{tikzcd} I \arrow[r, hook] \arrow[d, hook] & R \arrow[d, hook] \\ I\hat{R} \arrow[r, hook] & \hat{R} \end{tikzcd}$$

so we may assume R is complete. We begin with a lemma which is actually enough to show that colon-capturing implies the Direct Summand Conjecture.

Lemma 3.5. Let R be a complete local domain such that colon-capturing holds for ^{cl} and assume that every finite integral extension of R is equidimensional. Let $x_1, ..., x_n$ be a full system of parameters, let $r = x_1^{f_1}...x_n^{f_n}$ and $J = (x_1^{f_1+1}, ..., x_n^{f_n+1})R$. Then $(J : r) \subseteq (x_1, ..., x_n)^{cl}$.

Proof. We will prove it for rank one closure. Since J is primary, $p^N \in J$ for some N. Thus we may ignore p^n in the definition of the closure. We will allow R to vary if necessary and prove the result by induction on $e = f_1 + ... + f_n$. The case $e = 0$ is trivial. Assume the result is true for $e < k$ and now want to prove for $e = k > 0$. Some f_i must be positive. WLOG, assume $f_n > 0$. Assume $b \in (J : r)$. So :

$$bx_1^{f_1}...x_n^{f_n} = a_1x_1^{f_1+1} + ... + a_nx_n^{f_n+1}$$

or :

$$x_n^{f_n}(bx_1^{f_1}...x_{n-1}^{f_{n-1}} - a_nx_n) = a_1x_1^{f_1+1} + ... + a_{n-1}x_{n-1}^{f_{n-1}+1}$$

By the colon-capturing property, which we assume, $bx_1^{f_1}...x_n^{f_n} - a_n x_n \in (x_1^{f_1+1}, ..., x_{n-1}^{f_{n-1}+1})^{r1f}$. Thus $bx_1^{f_1}...x_{n-1}^{f_{n-1}} \in (x_1^{f_1+1}, ..., x_{n-1}^{f_{n-1}+1}, x_n)^{r1f}$. Take any valuation v of R, and any $\epsilon > 0$. We can find $d \in R^+$ with $v(d) < \epsilon/2$ and $dbx_1^{f_1}...x_{n-1}^{f_{n-1}} \in (x_1^{f_1+1}, ..., x_{n-1}^{f_{n-1}+1}, x_n)R^+$. Thus we can find a finite integral extension S of R containing d such that $dbx_1^{f_1}...x_{n-1}^{f_{n-1}} \in (x_1^{f_1+1}, ..., x_{n-1}^{f_{n-1}+1}, x_n)S$. By the induction hypothesis applied on S, we get $db \in ((x_1, ..., x_n)S)^{r1f}$. Hence there exists $c \in S^+$ with $v(c) < \epsilon/2$ and $cdb \in (x_1, ..., x_n)S^+$. But $S^+ = R^+$ and $v(cd) < 2\epsilon/2 = \epsilon$. So $b \in (x_1, ...x_n)^{1rf}$.

$\square$

Now, note that if R is regular, S is integral over R, and suppose :

$$(x_1...x_n)^k \in (x_1^{k+1}, ..., x_n^{k+1})S$$

then the above lemma says : $1 \in (x_1, ..., x_n)^{r1f} \subseteq \overline{(x_1, ..., x_n)}$, clearly impossible. Thus colon-capturing implies Lemma 2.3 and so also the Direct Summand Conjecture.

Next, we want to discuss how colon-capturing also implies the existence of balanced big Cohen-Maculay algebras for R. Recall that an R-algebra B is a balanced big Cohen-Maculay algebra for R if $mB \neq B$ and every system of parameters for R is a regular sequence on B. Hochster (2002) used the results in [Heitmann,2002] to show that for dimension 3, balanced big Cohen-Macaulay algebras exists in a weakly functorial sense :

Theorem 3.6. Let $(R, m) \to (S, n)$ be a local homomorphism of complete local domains of mixed characteristic and dimension at most 3. Then there is a commutative diagram :

$$\begin{tikzcd} B \arrow[r] & C \\ R \arrow[u] \arrow[r] & S \arrow[u] \end{tikzcd}$$

where B, C are balanced big Cohen-Macauley algebras over R, S respectively.

The existence of weakly functorial balanced big Cohen-Macaulay algebras is a very useful result. (See the lectures of Hochster in this summer course).

The proof of Theorem 3.6 starts with the idea of "partial algebra modifications". We will first try to create a balanced big Cohen-Macaulay algebra for R in a relatively free way. Consider a system of parameters $x_1, ..., x_n$ and an R-algebra T, if for some k, and α, $x_{k+1}\alpha \in (x_1, ...x_k)T$, we will force α into $(x_1, ..., x_k)T$ by letting $T' = T[X_1, ...X_k]/(\alpha - x_1X_1 - ... - x_kX_k)$. Start with $R_0 = R$ and repeat the process with careful indexing, the direct limit R_∞ has the property that $x_1, ..., x_n$ is a regular sequence. The problem is $(x_1, ..., x_n)R_\infty$ may be the whole ring !. If that is the case, at some step i we must have $1 \in mR_i$ (Here 1 is the image in R_i of $1 \in R$). First, observe that, at each step, R_i is a direct limit of modules $M_{ij} = \frac{R_{i-1}[X_1,...,X_k]_{degree\leq j}}{(\alpha - x_1X_1 - ... - x_kX_k)R_{i-1}[X_1,...,X_k]_{degree\leq j-1}}$. The violating condition $1 \in mR_i$ must also becomes $1 \in mM_{ij}$ for some j.

So we could consider a sequence of R-modules $M_0 = M$, $M_{i+1} = \frac{M_i[X_1,...,X_k]_{degree\leq N_i}}{(\alpha - x_1X_1 - ... - x_kX_k)R[X_1,...,X_k]_{degree\leq N_i - 1}}$. Such a sequence is called a *sequence of partial algebra modifications*. We can build a double sequence over $R \to S$ as $M_0 = R, M_1, ..., M_r, N_0 = S \otimes_R M_r, N_1, ..., N_s$. A double sequence is called *bad* if the image of $1 \in R$ in N_s is in nN_s.

It is not hard to show that weakly functorial balanced big Cohen-Macaulay algebras for R, S exist iff no bad double sequence of partial algebra modifications exists. The difficult part of the proof is to build, from a bad double sequence and a fixed rational number $\epsilon > 0$, a commutative diagram :

$$\begin{tikzcd}
R[\frac{1}{p}]^+ \arrow[r, "="] & R[\frac{1}{p}]^+ \arrow[r, "="] & \cdots \arrow[r, "="] & R[\frac{1}{p}]^+ \arrow[r] & S[\frac{1}{p}]^+ \arrow[r, "="] & S[\frac{1}{p}]^+ \arrow[r, "="] & \cdots \arrow[r, "="] & S[\frac{1}{p}]^+ \\
R \arrow[u] \arrow[r] & M_1 \arrow[u] \arrow[r] & \cdots \arrow[r] & M_r \arrow[u] \arrow[r] & S \otimes_R M_r \arrow[u] \arrow[r] & N_1 \arrow[u] \arrow[r] & \cdots \arrow[r] & N_s \arrow[u]
\end{tikzcd}$$

such that the first vertical map is the inclusion map, the vertical maps from M_i, N_i are R, S-linear respectively, and the image of each N_j is inside $p^{-D_{r+j}\epsilon}S^+$, where D_i's are positive integers formed recursively. Then, chasing 1 from R to the rightmost $S[\frac{1}{p}]^+$ in two ways, we can see that $1 \in np^{-D_{r+s}\epsilon}S^+$ (because of badness). Let $\epsilon = 1/N$, this means that $p^{D_{r+s}} \in n^N S^+$ for all $N > 0$, a contradiction !.

To be able to build the above commutative diagram, we need a key result from Heitmann's 2002 paper which showed that $p^\epsilon H^2_m(R^+) = 0$ for all rational $\epsilon > 0$ (So, roughly speaking, plus closure "almost" has colon-capturing property. See Lecture 4 of this note). So, it is natural to search for a closure whose colon-capturing property implies weakly functorial balanced big Cohen-Macaulay algebras. Also, it would be nice if the colon-capturing property of such a closure follows from colon-capturing property of closures defined at the beginning of the lecture. Of course, such a closure must be comparatively large. A somewhat technical candidate is discussed in the Appendix.

4. Direct Summand Conjecture in Dimension 3

Unlike tight closure, the extended plus closure does not let us bypass the understanding of the plus closure. In particular, when is an element $z \in I^+$?. In the case $I = (x, y)R$, we have a tool. Suppose $z \in I^+$, then $z = vx + wy$ with $v, w \in R^+$. Choosing v, w is the same as choosing their minimal monic polynomials $f(T), g(T)$. But since $v = \frac{z-vx}{y}$, f determines a polynomial for w, whose coefficients are in the quotient field of R, and the denominators are powers of y. For w to be in R^+, those coefficient must be in R. Specifically, if $f(T) = T^n + a_1T^{n-1} + ... + a_n$ we need : $\sum_{j=0}^{i} \binom{n-j}{i-j} a_j z^{i-j} x^j \in y^i R$ for $1 \leq i \leq n$. The nice thing is that the first condition depends on a_1, the second on a_1, a_2, the third on a_1, a_2, a_3, etc. Nothing like that work for 3-generators ideals. There is too much freedom !

Let (R, m) be an excellent domain of mixed characteristic p.

Theorem 4.1. (Unpublished) If x_1, x_2, x_3 are parameters, then $((x_1, x_2) : x_3) \subseteq (x_1, x_2)^{epf}$.

Theorem 4.2. If p, x, y are parameters, $((x, y) : p^N) \subseteq (x, y)^{epf}$ $\forall$ N. In fact $p^\epsilon((x, y) : p^N) \subseteq (x, y)R^+$ for all rational $\epsilon > 0$.

Theorem 4.3. If $\dim R = 3$, $p^\epsilon H^2_m(R^+) = 0$ for all rational $\epsilon > 0$.

Note that theorem 4.3 implies the Direct Summand Conjecture, and more, for dimension 3 (see Lecture 3).

We will ouline the proof of Theorem 4.3. We may assume R is integrally closed. Since R satisfy (S_2), a power of any element in m is a colon-killer.Assume p^N, x, y are colon-killing parameters. It suffices to prove $p^\epsilon((x,y):p^N) \subseteq (x,y)R^+$ for all $\epsilon > 0$. For simplicity, we also assume that $\frac{((x,y):p^N)}{(x,y)R}$ is cyclic, so $((x,y):p^N) = (x,y,z)R$ for some z. We want to show $p^\epsilon z \in (x,y)^+$ for all $\epsilon > 0$. It suffices to prove it for $\epsilon = \frac{1}{p^{K+1}}$, K any positive integer.

The colon-killing property says that $\forall\ m,k \geq N, ((x^m,y^m) : p^k) = ((x,y) : p^N)(xy)^{m-1} + (x^m,y^m)R$ (1). In our cyclic situation and $k = N$, it says: $((x^m,y^m):p^N) = (x^m,y^m,(xy)^{m-1}z)$ (2).

First, we adjoin p^ϵ to R. Now we want to construct v,w integral over R such that $p^\epsilon z = yv + xw$. We will pick v to be a root of a monic polynomial $f(T)$ of degree p^L, $L = N+K$. Then $w = \frac{p^\epsilon z - yv}{x}$. As above, w is integral over R provided that :

$$\sum_{j=0}^{i} \binom{N-j}{i-j} a_j p^{e(i-j)} z^{i-j} y^j \in (x^i)R$$

for each $i = 1, ..., p^L$. Viewed another way, for each such i, we want to find a_i, b_i so that :

$$\sum_{j=0}^{i-1} \binom{N-j}{i-j} a_j p^{e(i-j)} z^{i-j} y^j = b_i x^i - a_i y^i$$

We will illustrate the idea in the proof by working on a specific case : $p = 2, N = 2, L = 3$ (so $K = 1$ and $\epsilon = 1/4$).

$i = 1$: $8p^\epsilon z = b_1 x - a_1 y$. Solving this is trivial since we already have $4z + bx + ay = 0$ for some a, b. In fact we can choose a_1, b_1 to be multiples of $2^{1+\epsilon}$.

$i = 2$: $28p^{2\epsilon} z^2 + 7p^e z a_1 y = b_2 x^2 - a_2 y^2$. To do this we need a result that says essentially that we can't pick a_1 too badly :

Lemma 4.4. Let $z \in ((x,y) : p^N)$. Assume we have chosen $a_0 = 1, a_1, ..., a_{k-1} \in R$ so that :

$$\sum_{j=0}^{i} \binom{N-j}{i-j} a_j p^{e(i-j)} z^{i-j} y^j \in (x^i)R$$

for $i = 1, ..., k-1$. Then, for some M :

$$p^M \left(\sum_{j=0}^{k-1} \binom{N-j}{k-j} a_j p^{e(k-j)} z^{k-j} y^j \right) \in (x^k, y^k)R$$

Thus, in our case, $z_1 = 28p^{2\epsilon}z^2 + 7p^e z a_1 y \in ((x^2, y^2) : 4) = (x^2, y^2, xyz)R$ (from (2)). Therefore $z_1 = Ay^2 + Bx^2 + Cxyz$. Note that the terms in z_1 are divisible by $2^{1+2\epsilon}$ (remember $p = 2!$), so we can also make A, B, C to be divisible by $2^{1+2\epsilon}$. Finally, to get rid of C we replace a_1 by $a_1 - \frac{2^{-\epsilon}Cx}{7}$ (it is still in R because 7 is a unit and C is divisible by $2^{1+2\epsilon}$.

$i = 3 :\ 56p^{3\epsilon}z^3 + 21p^{2\epsilon}z^2 a_1 y + 6p^\epsilon z a_2 y^2 = b_3 x^3 - a^3 y^3$

This is handled similarly but there is a twist. The LHS can be written as $Ay^3 + Bx^3 + Cx^2y^2z$ with A, B, C divisible by $2^{1+3\epsilon}$. Again we get rid of C by "tweaking" a_2 but the number 6 complicates things: the new a_2 is only divisible by $2^{2\epsilon}$. We have lost a factor of 2!.

When we come to $i = 5$, we lose a second power of 2 and have to settle for a new a_4 which is divisible by $2^{4\epsilon-1}$. The fact that $\epsilon = \frac{1}{4}$ saves us! If we go through the entire process, we see that the powers we lose from $a_1, ..., a_8$ are 0,1,1,2,1,2,2,3 respectively (actually the last one isn't needed unless the degree is greater than 8. In general, it can be shown that the rate at which we lost powers is logarithmic. So we can offset it with the gain from p^ϵ, which is linear. Hence using higher degrees for $f(T)$ allows us to lower ϵ.

In the non-cyclic case, for example when $i = 2$ we have to use (1) to write $z_1 = Ay^2 + Bx^2 + Cxyz_2$ with $z_2 \in ((x,y) : 4)$. Monitoring the powers of p precisely becomes much more difficult (In fact, we had to use power of $\sigma = \sqrt[p-1]{p}$) and we will need higher degree polynomials.

References

Lecture 1:

M. Auslander, *Modules over unramified regular local rings*, Illinois J. Math. **5** (1961), 631-645.

D. Buchsbaum, D. Eisenbud, *Generic free resolutions and a family of generically perfect ideals*, Advances. in Math. **18** (1975), 245-301.

R. Heitmann, *A counterexample to the rigidity conjectures for rings*, Bull. Amer. Math. Soc. **29** (1993), 94-97.

S. Lichtenbaum, *On the vanishing of Tor in regular local rings*, Illinois J. Math. **10** (1966), 220-226.

Lecture 2:

M. Hochster, *Contracted ideals from integral extensions of regular rings*, Nagoya Math. J. **51** (1973), 25-43.

M. Hochster, C. Huneke, *Infinite integral extensions and big Cohen-Macaulay algebras*, Ann. of Math. **135** (1992), 53-89.

Lecture 3:

R. Heitmann, *The direct summand conjecture in dimension three*, Ann. of Math. **156** (2002), 695-712.

R. Heitmann, *Extensions of plus closure*, J. Algebra. **238** (2001), 801-826.

M. Hochster, *Big Cohen-Macaulay algebras in dimension three via Heitmann's theorem*, J. Algebra. **254** (2002), 395-408.

Lecture 4:

R. Heitmann, *The direct summand conjecture in dimension three*, Ann. of Math. **156** (2002), 695-712.