Paul Roberts - Lecture I

Several of the early homological conjectures were settled by the following result, and several later ones are meant to generalize it:

Theorem 1 (Peskine-Szpiro). *(Intersection Theorem) Let M, N be nonzero finitely generated modules over a local ring A such that $l(M \otimes_A N) < \infty$. Then* $\dim N \leq \operatorname{pd} M$.

Note that this theorem is really one about modules of finite projective dimension, since the assertion is trivial if M has infinite projective dimension. Furthermore, as phrased in Peskine and Szpiro's original paper, it is a statement about the topology of the support of a module of finite projective dimension (see Peskine-Szpiro [5]).

Example: For an easy example, let A be Cohen-Macaulay and $M = A/(x_1, \ldots, x_k)$ where the x_i form an A-regular sequence. Then $M \otimes N = N/(x_1, \ldots, x_k)N$. Note that since the x_is were A-regular, the Koszul complex on the x_is gives a minimal free resolution of M and hence $\operatorname{pd} M = k$. Hence the Intersection Theorem says that $\dim N \leq k$, and this case is a consequence of Krull's Principal Ideal Theorem.

Along a similar vein, we give an elementary fact about the support of modules of finite projective dimension that will be used throughout the lectures.

Proposition 2. *Suppose that A is a Noetherian ring and M a module of finite projective dimension over A. Suppose there exists $\mathfrak{p} \in \operatorname{Supp}(M) \cap \operatorname{Ass}(M)$. Then* $\operatorname{Supp}(M) = \operatorname{Spec}(A)$.

Proof. Let

$$F. : 0 \to F_k \to \cdots \to F_0 \to M \to 0$$

be a resolution of length k. Then $M_{\mathfrak{p}} \neq 0$, as $\mathfrak{p} \in \operatorname{Supp}(M)$, and also $A_{\mathfrak{p}}$ has depth 0 as $\mathfrak{p} \in \operatorname{Ass}(A)$. Now the Auslander-Buchsbaum formula tells us that $\operatorname{pd} M_{\mathfrak{p}} + \operatorname{depth} M_{\mathfrak{p}} = \operatorname{depth} A_{\mathfrak{p}} = 0$, and hence $\operatorname{pd} M_{\mathfrak{p}} = 0$ so that $M_{\mathfrak{p}}$ is free. So, localizing the above resolution at $\mathfrak{p}$ gives us that $\operatorname{rank}(M_{\mathfrak{p}}) = \sum_{i=0}^{k} (-1)^i \operatorname{rank}(F_i) > 0$. Therefore, we must have that $M_{\mathfrak{q}} \neq 0$ for all $\mathfrak{q} \in \operatorname{Spec} A$. Therefore, $\operatorname{Supp}(M) = \operatorname{Spec}(A)$. ∎

As somewhat of a motivation towards the multiplicity conjectures, also note that in certain 'nice' situations, $\operatorname{pd} M \leq \dim A - \dim M$, and so the intersection theorem says that $\dim M + \dim N \leq \dim A$. This is a result proved by Serre in the regular case and remains open in many other cases. However, this motivation does not work in all cases, since the inequality $\operatorname{pd} M \leq \dim A - \dim M$ does not always hold.

The following was once a conjecture of Auslander and is a consequence of Peskine and Szpiro's Intersection Theorem.

Theorem 3. *(Auslander's Zero-Divisor Conjecture) Suppose M is a nonzero module of finite projective dimension. If $x \in A$ is a nonzerodivisor on M then x is a nonzerodivisor on A.*

We give an equivalent formulation of the above theorem below. Also, we show why the zerodivisor conjecture is a consequence of the intersection theorem.

Theorem 4. *For all $\mathfrak{p} \in \operatorname{Ass}(A)$, there exists $\mathfrak{q} \in \operatorname{Ass}(M)$ with $\mathfrak{q} \supseteq \mathfrak{p}$.*

Proof. We proceed by induction on $\dim A$. Assume that there exists a counterexample $A, M \neq 0$, $\operatorname{pd} M < \infty$, and $\mathfrak{p} \in \operatorname{Ass}(A)$. We wish to find a prime $\mathfrak{q} \in \operatorname{Supp}(M)$, $\mathfrak{p} \subseteq \mathfrak{q}$ and $\mathfrak{q} \neq \mathfrak{m}$, so that $\dim A_q < \dim A$. Then we would have that $p_{\mathfrak{q}} \in \operatorname{Ass}(A_{\mathfrak{q}})$, and by induction we would have that there exists $\mathfrak{q}' \in \operatorname{Ass}(M_{\mathfrak{q}})$ such that $\mathfrak{q}' \supseteq \mathfrak{p}_{\mathfrak{q}}$. But then this $\mathfrak{q}'$ would correspond to an associated prime of M that contradicts the fact that A and M were supposed to be a counterexample. So, the above argument fails if we cannot find the $\mathfrak{q}$, i.e., if and only if $\operatorname{Supp}(A/\mathfrak{p}) \cap \operatorname{Supp}(M) = \{\mathfrak{m}\}$, i.e. if and only if $l(A/\mathfrak{p} \otimes M) < \infty$. Hence, the intersection theorem says that $\dim A/\mathfrak{p} \leq \operatorname{pd} M$. So $\mathfrak{p} \in \operatorname{Ass}(A)$ implies that $\operatorname{depth}(A) \leq \operatorname{pd}(M)$. However, the Auslander-Buchsbaum formula again tells us that $\operatorname{depth} M = 0$ hence $\mathfrak{m} \in \operatorname{Ass}(M)$, so $\mathfrak{m} \supset p$, hence $\mathfrak{m}$ satisfies the conditions of $\mathfrak{q}$ in the conjecture. $\square$

Another conjecture settled by the intersection theorem was one of Bass:

Theorem 5. *(Bass's Conjecture) If A has a finitely generated nonzero module N of finite injective dimension, then A is Cohen-Macualay.*

We sketch the proof of this conjecture below (for a complete proof, see Peskine-Szpiro [6], II.5). We will use the fact, proven in the cited reference, that if there is finitely generated module of finite injective dimension, then there is a finitely generated module of finite projective dimension with the same support.

Proof. If N has dimension zero, so has finite length, by the above remarks there is also a module M of finite length and finite projective dimension. The Intersection Theorem implies that the projective dimension of M is equal to the dimension of the ring A. On the other hand, the Auslander-Buchsbaum formula says that the projective dimension of M is equal to the depth of A, so A is Cohen-Macaulay.

In general, the proof is by induction on the dimension of N, which we now assume is at least 1. We know by the Bass formula that $\mathrm{injdim}(N) = \mathrm{depth}(A)$. We want a prime $\mathfrak{q}$ such that $\mathfrak{q} \in \mathrm{Supp}(N)$ and $\dim A_{\mathfrak{q}} = \dim A - 1$. We may assume that the ring A is catenary, so the only way this would not be possible is if it were true that for every $\mathfrak{q}$ in the support of N with $\dim(A/\mathfrak{q}) = 1$, and for every minimal prime $\mathfrak{p}$ of A contained in $\mathfrak{q}$, we had $\dim(A/\mathfrak{p}) < \dim(A)$. If this is the case, let $\mathfrak{p}$ be a minimal prime of A with $\dim(A/\mathfrak{p}) = \dim(A)$; the assumptions now imply that $A/\mathfrak{p} \otimes_A N$ has finite length. The Intersection Theorem then implies that the projective dimension of a module of finite projective dimension with the same support as N has projective dimension at least $\dim(A)$, and the Auslander-Buchsbaum formula again implies that A is Cohen-Macaulay.

Thus we may assume that there is a prime ideal $\mathfrak{q}$ in the support of N with $\dim(A_{\mathfrak{q}}) = \dim(A) - 1$. Then by a lemma of Bass, $\mathrm{depth}\, A_{\mathfrak{q}} = \mathrm{depth}\, A - 1$. By induction on dimension $A_{\mathfrak{q}}$ is Cohen-Macaulay, and the above equalities imply the A is Cohen-Macaulay.

□

The following is a conjecture(now a theorem), of Hochster that implies the Intersection Theorem:

Theorem 6. *(Homological Height Conjecture) Let M be a finitely generated A-module of finite projective dimension, and let $I = \mathrm{ann}\, M$. Further, suppose that $f : A \to B$ is a homomorphism of rings and $\mathfrak{p}$ a minimal prime over IB. Then* $\mathrm{ht}\, \mathfrak{p} \leq \mathrm{pd}\, M$.

A more general theorem was conjectured that is related to the intersection theorem and even implies the homological height conjecture.

Theorem 7. *(New Intersection Theorem) Let A be a local ring of dimension d, and*

$$F. : 0 \to F_k \to \cdots \to F_0 \to 0$$

be a complex of free modules with $l(H_i(F.)) < \infty$ *for all* i*. Then if* $k < d$*, the complex is exact.*

To see the connection to the original intersection theorem, consider an A-module M with minimal free resolution

$$F. : 0 \to F_k \to \cdots \to F_0 \to 0.$$

Set $N = A/\mathfrak{p}$ and assume $l(M \otimes A/\mathfrak{p}) < \infty$. Then

$$F. \otimes A/\mathfrak{p} : 0 \to F_k \otimes A/\mathfrak{p} \to \cdots \to F_0 \otimes A/\mathfrak{p} \to 0$$

is a complex of free $A/\mathfrak{p}$-modules satisfying $l(H_i(F. \otimes A/\mathfrak{p})) < \infty$ for all i. But the complex $F. \otimes A/\mathfrak{p}$ is not exact, hence by the contrapositive of the New Intersection Theorem, we must have that $k \geq \dim A/\mathfrak{p}$, as desired.

Now we prove the New Intersection Theorem in positive characteristic. Before we do so, we need a lemma, which we will assume:

Lemma 8. *Let* A *be a local ring that is a homomorphic image of a Gorenstein ring, and suppose* $\dim A = d$*. Then there are ideals* $\mathfrak{a}_1, \ldots, \mathfrak{a}_d$ *such that* $\operatorname{ht} \mathfrak{a}_i \geq i$ *and whenever*

$$0 \to F_d \to \cdots \to F_0 \to 0$$

is a complex of free modules with $l(H_i(F.)) < \infty$*, then* $\mathfrak{a}_i$ *annihilates* $H_i(F.)$*.*

Proof. A counterexample to the New Intersection Theorem would look like

$$0 \to F_d \to F_{d-1} \to \cdots \overset{(a_{ij})}{\to} F_1 \to 0$$

where the complex is not exact and we may take the $a_{ij} \in \mathfrak{m}$. Note that $\mathfrak{a}_1$ has height at least 1. Now, tensoring the above complex with the eth iteration of the Frobenius gives

$$0 \to F_d \to F_{d-1} \to \cdots \to F_2 \overset{(a_{ij}^{p^e})}{\to} F_1 \to 0.$$

Therefore, $\operatorname{coker}(a_{ij}^{p^e})$ is annihilated by $\mathfrak{a}_1$, and so $\mathfrak{a}_1 \subseteq \mathfrak{m}^{p^e}$ for all e. Since e was arbitrary, we deduce that $\mathfrak{a}_1 = 0$ by Krull's intersection theorem, contradicting that it has height at least 1. $\square$

A proof of this theorem in mixed characterisitc can be found in Roberts [7].

So where do the $\mathfrak{a}_i$ come from? A dualizing complex for A (with dimension of A equal to d) is a complex

$$0 \to D^0 \to \cdots \to D^d \to 0$$

such that $H_i(D^\cdot)$ is finitely generated for all i and $D^i \cong \bigoplus_{\dim A/\mathfrak{p}=d-i} E(A/\mathfrak{p})$ where $E(A/\mathfrak{p})$ denotes the injective hull of $A/\mathfrak{p}$. In the example of a Gorenstein ring, the above is just given by a minimal injective resolution of the ring itself. Define $\tilde{\mathfrak{a}}_i = \operatorname{ann} H_i(D^\cdot)$, hence $\operatorname{ht} \tilde{\mathfrak{a}}_i \geq i$. Set $\mathfrak{a}_i = \tilde{\mathfrak{a}}_i \cdots \tilde{\mathfrak{a}}_d$. Note that the above complex is exact everwhere except at degree 0 if and only if A is Cohen-Macaulay. Hence, the $\mathfrak{a}_i$ give a measure of how far the ring is from being Cohen-Macaulay.

To end this lecture, we give another conjecture (proven in equicharacteristic, still open in mixed characteristic) that is of a similar flavor to the New Intersection Theorem. Appropriately, Hochster coined it the Improved New Intersection Theorem:

Conjecture 1. *(Improved New Intersection Conjecture) Let $(R, \mathfrak{m})$ be a local ring and let*

$$0 \to F_k \to F_{k-1} \to \cdots \to F_0 \to 0$$

be a complex of free R-modules. Assume now that $\ell(H_i(F.)) < \infty$ for all $i > 0$ and that F_0 has a minimal generator whose image in $H_0(F.)$ is annihilated by a power of $\mathfrak{m}$. Then $F.$ is exact.

Paul Roberts - Lecture II

Local Chern Characters

Throughout this section, let $(A, \mathfrak{m})$ denote a local ring of dimension d.

Recall that the component of dimension i of the Chow group of a local ring A is

$$CH_i(A) = \frac{\text{cycles}}{\text{rational equivalence}}.$$

The cycles form the free abelian group on $\left[\frac{A}{\mathfrak{p}}\right]$ where $\mathfrak{p}$ is a prime ideal of A with $\dim(A/\mathfrak{p}) = i$. We define rational equivalence by killing elements of the form

$$\operatorname{div}(x, \mathfrak{q}) = \left[\frac{A}{(\mathfrak{q}, x)}\right]_i$$

for all prime ideals $\mathfrak{q}$ of A with $\dim(A/\mathfrak{q}) = i+1$ and all elements $x \in A - \mathfrak{q}$. Also recall that

$$\left[\frac{A}{(\mathfrak{q}, x)}\right]_i = \sum_{\dim(A/\mathfrak{p})=i} \ell\left(\frac{A}{(\mathfrak{q}, x)}\right)_{\mathfrak{p}} \cdot \left[\frac{A}{\mathfrak{q}}\right]$$

We define the Chow group of A to be the direct sum of the components

$$CH_*(A) = \bigoplus_i CH_i(A)$$

We call the group $CH_*(A) \otimes_{\mathbb{Z}} \mathbb{Q}$ the rational Chow group of A.

Local Chern characters are operations on the Chow group defined by perfect complexes (bounded complexes of free modules). Let $F.$ be a perfect complex. Now define $\operatorname{Supp}(F.)$ to be the set of prime ideals $\mathfrak{p}$ of A such that $(F.)_{\mathfrak{p}}$ is not exact. Note that since localization is exact we have that $\operatorname{Supp}(F.) = \bigcup_i \operatorname{Supp}(H_i(F.))$.

Let $Z = \operatorname{Supp}(F.)$. The local Chern character defined by $F.$, denoted $\operatorname{ch}(F.)$, defines for each closed set $Y \subseteq \operatorname{Spec}(A)$, a map

$$CH_*(Y) \longrightarrow CH_*(Y \cap Z)$$

The local Chern character defined by $F.$ is actually a sum of the local Chern characters in each degree:

$$\operatorname{ch}(F.) = ch_0(F.) + \operatorname{ch}_1(F.) + \cdots + \operatorname{ch}_d(F.)$$

where

$$ch_i(F.) : CH_k(Y) \longrightarrow CH_{k-i}(Y \cap Z)$$

For a description of the construction of local Chern characters, see Fulton [3] or Roberts [8].

Examples

1. The map $ch_0(F.) : CH_k(Y) \to CH_k(Y)$ is multiplication by $\sum_j (-1)^j \operatorname{rank}(F_j)$.

2. Let $F. = 0 \to A \xrightarrow{\cdot x} A \to 0$. Then $ch(F.)$ acts on $CH_*(A)$ as intersection with the divisor defined by x. That is,

$$ch_1(F.) : \left[\frac{A}{\mathfrak{p}}\right]_i \mapsto \begin{cases} 0, & x \in \mathfrak{p} \\ \left[\frac{A}{(\mathfrak{p},x)}\right]_{i-1}, & x \notin \mathfrak{p} \end{cases}$$

   The higher local Chern characters vanish and ch_0 vanishes by the previous example.

3. Let $K.(x_1, \ldots, x_j)$ denote the Koszul complex on $x_1, \ldots, x_j$. Then $\mathrm{ch}_i(K.(x_1, \ldots, x_j))$ acts as the composition of the intersections with each of the x_i, $1 \leq i \leq j$.

Properties of local Chern characters

1. Given an exact sequence

$$0 \to F.' \to F. \to F.'' \to 0$$

   of complexes (exact in each degree) we have that

$$\mathrm{ch}(F.) = \mathrm{ch}(F.') + \mathrm{ch}(F.'')$$

   on the union of the supports of $F.$, $F.'$ and $F.''$.

2. Given two complexes $F.$ and $G.$,

$$\mathrm{ch}(F. \otimes G.) = \mathrm{ch}(F.) \cdot \mathrm{ch}(G.)$$

   That is,

$$\mathrm{ch}_k(F. \otimes G.) = \sum_{i+j=k} \mathrm{ch}_i(F.) \cdot \mathrm{ch}_j(G.).$$

   Note that $\mathrm{Supp}(F. \otimes G.) = \mathrm{Supp}(F.) \cap \mathrm{Supp}(G.)$.

   Suppose that $\mathrm{Supp}(F.) = Z$ and $\mathrm{Supp}(G.) = Z'$. Then geometrically, this property translates as

$$\mathrm{ch}(F. \otimes G.) : CH_*(Y) \to CH_*(Y \cap Z') \to CH_*(Y \cap Z' \cap Z)$$

3. Local Chern characters commute with the intersection with divisors.

4. Given complexes $F.$ and $G.$,

$$\mathrm{ch}_i(F.) \cdot \mathrm{ch}_j(G.) = \mathrm{ch}_j(G.) \cdot \mathrm{ch}_i(F.)$$

Local Riemann-Roch Formula

Given a bounded complex $M.$ there is a class $\tau(M.) \in CH_*(\mathrm{Supp}(M.))$ satisfying the following properties.

1. If $M.$ is a module M of dimension at most i then $\tau(M) = [M]_i +$ terms of lower degree.

2. If $M.$ has homology of finite length then

$$\tau(M.) = \chi(M.) = \sum_i (-1)^i \ell(H_i(M.))$$

3. (Local Riemann-Roch Formula) Given a perfect complex $F.$ and $M.$ an arbitrary bounded complex,

$$\tau(F. \otimes M.) = \mathrm{ch}(F.)\tau(M.)$$

The Serre Vanishing Theorem

Let A be a local domain of dimension d. From the properties above, $\tau(A) = [A]_d +$ terms of lower degree. It can be shown that if A is a complete intersection then $\tau(A) = [A]_d$. The following theorem thus gives the vanishing Theorem for complete intersections.

Theorem 9. *Suppose that for a local ring $(A, \mathfrak{m})$ of dimension d, $\tau(A) = [A]_d$. Let M and N be A-modules, each of finite projective dimension, such that $\ell(M \otimes N) < \infty$. Then $\dim(M) + \dim(N) < \dim(A)$ implies that $\chi(M, N) = 0$.*

Proof. Let $F. \to M$ and $G. \to N$ be free resolutions of M and N, respectively. Note that this implies that $F.$ and $G.$ are both perfect complexes. Furthermore, $\mathrm{Supp}(F.) = \mathrm{Supp}(M)$ and $\mathrm{Supp}(G.) = \mathrm{Supp}(N)$. Since $\ell(M \otimes N) < \infty$ we have from property 2 above that

$$\chi(M, N) = \chi(F. \otimes G.) = \sum_i (-1)^i \ell(H_i(F. \otimes G.)) = \tau(F. \otimes G.)$$

Now by the local Riemann-Roch Theorem and since the lower terms vanish, this is equal to

$$\mathrm{ch}(F. \otimes G.)\tau(A) = \mathrm{ch}_d(F. \otimes G.)[A]_d = \sum_{i+j=d} \mathrm{ch}_i(F.)\,\mathrm{ch}_j(G.)[A]_d = 0$$

The last equality deserves an explanation. Suppose that $j < \dim(A) - \dim(N)$. Then $\mathrm{Supp}(G.)$ has dimension strictly less than $d - j$ because $\mathrm{ch}_j(G.)[A]_d \in CH_{d-j}(\mathrm{Supp}(N)) = 0$. Similarly (using the commutativity described above) we see that if $i < \dim(A) - \dim(M)$, $\mathrm{Supp}(F.)$ has dimension strictly less that $d - i$. Now if $i \geq \dim(A) - \dim(M)$ and $j \geq \dim(A) - \dim(N)$ then $d = i + j \geq 2\dim(A) - \dim(M) - \dim(N)$, a contradiction. Thus we must be in one of the first two cases and the final equality is justified. □

Paul Roberts - Lecture III

Multiplicity Conjectures

Throughout, assume that R is a local ring, that M and N are finitely generated R-modules such that M has finite projective dimension and $\ell(M \otimes N) < \infty$. Recall that

$$\chi(M, N) = \sum_i (-1)^i \ell(\mathrm{Tor}_i(M, N))$$

Conjectures

M_0 $\dim(M) + \dim(N) \leq \dim(R)$

M_1 If $\dim(M) + \dim(N) < \dim(R)$ then $\chi(M, N) = 0$

- $\chi(M,N) \geq 0$

M_2 If $\dim(M) + \dim(N) = \dim(R)$ then $\chi(M,N) > 0$

conjecture	R regular	$\mathrm{pd}(M), \mathrm{pd}(N) < \infty$	$\mathrm{pd}(M) < \infty$
M_0	true (Serre)	open	open (true for $k[[\underline{x}]]/(f)$)
M_1	true (Roberts, Gillet-Soule)	open (true for C.I.'s)	false
•	true (Gabber)	open	false
M_3	open	open	false

Recall that

$$\chi_i(M,N) = \sum_{j=0}^{d} (-1)^j \ell(\mathrm{Tor}^R_{i+j}(M,N))$$

Conjecture 2. • $\chi_i(M,N) \geq 0$

- $\chi_i(M,N) = 0$ *only if* $\mathrm{Tor}_i(M,N) = \mathrm{Tor}_{i+1}(M,N) = \cdots = 0$.

This conjecture is open for ramified local rings but is false in general (cf. S. Dutta's Lecture II).

Theorem 10. *(de Jong's Theorem on Regular Alterations, de Jong [2]) Let X be a scheme, reduced and irreducible, essentially of finite type over a field or a discrete valuation ring. Then there exists a projective generically finite map $Y \to X$ where Y is regular (i.e. the extension of function fields $K(X) \to K(Y)$ is finite in the sense of field extensions).*

Using this result, Gabber was able to prove the non-negativity theorem.

Theorem 11. *(Gabber) Let R be a regular local ring, M and N finitely generated R-modules with $\ell(M \otimes N) < \infty$. Then $\chi(M,N) \geq 0$.*

Proof. We give here an outline of the proof of this theorem. For more details, see Berthelot [1], Hochster,[4], or Roberts [9].

We'll prove the result for complete regular local rings. In fact, there are assumptions on R that are needed to apply de Jong's theorem, but the theorem can be reduced to the case in which they hold.

Since χ is additive on short exact sequences, it suffices to prove the result for $M = R/\mathfrak{p}$ and $N = R/\mathfrak{q}$ by taking a filtration of M and N respectively.

We first assume that $R/\mathfrak{p}$ is regular. In this case the result follows immediately from results of Serre, but we present an discussion of this case that parallels the general proof. The last steps are not necessary to prove the result in this case, but this case is easier to understand, and it is hoped that this discussion makes it easier to follow the general case.

Since $R/\mathfrak{p}$ is regular, $\mathfrak{p}$ is generated by part of a regular s.o.p, say $\mathfrak{p} = (x_1, \ldots, x_k)$. In this case, by a theorem of Serre, we have

$$\chi(R/\mathfrak{p}, R/\mathfrak{q}) = e_{(x_1,\ldots,x_k)}(R/\mathfrak{q}, k),$$

where the right hand side denotes $k!$ times the degree k coefficient of the Hilbert-Samuel polynomial of $(\underline{x})$. (This quantity is clearly nonnegative; the remainder of this construction gives another interpretation that will be used in the general case.)

Let $S = \mathrm{gr}_{\underline{x})} = R/(\underline{x}) \oplus (\underline{x})/(\underline{x})^2 \oplus \cdots$ be the associated graded ring of $(\underline{x})$ on R. We next show that $\chi(R/\mathfrak{p}, R/\mathfrak{q})$ can be computed as a multiplicity on S. Define two ideals of S, $I = \bigoplus_{i=1}^{\infty} (\underline{x})^i/(\underline{x})^{i+1}$ the irrelevant ideal of S and $J = \ker(S \to \mathrm{gr}_{(\underline{x})} R/\mathfrak{q})$ where the map is induced by the surjection $R \to R/\mathfrak{q}$. Now, it is a fairly easy exercise using computations with the gradings to show that $e_{(x_1,\ldots,x_k)}(R/\mathfrak{q}, k)$, and hence $\chi(R/\mathfrak{p}, R/\mathfrak{q})$, is equal to $\chi_S(S/I, S/J)$.

Note that S/J is a graded ring with $(S/J)_0 = R/(\mathfrak{p} + \mathfrak{q})$. Since $R/(\mathfrak{p} + \mathfrak{q})$ has finite length, the ring S/J is annihilated by some power of the maximal ideal $\mathfrak{m}$ of R. Hence, using a filtration on S/J, one can show that $\chi_S(S/I, S/J) \geq 0$ by showing that $\chi_S(S/I, M) \geq 0$ fo all S-modules annihilated by $\mathfrak{m}$.

Finally, one can map S into the associated graded ring of $\mathfrak{m}$ on R by sending each of the x_i to its image in $\mathfrak{m}/\mathfrak{m}^2$. Denote the graded ring of $\mathfrak{m}$ on R by T, and denote the ideal of elements of positive degree by K. It is then easy to show that if M is a graded S-module annihilated by $\mathfrak{m}$, we have

$$\chi_S(S/I, M) = \chi_T(T/K, M \otimes_S T).$$

Thus the original problem is expressed in terms of intersection multiplicities of graded module over a graded polynomial ring over a field.

If $R/\mathfrak{p}$ is not regular, then we must use the theorem of de Jong on regular alterations.

Indeed, take $R, R/\mathfrak{p}$ and $R/\mathfrak{q}$ as above, where $R/\mathfrak{p}$ is not necessarily regular. Take a regular alteration of $R/\mathfrak{p}$. Hence, you have an n and a graded ideal I of $A = R[x_0, \dots, x_n]$ such that $\operatorname{Proj}(A/I)$ is regular, $I \cap R = \mathfrak{p}$, and $\phi : \operatorname{Proj}(A/I) \to \operatorname{Spec}(R/\mathfrak{p})$ is a generically finite map. Now set $S_0 = \operatorname{gr}_I(A)$, a bigraded ring, and J_0 be the irrelevant ideal with respect to the grading from I of S_0. Let $\bar{A} = R/\mathfrak{q}[x_0, \dots, x_n]$, and let $\bar{I}$ be the image of I in $\bar{A}$. Let $\bar{S_0} = \operatorname{gr}_{\bar{I}} \bar{A}$. So, we have a surjection $S_0 \to \bar{S_0}$. Let K_0 be the kernel of this surjection. So now, by a generalization of the theorem of Serre used above, we can reduce to computing $\chi_{S_0}(S_0/J_0, S_0/K_0)$.

In the case in which $R/\mathfrak{p}$ was regular, the associated graded ring S was a nice polynomial ring over the component of degree zero. In this case, since A/I is only regular locally, the situation is not so nice. However, we have a surjection $S = \operatorname{Sym}_A(I/I^2) \to S_0$, and this defines an isomorphism locally. Let J and K be the inverse images of J_0 and K_0 respectively. The computation can then be reduced to computing $\chi_S(S/J, S/K)$.

As in the previous case, the question can be reduced to the case of graded modules annihilated by the maximal ideal of R. And as in that case, we wish to reduce to an intersection in a graded polynomial ring over a field. However, since we do not know I explicitly, the definition is not as simple.

We wish to define a map $S \to (A/I \otimes k)[s_1, \dots, s_d, T_0, \dots, T_n]$, where the s_i are in degree 0 and the T_i are in degree 1. This amounts to defining a map $\phi : I/I^2 \to (A/I \otimes k)^d \oplus (A/I \otimes k)[-1]^{n+1} = F$. R is assumed ramified, hence $R/\mathfrak{m}^2 \cong R/\mathfrak{m}[Y_1, \dots, Y_d]/(Y_1, \dots, Y_d)^2$. Let $t_1, \dots, t_d$ be a regular s.o.p in R. One can define $\frac{\partial}{\partial t_i} : R \to R/m$ in terms of the Y_i that we get since R is ramified. Extend this map to $\frac{\partial}{\partial t_i} : A \to A/\mathfrak{m}A$. Note that $I \subset A$, and since, by the Leibniz rule, we have that $I^2 \subseteq \ker \frac{\partial}{\partial t_i}$ for all i, we get an induced map from I/I^2 to $A/I \otimes k$ for all i. So, the ϕ above is given componentwise by $(\frac{\partial}{\partial t_1}, \dots, \frac{\partial}{\partial t_d}, \frac{\partial}{\partial x_0}, \dots, \frac{\partial}{\partial x_n})$. Now, as $\operatorname{Proj}(A/I)$ is regular, the map ϕ above is a locally split injection. So, let $(A/I \otimes k)[s_1, \dots, s_d, T_0, \dots, T_n] = \tilde{S}$, $\tilde{J} = (s_1, \dots, s_d, T_0, \dots, T_n)$, and $\tilde{K}$ the image of K in S. Now compute $\chi_{\tilde{S}}(\tilde{S}/\tilde{J}, \tilde{S}/\tilde{K})$, and show that this is nonnegative. □

Paul Roberts - Lecture IV

Local Chern Characters

Assume now that $(R, \mathfrak{m})$ is a local ring, that M and N are finitely generated R-modules with M having finite projective dimension. Suppose further that $\dim(M) + \dim(N) < \dim(R)$ and that $\chi(M, N) \neq 0$.

Let

$$F. = 0 \to F_d \to \cdots \to F_0 \to M \to 0$$

be a free resolution of M. Now

$$\mathrm{ch}(F.) : CH_*(R) \to CH_0(R/\mathfrak{m}) \cong \mathbb{Q}$$

Recall that for any module N we have

$$\chi(F. \otimes N) = \mathrm{ch}(F.)\tau(N)$$

In particular, $\chi(F.) = \mathrm{ch}(F.)\tau(R)$ where $\tau(R) = [R]_d +$terms of lower degree

This means that

$$\chi(F.) = \mathrm{ch}_d(F.)[R]_d + \mathrm{ch}_{d-1}(F.)\tau_{d-1}(R) + \cdots$$

The Dutta multiplicity is $\chi(F.) = \mathrm{ch}_d(F.)[R]_d$. In positive characteristic this is equal to $\lim_{n\to\infty} \frac{\chi(F.\otimes^{f^n} R)}{p^{dn}}$.

Question: Is Dutta multiplicity equal to the ordinary multiplicity for Gorenstein rings? Note that there are two issues here, one is whether $\tau(A) = [A]_d$, and the other is the existence of a module of finite length and finite projective dimension whose local Chern character does not vanish on a term of degree lower than d.

Note that if R is a complete intersection, $\tau(R) = [R]_d$.

Note that for Gorenstein rings, $\tau_i(R)$ need not be zero for $i < d$ as is evidenced by the following two examples.

1. (Kurano) Let
$$R = \frac{k[x_{ij}]_{\substack{0\le i\le n\\ 0\le j\le m}}}{I_2((x_{ij}))}$$
Then $\tau_4(R) = 0$ but $\tau_3(R) \neq 0$.

2. (C. Miller, A. Singh) A complex for which Dutta and ordinary multi[licity are not equal over a finite Gorenstein extension of
$$\frac{k[x, y, z, w, u, v]}{(xu + yv + zw)}.$$

We now make the following assumptions: A is a localization at the graded maximal ideal of a standard graded ring R and $Q = \mathrm{Proj}(R)$ is smooth.

For a projective scheme $Q = \mathrm{Proj}(R)$ the rational Chow group of Q is a free $\mathbb{Q}$-module on $[R/\mathfrak{p}]$ (where $\mathfrak{p}$ is a graded prime ideal of R) modulo the rational equivalence relations generated by the relations $[R/(\mathfrak{q}, x)]_i - [R/(\mathfrak{q}, y)]_i$ where x and y are homogeneous elements of the same degree not in $\mathfrak{q}$. Let h be a hyperplane section acting on $CH_*(Q)$ via the map

$$CH_i(Q) \to CH_{i-1}(Q)$$

defined by

$$[R/\mathfrak{p}]_i \mapsto [R/(\mathfrak{p}, x)]_{i-1}$$

where x is an element of degree one not in $\mathfrak{p}$.

Note: h does not depend on the choice of the element x.

Paul Roberts - Lecture V

Chow Groups of Projective Schemes

Throughout, let R be a standard graded ring $R = R_0 \oplus R_1 \oplus R_2 \oplus \cdots$ where R_0 is a field and let $Q = \mathrm{Proj}(R)$. Recall that in this case the rational Chow group of Q is defined as the direct sum $CH_*(Q) = \bigoplus_i CH_i(Q)$ where

$$CH_i(Q) = \frac{\text{cycles}}{\text{rational equivalence}} = \frac{\text{free } \mathbb{Q} - \text{ module on } [R/\mathfrak{p}] \text{ with } \mathfrak{p} \text{ graded}}{\langle [R/(\mathfrak{q}, x)]_i - [R/(\mathfrak{q}, y)]_i}\rangle$$

for homogeneous elements x, y of the same degree not in $\mathfrak{q}$ a graded prime of R and such that $\dim(R/\mathfrak{q}) = i + 1$

Define a hyperplane section to be

$$\left[\frac{R}{\mathfrak{p}}\right]_i \mapsto \left[\frac{R}{(\mathfrak{p}, x)}\right]_{i-1}$$

where x is a homogeneous element of degree one.

Examples

1. Let $R = k[x_0, \ldots, x_n]$ be the coordinate ring of projective n-space $\mathbb{P}^n_k$. Let a denote a generic hyperplane H. Then

$$CH_*(\mathbb{P}^n_k) \cong Q[a]/(a^{n+1})$$

We now list the correspondence between the basis elements of $CH_*(\mathbb{P}^n_k)$ and subvarieties of $\mathbb{P}^n_k$.

$$\begin{array}{ccc} 1 & \longleftrightarrow & \mathbb{P}^n \\ a & \longleftrightarrow & H \\ a^2 & \longleftrightarrow & H \cap \widetilde{H} \\ & \vdots & \\ a^n & \longleftrightarrow & * \end{array}$$

where H and $\widetilde{H}$ denote hyperplanes in $\mathbb{P}^n$ and $*$ denotes a single point in $\mathbb{P}^n$.

2. Now consider the space $\mathbb{P}^n \times \mathbb{P}^m$ and hyperplanes $H \subseteq \mathbb{P}^n$ and $K \subseteq \mathbb{P}^m$. Letting $a = H \times \mathbb{P}^m$ and $b = \mathbb{P}^n \times K$ we have that

$$CH_*(\mathbb{P}^n \times \mathbb{P}^m) \cong \mathbb{Q}[a, b]/(a^{n+1}, b^{m+1})$$

Note that the coordinate ring of $\mathbb{P}^n \times \mathbb{P}^m$ is isomorphic to the ring

$$R = \frac{k[x_{ij}]_{\substack{0 \le i \le n \\ 0 \le j \le m}}}{I_2((x_{ij}))}$$

3. (Continuation of 2) Recall that one can embed $\mathbb{P}^n \times \mathbb{P}^m$ into $\mathbb{P}^{nm}$ via the Segre embedding

$$((a_0, \ldots, a_n), (b_0, \ldots, b_m)) \mapsto \begin{bmatrix} a_0 b_0 & \ldots & a_0 b_m \\ \vdots & \ddots & \vdots \\ a_n b_0 & \ldots & a_n b_m \end{bmatrix}$$

Let h be the hyperplane associated to (x_{00}) and let a and b be as in the previous example. It is an easy exercise to check that

$$(x_{00}) = (x_{00}, \ldots, x_{0m}) \cap (x_{00}, \ldots, x_{n0})$$

and thus that $h = a+b$ in $CH_*(\mathbb{P}^n \times \mathbb{P}^m)$, identified with $\mathbb{Q}[a, b]/(a^{n+1}, b^{m+1})$ as above.

Theorem 12. *(Kurano) Let R and Q be as above. Then*

$$CH_*(R_\mathfrak{m}) \cong \frac{CH_*(Q)}{hCH_*(Q)}.$$

Corollary 13. *$CH_*(k[x_0, \dots x_n]) \cong \mathbb{Q}$ in dimension $n+1$.*

Corollary 14.

$$CH_*(\frac{k[x_{ij}]_{\substack{0\le i\le n\\ 0\le j\le m}}}{I_2((x_{ij}))}) \cong \mathbb{Q}[a,b]/(a^{n+1}, b^{m+1}, a+b) \cong \mathbb{Q}[a]/(a^{\min\{n+1,m+1\}})$$

In particular,

$$CH_*(k[x,y,z,w]/(xy-zw)) \cong \mathbb{Q}[a]/(a^2)$$

Counterexamples to Vanishing

Let R be a standard graded ring, $A = R_\mathfrak{m}$ where $\mathfrak{m}$ is the graded maximal ideal, and let M be a module of finite length and finite projective dimension. Let

$$0 \to F_d \to \cdots \to F_0 \to M \to 0$$

be a finite free resolution of M. We now consider $\mathrm{ch}(F.)$ acting on the Chow group as

$$\mathrm{ch}(F.) : CH_*(A) \to CH_*(A/\mathfrak{m}) \cong \mathbb{Q}$$

Note that this map is defined for any bounded complex of free modules with homologies of finite length. The function is also additive on short (split in each degree) exact sequences of complexes and is independent of quasi-isomorphisms.

Thus there exists a homomorphism

$$K_0(\mathcal{C}) \to CH_*(A)^*$$

where $*$ denotes $\hom_\mathbb{Q}(-, \mathbb{Q})$ and $\mathcal{C}$ denotes the category of all perfect complexes.

Assume that Q is smooth. Since $CH_*(A) \cong CH_*(Q)/hCH_*(Q)$ we have the following diagram:

$$\begin{array}{ccccccc}
0 & \longrightarrow & \ker(h) & \longrightarrow & CH_*(Q) & \xrightarrow{h} & CH_*(Q) \\
 & & \downarrow{\scriptstyle\phi} & & \downarrow & & \downarrow \\
0 & \longrightarrow & CH_*(A)^* & \longrightarrow & CH_*(Q) & \xrightarrow{h^*} & CH_*(Q)
\end{array}$$

Theorem 15. *(Roberts, Srinivas) The image of $K_0(\mathcal{C}) \to CH_*(A)^*$ is the image of ϕ in the above diagram.*

This theorem allows one to construct many counterexamples to vanishing.

1. Let
$$A = (k[x, y, z, w]/(xy - zw))_{(x,y,z,w)}$$
and let $Q = \mathbb{P}^1 \times \mathbb{P}^1$. Then
$$CH_*(Q) \cong \mathbb{Q}[a, b]/(a^2, b^2)$$
The set $\{1, a, b, ab\}$ is a $\mathbb{Q}$-basis for $CH_*(Q)$ as a $\mathbb{Q}$-vector space. Consider the hyperplane associated to $a + b$. The kernel of multiplication by $a + b$ has basis $\{a - b, ab\}$.

   Thus there exists a module M of finite projective dimension and of finite length with $\chi(M, A/(x, z)) = -1$ but with $\chi(M, A/(x, u)) = 1$.

2. Let
$$A = \left(k\frac{\begin{bmatrix} x_{11} & x_{12} & x_{13} \\ x_{21} & x_{22} & x_{23} \\ x_{31} & x_{32} & x_{33} \end{bmatrix}}{I_2(x_{ji})} \right)_{(x_{ij})}$$
and let $Q = \mathbb{P}^2_k \times \mathbb{P}^2_k$. Then $CH_*(Q) \cong \mathbb{Q}[a, b]/(a^3, b^3)$. In this case the kernel has basis $\{a^2 - ab + b^2, a^2b - ab^2, a^2b^2\}$. There are lots of examples like this one.

From the second example we see also that there are modules for which the Dutta multiplicity does not equal the ordinary multiplicity.

Note: If the intersection pairing is perfect, then the map $K_0(\mathcal{C}) \to CH_*(A)^*$ is surjective.

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