Commutative Algebra Mini-Course

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Koszul Cohomology, Cohen Structure Theorems, and Intersection Multiplicities

Goal: Give enough information on each topic to state the following results:

1. The realization of the Hilbert-Samuel multiplicity as an alternating sum of lengths of Koszul homology modules;

2. Cohen's structure theorems for complete local rings;

3. Serre's results on intersection multiplicities for equicharacteristic and unramified regular local rings.

Proofs will be sketches due to time restrictions.

General references for the material presented here are:

Koszul homology and cohomology. Bruns-Herzog [2, Section 1.6] and Matsumura [10, Section 16];

Cohen structure theorems. Matsumura [10, Sections 28,29];

Intersection multiplicities. Serre [14].

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Chapter 1

Koszul cohomology

1.1 Motivating Basics

A *chain complex* is a sequence of R-module homomorphisms

$$M_\bullet = \cdots \xrightarrow{\partial_{i+1}} M_i \xrightarrow{\partial_i^M} M_{i-1} \xrightarrow{\partial_{i-1}^M} \cdots$$

such that $\partial_i^M \partial_{i+1}^M = 0$ for each integer i, that is, $\operatorname{Im} \partial_{i+1}^M \subseteq \operatorname{Ker} \partial_i^M$. The ith *homology module* of $M_\bullet$ is $\mathrm{H}_i(M_\bullet) = \operatorname{Ker} \partial_i^M / \operatorname{Im} \partial_{i+1}^M$. This measures how close $M_\bullet$ is to being exact at the degree-i spot. If $M_i = 0$ for each $i < 0$, i.e., $M_\bullet$ is of the form

$$M_\bullet = \cdots \xrightarrow{\partial_2} M_1 \xrightarrow{\partial_1^M} M_0 \to 0 \to \cdots$$

then $M_\bullet$ is *acyclic* if $\mathrm{H}_i(L) = 0$ for each $i \neq 0$. In other words, $M_\bullet$ is acyclic if and only if it is exact everywhere except possibly at M_0.

A *cochain complex* is a sequence of R-module homomorphisms

$$N^\bullet = \cdots \xrightarrow{\partial_N^{i-2}} N^{i-1} \xrightarrow{\partial_N^{i-1}} N^i \xrightarrow{\partial_N^i} N^{i+1} \to \cdots$$

such that $\partial_N^i \partial_N^{i-1} = 0$ for each i, and the ith *cohomology module* of $N^\bullet$ is $\mathrm{H}^i(N^\bullet) = \operatorname{Ker} \partial_N^i / \operatorname{Im} \partial_N^{i-1}$.

Here is the basic idea of the Koszul complex: If R is a ring and x an element of R, and M an R-module, consider the homothety (i.e., multiplication) map $M \xrightarrow{\cdot x} M$. By definition, one has

$$\operatorname{Ker}(M \xrightarrow{\cdot x} M) = \{m \in M | xm = 0\} = (0 :_M x)$$

so this map is injective if and only if x is weakly M-regular, i.e., a non-zero-divisor on M. Furthermore, we have

$$\operatorname{Im}(M \xrightarrow{\cdot x} M) = xM \qquad \operatorname{Coker}(M \xrightarrow{\cdot x} M) = M/xM$$

so the surjectivity of this map is related to the other half of the definition of x being an M-regular element. We add 0's to obtain a chain complex concentrated in degrees 0 and 1:

$$K_\bullet(x; M) = 0 \to M \xrightarrow{\cdot x} M \to 0$$

This is the Koszul complex on x with coefficients in M. One verifies easily the isomorphisms

$$\mathrm{H}_0(K_\bullet(x;M)) \cong M/xM \qquad \mathrm{H}_1(K_\bullet(x;M)) \cong (0:_M x)$$

and it follows that x is M-regular if and only if $\mathrm{H}_0(K_\bullet(x;M)) \neq 0$ and $\mathrm{H}_1(K_\bullet(x;M)) = 0$.

In the following two sections we outline a generalization of this construction that gives, among other things, information about when a longer sequence $\mathbf{x} = x_1, \ldots, x_n$ is M-regular.

1.2 Construction of the Koszul complex: Method 1

Fix a sequence $\mathbf{x} = x_1, \ldots, x_n \in R$. Set $K_0 = R$ and $K_1 = R^n$, and let $e_1, \ldots, e_n \in K_1$ be a basis. For each $i \geq 2$ set $K_i = \wedge^i(K_1) = \wedge^i(R^n)$ which is a free R-module of rank $\binom{n}{i}$ with basis

$$\{e_{j_1} \wedge e_{j_2} \wedge \ldots \wedge e_{j_i} | 1 \leq j_1 < j_2 < \ldots < j_i \leq n\}$$

Observe that $K_i = 0$ for each $i > n$, and $K_n \cong R$ with basis $\{e_1 \wedge \ldots \wedge e_n\}$. For $i < 0$ set $K_i = 0$. These are the modules in our Koszul complex.

Next, we define the differentials. For $i = 1, \ldots, n$ let $\partial_i^K \colon K_i \to K_{i-1}$ be given by

$$e_{j_1} \wedge \ldots \wedge e_{j_i} \mapsto \sum_{k=1}^{i} (-1)^{k+1} x_k e_{j_1} \wedge \ldots \wedge \hat{e_{j_k}} \wedge \ldots \wedge e_{j_i}.$$

For $i > n$ or $i < 1$ the map $\partial_i^K = 0$. Since the modules K_i are free with distinguished bases, the maps ∂_i may be written as matrices. Before doing so, let us be clear about our notational conventions.

Once a basis $e_1, \ldots, e_n$ for R^n is specified, we can think of the elements of R^n as column vectors of length n with entries in R. Under this identification, the basis vector e_i corresponds to the ith standard basis vector:

$$e_i \sim \begin{pmatrix} 0 \\ \vdots \\ 1 \\ \vdots \\ 0 \end{pmatrix}$$

Given an R-linear map $\phi \colon R^n \to R^m$ where bases have been fixed for R^n and R^m, the identification with column vectors allows us to write ϕ as an $m \times n$ matrix whose jth column is the image of the jth basis vector of R^n.

It follows readily that the matrix representing the map ∂_i^K consists of 0's and $\pm x_j$'s. Let us be more specific in two cases. The map $\partial_1^K \colon K_1 \to K_0$ maps $e_j \mapsto x_j$, and therefore the matrix is $A = (x_1 \ \cdots \ x_n)$. On the other side, $\partial_n^K \colon K_n \to K_{n-1}$ maps

$$e_1 \wedge \ldots \wedge e_n \mapsto x_1 e_2 \wedge \ldots \wedge e_n - x_2 e_1 \wedge e_3 \wedge \ldots \wedge e_{n-1} + \cdots + (-1)^{n+1} x_n e_1 \wedge e_2 \wedge \ldots \wedge e_{n-1}$$

and thus the matrix is

$$B = \begin{pmatrix} x_1 \\ -x_2 \\ \vdots \\ (-1)^{n+1} x_n \end{pmatrix}$$

and this sequence of homomorphisms is of the form

$$K_\bullet = 0 \to R \xrightarrow{B} R^n \to \cdots \to R^n \xrightarrow{A} R \to 0.$$

Exercise 1.2.1. Verify the following.

(a) $\partial_i^K \partial_{i+1}^K = 0$ for each integer i. Thus, $K_\bullet$ is a chain complex.

(b) $\mathrm{H}_0(K_\bullet) \cong R/(\mathbf{x})$ and $\mathrm{H}_n(K_\bullet) \cong \{r \in R | x_i r = 0, \forall i = 1, \dots, n\} = \bigcap_{i=1}^{n} (0 :_R x_i)$.

Definition 1.2.2. The complex $K_\bullet$ constructed above is the *Koszul complex* on $\mathbf{x}$, which we denote $K_\bullet(\mathbf{x})$ or $K_\bullet(\mathbf{x}; R)$.

For an R-module M, set $K_\bullet(\mathbf{x}; M) = K_\bullet(\mathbf{x}) \otimes_R M$. This is a chain complex by Exercise 1.2.1 and the functoriality of $(-) \otimes_R M$. It is concentrated in degrees 0 to n with the following form

$$0 \to M \to M^n \to M^{\binom{n}{2}} \to \cdots \to M^n \to M \to 0.$$

The ith homology of this complex $\mathrm{H}_i(K_\bullet(\mathbf{x}; M))$ is denoted $\mathrm{H}_i(\mathbf{x}; M)$.

In a dual manner, set $K^\bullet(\mathbf{x}; M) = \mathrm{Hom}_R(K_\bullet(\mathbf{x}), M)$ This is a cochain complex concentrated in degrees 0 to n with the following form

$$0 \to M \to M^n \to M^{\binom{n}{2}} \to \cdots \to M^n \to M \to 0.$$

The ith cohomology of this complex $\mathrm{H}^i(K^\bullet(\mathbf{x}; M))$ is denoted $\mathrm{H}^i(\mathbf{x}; M)$. When $M = R$, we write $K^\bullet(\mathbf{x})$ and $\mathrm{H}^i(\mathbf{x})$.

Exercise 1.2.3. Let $R = k[X, Y]$ and compute $\mathrm{H}_i(\mathbf{x})$ and $\mathrm{H}^i(\mathbf{x})$ for the following sequences.

(a) $\mathbf{x} = X, Y$

(b) $\mathbf{x} = X, Y, X + Y$

Exercise 1.2.4. Verify the following.

(a) $\mathrm{H}_0(\mathbf{x}; M) \cong M/\mathbf{x}M \cong \mathrm{H}^n(\mathbf{x}; M)$

(b) $\mathrm{H}_n(\mathbf{x}; M) \cong \bigcap_{i=1}^{n} (0 :_M x_i) \cong \mathrm{H}^0(\mathbf{x}; M)$

(c) If R is Noetherian and M is finitely generated, then $\mathrm{H}_i(\mathbf{x}; M)$ and $\mathrm{H}^i(\mathbf{x}; M)$ are finitely generated.

1.3 Construction of the Koszul complex: Method 2

We define the *tensor product* of two chain complexes $X_\bullet$ and $Y_\bullet$. For each integer i set

$$(X_\bullet \otimes_R Y_\bullet)_i = \oplus_{p+q=i} (X_p \otimes_R Y_q)$$

and let $\partial_i^{X\otimes_R Y}\colon (X_\bullet \otimes_R Y_\bullet)_i \to (X_\bullet \otimes_R Y_\bullet)_{i-1}$ be given by

$$x_p \otimes y_q \mapsto (\partial_p^X x_p)\otimes y_q + (-1)^p x_p \otimes (\partial_q^Y y_q).$$

The relevant facts about the tensor product, including its connection to the Koszul complex, are collected in the following exercise.

Exercise 1.3.1. Verify the following.

(a) $X_\bullet \otimes_R Y_\bullet$ is a chain complex.

(b) There exist natural isomorphisms of chain complexes

$$X_\bullet \otimes_R Y_\bullet \cong Y_\bullet \otimes_R X_\bullet \qquad (X_\bullet \otimes_R Y_\bullet)\otimes_R Z_\bullet \cong X_\bullet \otimes_R (Y_\bullet \otimes_R Z_\bullet).$$

(Careful of the signs $(\pm)$.)

(c) For $x \in R$, there is an isomorphism $(X_\bullet \otimes_R K_\bullet(x))_i \cong X_i \oplus X_{i-1}$ for each integer i. Furthermore, the map $\partial_i^{X_\bullet\otimes_R K_\bullet(x)}$ is given by

$$\begin{pmatrix}\alpha_i\\ \alpha_{i-1}\end{pmatrix} \mapsto \begin{pmatrix}\partial_i^X\alpha_i + (-1)^{i-1}x\cdot\alpha_{i-1}\\ \partial_{i-1}^X\alpha_{i-1}\end{pmatrix} = \begin{pmatrix}\partial_i^X & (-1)^{i-1}x\cdot\\ 0 & \partial_{i-1}^X\end{pmatrix}\begin{pmatrix}\alpha_i\\ \alpha_{i-1}\end{pmatrix}$$

In other words, the complex $X_\bullet \otimes_R K_\bullet(x)$ is the mapping cone of the homothety $X_\bullet \xrightarrow{x\cdot} X_\bullet$.

(d) For $\mathbf{x} = x_1,\dots,x_n$, there is a natural isomorphism of chain complexes

$$K_\bullet(\mathbf{x}) \cong K_\bullet(x_1)\otimes_R \cdots \otimes_R K_\bullet(x_n).$$

(e) The *suspension* (or *shift*) of $X_\bullet$ is the chain complex $\Sigma X_\bullet$ given by the data $(\Sigma X)_i = X_{i-1}$ and $\partial_i^{\Sigma X} = -\partial_{i-1}^X\colon (\Sigma X)_i \to (\Sigma X)_{i-1}$. For each integer i, let $\epsilon_i\colon X_i \to X_i\oplus X_{i-1}$ and $\tau_i\colon X_i\oplus X_{i-1} \to X_{i-1}$ be given by

$$\alpha_i \mapsto \begin{pmatrix}\alpha_i\\ 0\end{pmatrix} \qquad\qquad \begin{pmatrix}\alpha_i\\ \alpha_{i-1}\end{pmatrix} \mapsto (-1)^{i-1}\alpha_{i-1}$$

respectively. These maps describe chain maps $\epsilon\colon X_\bullet \to X_\bullet \otimes K_\bullet(x)$ and $\tau\colon X_\bullet \otimes K_\bullet(x) \to \Sigma X_\bullet$ that fit into a (degreewise split) short exact sequence of chain complexes

$$0 \to X_\bullet \to X_\bullet \otimes K_\bullet(x) \to \Sigma X_\bullet \to 0.$$

The associated long exact sequence on homology has the form

$$\cdots \to \mathrm{H}_i(X_\bullet) \xrightarrow{\cdot x} \mathrm{H}_i(X_\bullet) \to \mathrm{H}_i(X_\bullet \otimes K_\bullet(x)) \to \mathrm{H}_{i-1}(X_\bullet) \xrightarrow{\cdot x} \mathrm{H}_{i-1}(X_\bullet) \to \cdots$$

which induces short exact sequences

$$0 \to \frac{\mathrm{H}_i(X_\bullet)}{x\,\mathrm{H}_i(X_\bullet)} \to \mathrm{H}_i(X_\bullet \otimes K_\bullet(x)) \to (0 :_{\mathrm{H}_{i-1}(X_\bullet)} x) \to 0.$$

(f) Let $\mathbf{x} = x_1, \ldots, x_n \in R$ and $\mathbf{x}' = x_1, \ldots, x_n, x_{n+1} \in R$. There is an exact sequence of chain complexes

$$0 \to K_\bullet(\mathbf{x}; M) \to K_\bullet(\mathbf{x}'; M) \to \Sigma K_\bullet(\mathbf{x}' M) \to 0$$

and the associated long exact sequence on homology looks like

$$\cdots \to \mathrm{H}_i(\mathbf{x}; M) \to \mathrm{H}_{i-1}(\mathbf{x}; M) \xrightarrow{x_{n+1}} \mathrm{H}_{i-1}(\mathbf{x}; M) \to \mathrm{H}_{i-1}(\mathbf{x}'; M) \to \cdots$$

which gives rise to short exact sequences

$$0 \to \frac{\mathrm{H}_i(\mathbf{x}; M)}{x_{n+1}\,\mathrm{H}_i(\mathbf{x}; M)} \to \mathrm{H}_i(\mathbf{x}'; M) \to (0 :_{\mathrm{H}_{i-1}(\mathbf{x}; M)} x_{n+1}) \to 0.$$

(g) $\mathrm{Supp}(\mathrm{H}_i(\mathbf{x}; M)) \subseteq \mathrm{Supp}(M) \cap V(\mathbf{x})$

(h) $x_i^2 \in \mathrm{Ann}(\mathrm{H}_i(\mathbf{x}; M))$ for $i = 1, \ldots, n$. (Actually, $x_i \in \mathrm{Ann}(\mathrm{H}_i(\mathbf{x}; M))$. See Corollary 1.4.9.) So $(\mathbf{x}) + \mathrm{Ann}(M) \subseteq \mathrm{Ann}(\mathrm{H}_i(\mathbf{x}; M))$.

1.4 Properties of the Koszul complex

Proposition 1.4.1. If $L_\bullet = \ldots \to L_i \to L_{i-1} \to \ldots \to L_1 \to L_0 \to 0$ is acyclic and x is regular on $\mathrm{H}_0(L_\bullet) = \mathrm{Coker}(\partial_1^L)$, then $L_\bullet \otimes_R K_\bullet(x)$ is acyclic and $\mathrm{H}_0(L_\bullet \otimes_R K_\bullet(x)) \cong \mathrm{H}_0(L_\bullet)/x\,\mathrm{H}_0(L_\bullet)$.

Proof. We only need $\mathrm{H}_i(L_\bullet \otimes_R K_\bullet(x)) = 0$ for each $i \neq 0$. Exercise 1.3.1(e) gives the short exact sequence

$$0 \to \frac{\mathrm{H}_i(L_\bullet)}{x\,\mathrm{H}_i(L_\bullet)} \to \mathrm{H}_i(L_\bullet \otimes_R K_\bullet(x)) \to (0 :_{\mathrm{H}_{i-1}(L_\bullet)} x) \to 0.$$

When $i \neq 0$, our assumptions yield $\mathrm{H}_i(L_\bullet)/x\,\mathrm{H}_i(L_\bullet) = 0 = (0 :_{\mathrm{H}_{i-1}(L_\bullet)} x)$. Thus, the displayed sequence implies $\mathrm{H}_i(L_\bullet \otimes_R K_\bullet(x)) = 0$. □

Corollary 1.4.2. If $L_\bullet$ is a (minimal) R-free resolution of M and x is M-regular, then $L_\bullet \otimes_R K_\bullet(x)$ is a (minimal) R-free resolution of M/xM. □

Corollary 1.4.3. If $\mathbf{x}$ is an R-regular sequence, then $K_\bullet(\mathbf{x})$ is a free resolution of $R/(\mathbf{x})$. When R is local, this resolution is minimal. □

Corollary 1.4.4. If $\mathbf{x}$ is (weakly) M-regular, then $\mathrm{H}_i(\mathbf{x}; M) = 0$ for each integer $i > 0$.

Proof. By induction on n, the length of the sequence. The case $n = 1$ is easy. Use Proposition 1 for the induction step. □

It is natural to ask whether the converse of Corollary 1.4.4 holds. Note that the statement "$\mathrm{H}_i(\mathbf{x}; M) = 0$ for each integer $i > 0$" is independent of the order of the sequence by Exercise 1.3.1 parts (b) and (d). However, the statement "$\mathbf{x}$ is weakly M-regular" is not independent of the order. So, we should not expect the converse to hold in general. However, we have the following.

Proposition 1.4.5. Assume R is Noetherian and M is a finitely generated nonzero R-module. For a sequence $\mathbf{x} = x_1, \ldots, x_n \in \mathrm{Jac}(R)$, the following conditions are equivalent:

(i) $\mathrm{H}_i(\mathbf{x}; M) = 0$ for each $i > 0$;

(ii) $\mathrm{H}_1(\mathbf{x}; M) = 0$;

(iii) $\mathbf{x}$ is weakly M-regular;

(iv) $\mathbf{x}$ is M-regular.

Proof. The implications (i) $\Longrightarrow$ (ii) and (iii) $\Longrightarrow$ (iv) are trivial, while (iv) $\Longrightarrow$ (i) holds by Corollary 1.4.4.

(ii) $\Longrightarrow$ (iii). Since $\mathbf{x}$ is a sequence in the Jacobson radical of R and M is nonzero, Nakayama's Lemma implies $M/xM \neq 0$. So it suffices to show that $\mathbf{x}$ is weakly M-regular. We prove this by induction on n where $\mathbf{x} = x_1, \ldots, x_n$. The case $n = 1$ is straightforward, so assume $n > 1$ and let $\hat{\mathbf{x}} = x_1, \ldots, x_{n-1}$. The exact sequence

$$0 \to \frac{\mathrm{H}_1(\hat{\mathbf{x}}; M)}{x_n \mathrm{H}_1(\hat{\mathbf{x}}; M)} \to \mathrm{H}_1(\hat{\mathbf{x}}; M) \to (0 :_{\mathrm{H}_0(\hat{\mathbf{x}};M)} x_n) \to 0$$

and the assumption $\mathrm{H}_1(\mathbf{x}; M) = 0$ imply that $\mathrm{H}_1(\hat{\mathbf{x}}; M) = 0 = (0 :_{\mathrm{H}_0(\hat{\mathbf{x}};M)} x_n)$. The second of these equalities shows that x_n is a nonzero divisor on $M/(\hat{\mathbf{x}})M$, while the first gives $\mathrm{H}_1(\hat{\mathbf{x}}; M) = 0$ by Nakayama's Lemma. By induction, the sequence $\hat{\mathbf{x}}$ is weakly M-regular, and the proof is complete. $\square$

Remark 1.4.6. This shows that the Koszul complex has the ability to detect when a sequence with more than one element is M-regular, as was promised in Section 1.1. Also, it shows that when $\mathbf{x} \in \mathrm{Jac}(R)$, M is finitely generated, and $\mathbf{x}$ is an M-regular sequence, then any permutation of $\mathbf{x}$ is M-regular.

Proposition 1.4.7. Fix $\mathbf{x} \in R$ and let M be an R-module with $L_\bullet \to R/(x) \to 0$ a free resolution. If $\pi\colon R/(\mathbf{x}) \to M$ is an R-linear map, then there exist a chain map $K_\bullet(\mathbf{x}) \xrightarrow{\phi} L_\bullet$ such that $\mathrm{H}_0(\phi) = \pi$. In other words, there exists a commutative diagram

$$\begin{tikzcd}
\cdots \arrow[r] & 0 \arrow[r] \arrow[d, "\phi_{n+1}=0"'] & K_n(\mathbf{x}) \arrow[r, "\partial_n^{K(x)}"] \arrow[d, "\phi_n"'] & K_{n-1}(\mathbf{x}) \arrow[r] \arrow[d, "\phi_{n-1}"'] & \cdots \arrow[r] & K_0(\mathbf{x}) \arrow[r] \arrow[d, "\phi_0"'] & R/(x) \arrow[r] \arrow[d, "\pi"'] & 0 \\
\cdots \arrow[r] & L_{n+1} \arrow[r] & L_n \arrow[r, "\partial_n^L"] & L_{n-1} \arrow[r] & \cdots \arrow[r] & L_0 \arrow[r] & M \arrow[r] & 0.
\end{tikzcd}$$

Proof. Exercise. Move right-to-left, "lifting" the previous map. Only uses the fact that the bottom row is exact and the $K_i(\mathbf{x})$ are free. $\square$

Corollary 1.4.8. There exist natural maps

$$\mathrm{H}_i(\mathbf{x}; M) \to \mathrm{Tor}_i^R(R/(\mathbf{x}), M) \qquad \mathrm{Ext}_R^i(R/(\mathbf{x}), M) \to \mathrm{H}^i(\mathbf{x}; M).$$

When $\mathbf{x}$ is an R-regular sequence, these maps are isomorphisms.

Proof. Let π be the identity on $R/(\mathbf{x})$ and apply Proposition 1.4.7 to obtain $\phi\colon K_\bullet(\mathbf{x}) \to L_\bullet$ where $L_\bullet \to R/(\mathbf{x}) \to 0$ is a free resolution. The first map is $\mathrm{H}_i(\phi \otimes_R M)$ and the second is $\mathrm{H}^i(\mathrm{Hom}_R(\phi, M))$. When $\mathbf{x}$ is R-regular, ϕ is a homotopy equivalence and therefore these maps are isomorphisms. $\square$

Corollary 1.4.9. $\mathbf{x} \in \mathrm{Ann}(\mathrm{H}_i(\mathbf{x}; M))$.

Proof. Let $A = \mathbb{Z}[X_1, \dots, X_n]$ and let $\psi\colon A \to R$ be given by $X_i \mapsto x_i$. This makes M into an A-module with $X_i m = x_i m$. One checks easily the isomorphism of complexes $K^A_\bullet(\mathbf{X}; M) \cong K^R_\bullet(\mathbf{x}; M)$. This yields the first of the following isomorphisms

$$\mathrm{H}^R_i(\mathbf{x}; M) \cong \mathrm{H}^A_i(\mathbf{X}; M) \cong \mathrm{Tor}^A_i(A/(\mathbf{x}), M)$$

where the second isomorphism is by Corollary 1.4.8, since $\mathbf{X}$ is A-regular. Therefore

$$x_i\,\mathrm{H}^R_i(\mathbf{x}; M) \cong X_i\,\mathrm{H}^A_i(\mathbf{X}; M) \cong X_i\,\mathrm{Tor}^A_i(A/(\mathbf{x}), M) = 0$$

as desired. $\square$

Conjecture 1.4.10 (Canonical Element Conjecture, (CEC)). Let (R, m, k) be a local Noetherian ring with a system of parameters $\mathbf{x} \in m$. Let $F_\bullet \to k \to 0$ be a free resolution of k. As in Proposition 1.4.7, construct a commutative diagram

$$\begin{tikzcd}
\cdots \arrow[r] & 0 \arrow[r] & K_n(\mathbf{x}) \arrow[r] \arrow[d, "\phi_n"'] & K_{n-1}(\mathbf{x}) \arrow[r] \arrow[d, "\phi_{n-1}"'] & \cdots \arrow[r] & K_0(\mathbf{x}) \arrow[r] \arrow[d, "\phi_0"'] & R/(\mathbf{x}) \arrow[r] \arrow[d, "\pi"'] & 0 \\
\cdots \arrow[r] & F_{n+1} \arrow[r] & F_n \arrow[r] & F_{n-1} \arrow[r] & \cdots \arrow[r] & F_0 \arrow[r] & k \arrow[r] & 0
\end{tikzcd}$$

where π is the natural surjection. Then $\phi_n \neq 0$.

Remark 1.4.11. This was conjectured by Hochster [8] and proved for rings containing a field (i.e. equicharacteristic). This is easily checked when $\dim R \leq 2$ or R is Cohen Macaulay. This was recently verified by Heitmann [6] for $\dim R = 3$; see also [9, 11].

Why has there been so little discussion of $\mathrm{H}^i(\mathbf{x}; M)$?

Proposition 1.4.12. There are isomorphisms $\mathrm{H}^i(\mathbf{x}; M) \cong \mathrm{H}_{n-i}(\mathbf{x}; M)$.

Proof. Map $K_{n-i} \otimes_R K_i \to R$ by $u \otimes v \mapsto u \wedge v$. This describes a perfect pairing and therefore induces an isomorphism $K_{n-i} \to \mathrm{Hom}_R(K_i, R)$ which can be described as $e_{j_1} \wedge \dots \wedge e_{j_{n-i}} \mapsto \pm e_{k_1} \wedge \dots \wedge e_{k_i}$ where $\{j_1, \dots, j_{n-i}\} \sqcup \{k_1, \dots, k_i\} = \{1, \dots, n\}$ and the sign $(\pm)$ is the sign of the permutation

$$\begin{pmatrix} 1 & \cdots & n-i & n-i+1 & \cdots & n \\ j_1 & \cdots & j_{n-1} & k_1 & \cdots & k_i \end{pmatrix}.$$

This choice of sign makes the following diagram commute.

$$\begin{tikzcd}
K_{n-i} \arrow[r] \arrow[d, "\cong"'] & K_{n-i-1} \arrow[d, "\cong"'] \\
\mathrm{Hom}_R(K_i, R) \arrow[r] & \mathrm{Hom}_R(K_{i+1}, R)
\end{tikzcd}$$

Now tensor with M and use the fact that K_i is free to obtain a commutative diagram

$$\begin{tikzcd}
\cdots \arrow[r] & K_{n-i}\otimes_R M \arrow[r] \arrow[d, "\cong"'] & K_{n-i-1}\otimes_R M \arrow[r] \arrow[d, "\cong"'] & \cdots \\
\cdots \arrow[r] & \operatorname{Hom}_R(K_i, M) \arrow[r] & \operatorname{Hom}_R(K_{i+1}, M) \arrow[r] & \cdots
\end{tikzcd}$$

and therefore the desired isomorphisms □

We have two more beautiful results with no time to prove them.

Theorem 1.4.13 (Depth Sensitivity of the Koszul Complex). Let R be a ring, I a finitely generated ideal and M an R-module. Suppose $\mathbf{x} = x_1, \ldots, x_n$ and $\mathbf{y} = y_1, \ldots, y_p$ are generating sequences for I, and fix $g \in \mathbb{N}$.

(a) $\mathrm{H}_i(\mathbf{x}; M) = 0$ for each $i = n - g + 1, \ldots, n$ if and only if $\mathrm{H}_j(\mathbf{y}; M) = 0$ for each $j = p - g + 1, \ldots, p$.

(b) If R is Noetherian, M is nonzero and finitely generated, and $g = \operatorname{depth}_I(M)$, then

$$\mathrm{H}_i(\mathbf{x}; M) \text{ is } \begin{cases} = 0 & \text{for } i = n - g + 1, \ldots, n \\ \neq 0 & \text{for } i = 0, \ldots, n - g \end{cases}$$

so there is are equalities

$$\operatorname{depth}_I(M) = g = n - \sup\{i | \, \mathrm{H}_i(\mathbf{x}; M) \neq 0\} = \inf\{i | \, \mathrm{H}^i(\mathbf{x}; M) \neq 0\}.$$

It may be helpful to keep track of the vanishing and nonvanishing of the Koszul homologies and cohomologies visually:

$$\underbrace{\mathrm{H}_n(\mathbf{x}; M), \ldots, \mathrm{H}_{n-g+1}(\mathbf{x}; M)}_{=0 \text{ for } g \text{ values}}, \underbrace{\mathrm{H}_{n-g}(\mathbf{x}; M), \ldots, \mathrm{H}_0(\mathbf{x}; M)}_{\neq 0}$$
$$\underbrace{\mathrm{H}^0(\mathbf{x}; M) \cdots \mathrm{H}^{g-1}(\mathbf{x}; M)}_{=0}, \underbrace{\mathrm{H}^g(\mathbf{x}; M) \cdots \mathrm{H}^n(\mathbf{x}; M)}_{\neq 0}.$$

Proof. See [2, (1.6.22) and (1.6.31)]. □

Remark 1.4.14. This motivates the definition of depth used for non-finitely generated modules. See [2, Section 9.1].

Before the last theorem, we make some observations and definitions.

Definition 1.4.15. Let (R, m, k) be local and $\mathbf{x} = x_1, \ldots, x_n \in m$ and M a nonzero finitely generated R-module such that the length of $M/(\mathbf{x})M$ is finite. Let $I = (\mathbf{x})R$ and $d = \dim(M)$. Then there exist polynomials $P(T) \in \mathbb{Q}[T]$ of degree d such that $P(t) = \operatorname{length}_R(M/I^{t+1}M)$ for $t >> 0$. Furthermore

$$P(T) = \frac{e_I(M)}{d!} T^d + \text{lower degree terms}$$

with $e_I(M) \in \mathbb{N}$. This is the *Hilbert-Samuel multiplicity* of M with respect to the ideal I. More generally, write $e_I(M, d) = e_I(M)$ and $e_I(M, d') = 0$ for each $d' > d$.

Observe that $\operatorname{Supp}(\mathrm{H}_i(\mathbf{x}; M)) \subseteq \operatorname{Supp}(M) \cap V(\mathbf{x}) \subseteq \{m\}$ since $M/(\mathbf{x})M$ has finite length, using Exercise 1.3.1(g). Since $\mathrm{H}_i(\mathbf{x}; M)$ is finitely generated, by Exercise 1.2.4(c), it follows that $\operatorname{length}_R(\mathrm{H}_i(\mathbf{x}; M)) < \infty$ for each $i \geq 0$.

The final result of this chapter will be the key to some facts about intersection multiplicities.

Theorem 1.4.16. With notation as above, $e_I(M, n) = \sum_{i=1}^{n} (-1)^i \operatorname{length}_R(\mathrm{H}_i(\mathbf{x}; M))$.

Proof. See [14, (4.3)]. □

Chapter 2

Cohen Structure Theorems

Motivating Question: Completions are hard, why do we study them?

Partial Answer: Many questions reduce easily to the complete case since $\hat{R}$ is faithfully flat over R (assuming R is local), and complete rings are really nice.

Follow-up Question: How nice are they?

2.1 Fundamentals

Let (R, m, k) be a local ring and $\eta_R \colon \mathbb{Z} \to R$ the natural ring homomorphism given by mapping $1 \mapsto 1_R$.

Exercise 2.1.1. Since R is local, either $\mathrm{Ker}(\eta_R) = (0)$ or $\mathrm{Ker}(\eta_R) = (p^e)$ where p is a prime number and $e \in \mathbb{N}$. If R is reduced, then $\mathrm{Ker}(\eta_R) = (0)$ or (p).

Definition 2.1.2. The *characteristic* of a ring R, denoted $\mathrm{char}(R)$, is the unique nonnegative generator of $\mathrm{Ker}(\eta_R)$.

Exercise 2.1.3. The contraction $\eta_R^{-1}(m)$ is either (0) or (p). Also, the map η_R factors through the localization $\mathbb{Z} \to \mathbb{Z}_{\eta_R^{-1}(m)}$, i.e., there is a commutative diagram:

$$\begin{tikzcd} \mathbb{Z} \arrow[r, "\eta_R"] \arrow[d] & R \\ \mathbb{Z}_{\eta_R^{-1}(m)} \arrow[ur, dotted] & \end{tikzcd}$$

Exercise 2.1.4. The following conditions on the local ring R are equivalent:

(i) R contains a copy of $\mathbb{Q}$ (as a subring);

(ii) $\mathrm{char}(k) = 0$;

(iii) $\mathrm{char}(k) = 0 = \mathrm{char}(R)$.

Definition 2.1.5. The local ring R has *equal characteristic 0* when the equivalent conditions of Exercise 2.1.4 are satisfied.

Example 2.1.6. The local ring $\mathbb{Q}[\![X_1, \ldots, X_n]\!]$ has equal characteristic 0.

Exercise 2.1.7. The following conditions on the local ring R and a prime number p are equivalent:

(i) R contains a copy of $\mathbb{Z}/p\mathbb{Z}$ (as a subring);

(ii) $\operatorname{char}(R) = p$;

(iii) $\operatorname{char}(R) = p = \operatorname{char}(k)$.

Definition 2.1.8. The local ring R has *(equal) characteristic* p when the equivalent conditions of Exercise 2.1.7 are satisfied.

Example 2.1.9. The local ring $\mathbb{Z}/p\mathbb{Z}[\![X_1, \ldots, X_n]\!]$ has equal characteristic p.

Exercise 2.1.10. The following conditions on the local ring R are equivalent:

(i) R does not contain a field as a subring.

(ii) $\operatorname{char}(R) \neq \operatorname{char}(k)$

Definition 2.1.11. The local ring R has *mixed characteristic* when the equivalent conditions of Exercise 2.1.10 are satisfied. There are two cases:

(a) $\operatorname{char}(R) = p^n > p = \operatorname{char}(k) > 0$;

(b) $\operatorname{char}(R) = 0 < p = \operatorname{char}(k)$.

The characteristic of k is the *residual characteristic* of R.

Example 2.1.12. The local ring $\mathbb{Z}/p^n\mathbb{Z}$ satisfies condition (a) of Definition 2.1.11. Condition (b) is satisfied by the local rings $\mathbb{Z}_{(p)}$ and $\widehat{\mathbb{Z}_{(p)}}$.

Remark 2.1.13. When R has mixed characteristic and $p = \operatorname{char}(k) > 0$, we identify p with its image $\eta_R(p)$ in R, and it follows that $0 \neq p \in m$. Furthermore, the quotient $R/(p)R$ has equal characteristic p.

Definition 2.1.14. A local ring (R, m, k) is *regular* if its maximal ideal can be generated by a system of parameters for R. By the theorem of Auslander, Buchsbaum and Serre, the ring R is regular if and only if every finitely generated R-module has finite projective dimension. A *discrete valuation ring* is a regular local ring of dimension 1.

Definition 2.1.15. Let R be a regular local ring of mixed characteristic with residual characteristic p.

(a) R is *unramified* if $p \in m \setminus m^2$, i.e., if p is a part of a regular system of parameters for R, i.e., the quotient $R/(p)R$ is a regular local ring.

(b) R is *ramified* if $p \in m^2$, i.e., the quotient $R/(p)R$ is not a regular local ring.

Example 2.1.16. The rings $\mathbb{Z}_{(p)}$ and $\widehat{\mathbb{Z}_{(p)}}$ are unramified regular local rings. For each $i \geq 2$, the local rings $\mathbb{Z}_{(p)}[\![X]\!]/(p - X^i)$ and $\widehat{\mathbb{Z}_{(p)}}[\![X]\!]/(p - X^i)$ are ramified regular local rings.

2.2 The Equal Characteristic Case

Let (R, m, k) be a ring containing a field L.

Definition 2.2.1. A *coefficient field* for R (if it exists) is a subfield $k_0 \subseteq R$ such that the composition $k_0 \hookrightarrow R \twoheadrightarrow k$ is an isomorphism.

Warning! If k_0 is a coefficient field for R, there are times when k and k_0 can be identified and times when they cannot be identified.

Example 2.2.2. The subring k_0 is not usually an R-module in the natural way. If $R = k_0[\![X]\!]$, the field k_0 is not closed under multiplication by elements in R. On the other hand, $k = R/m$ is an R-module. Another way to think of it is that m annihilates $k = R/m$, but m does not annihilate k_0 (unless $m = 0$).

Theorem 2.2.3. With (R, m, k) and L as above, assume R is complete.

(a) R admits a coefficient field.

(b) If the composition $L \hookrightarrow R \twoheadrightarrow k$ is a separable field extension, then R admits a coefficient field that contains L.

Proof. See [10, (28.3)]. The proof uses the theory of differential bases. □

Exercise 2.2.4. $L \subseteq R$ is a coefficient field if and only if $R = L + m$.

The last properties in this section follow from Theorem 2.2.3. They compare directly to properties of rings that are essentially of finite type over a field.

Corollary 2.2.5. With (R, m, k) and L as above, assume R is complete, k_0 is a coefficient field, and $\mathbf{x} = x_1, \ldots, x_n \in m$.

(a) There is a well-defined local ring homomorphism $\Phi \colon A = k_0[\![X_1, \ldots, X_n]\!] \to R$ given by $X_i \mapsto x_i$.

(b) If $(\mathbf{x}) = m$, then Φ is surjective.

(c) If $\mathbf{x}$ is a system of parameters for R, then Φ is injective and module-finite.

(d) If R is a regular local ring and $\mathbf{x}$ is a regular system of parameters, then Φ is an isomorphism. □

2.3 The Mixed Characteristic Case

Let (R, m, k) be a local ring of mixed characteristic with $\mathrm{char}(k) = p > 0$.

Definition 2.3.1. A *(complete) p-ring* is a (complete) discrete valuation ring (A, pA) of characteristic 0.

Remark 2.3.2. A p-ring has mixed characteristic since the characteristic of the residue field of A is $\mathrm{char}(A/pA) = p$.

Theorem 2.3.3. Let (R, m, k) and p be as above.

(a) If l is a field of characteristic p, then there exists a complete p-ring (A, pA, l). Such a ring is unique up to (non-unique) isomorphism.

(b) If R is complete and (A_0, pA_0, k) is a complete p-ring as in (a), then there exists a local ring homomorphism $\phi\colon A_0 \to R$ inducing an isomorphism on residue fields:

$$\begin{tikzcd} A \arrow[r, "\Phi"] \arrow[d] & R \arrow[d] \\ k \arrow[r, "\cong"] & k. \end{tikzcd}$$

If p is not nilpotent in R, then ϕ is injective. If p is R-regular, then ϕ is faithfully flat.

Proof. See [10, (29.1) and (29.2)]. The proof relies on the theory of smoothness. □

Definition 2.3.4. Assume R is complete. A subring $R_0 \subset R$ is a *coefficient ring* for R if (R_0, pR_0, k) is complete and the inclusion induces an isomorphism of residue fields:

$$\begin{tikzcd} R_0 \arrow[r, hook] \arrow[d] & R \arrow[d] \\ k \arrow[r, "\cong"] & k \end{tikzcd}$$

i.e., $R = R_0 + m$.

Corollary 2.3.5. Assume that R as above is complete. Let $\mathbf{x} = x_1, \dots, x_n \in m$ and fix $\phi\colon A_0 \to R$ as in Theorem 2.3.3(b).

(a) The ring $R_0 = \mathrm{Im}(\phi)$ is a coefficient ring for R.

(b) There is a well-defined local ring homomorphism $\Phi\colon A = A_0[\![X_1, \dots, X_n]\!] \to R$ such that $\psi|_{A_0} = \phi$ and $X_i \mapsto x_i$.

(c) If $(p, \mathbf{x})R = m$, then Φ is surjective.

(d) If $p, x_1, \dots, x_n$ is a system of parameters for R, then Φ is injective and module-finite.

(e) If R is an unramified regular local ring and $p, x_1, \dots, x_n$ is a regular system of parameters, then Φ is an isomorphism.

(f) If R is a ramified regular local ring and $x_1, \dots, x_n$ is a regular system of parameters, then Φ is surjective with its kernel generated by an element of the form $p - f(X_1, \dots, X_n) \in A$ with $f(X_1, \dots, X_n) \in (X_1, \dots, X_n)^2 A$. □

2.4 Consequences

Corollary 2.4.1. A complete local ring R is a homomorphic image of a regular local ring. In particular, R is universally catenary.

Proof. The first statement follows from Corollaries 2.2.5(b) and 2.3.5(c). Let A be a complete regular local ring surjecting onto R. The ring A is Cohen-Macaulay and therefore universally catenary. Hence, R is universally catenary. □

Corollary 2.4.2. A complete regular local ring has one of the following forms:

1. (Equal characteristic) $k_0[\![X_1, \ldots, X_n]\!]$ with k_0 a field;
2. (Mixed characteristic, unramified) $A_0[\![X_1, \ldots, X_n]\!]$ with A_0 a complete p-ring;
3. (Mixed characteristic, ramified) $A_0[\![X_1, \ldots, X_n]\!]/(p - f(X_1, \ldots, X_n))$ with A_0 a complete p-ring and $f(X_1, \ldots, X_n) \in (X_1, \ldots, X_n)^2 A_0[\![X_1, \ldots, X_n]\!]$. □

Chapter 3

Intersection Multiplicities

3.1 Motivation: Intersections in projective space

Let k be an algebraically closed field and X, Y subvarieties of $\mathbb{P}^d_k$ such that $X \cap Y$ is a finite set of (closed) points. As in differential geometry, we would like, for each $Q \in X \cap Y$ an "intersection multiplicity" of X and Y at Q, denoted $e(Q; X \cap Y)$, with the following properties:

(a) "dimension inequality": $\dim X + \dim Y \leq d = \dim \mathbb{P}^d_k$.

(b) "nonnegativity": $e(Q; X \cap Y)$ is a nonnegative integer.

(c) "vanishing": If $\dim X + \dim Y < d$, then $e(Q; X \cap Y) = 0$.

(d) "positivity": If $\dim X + \dim Y = d$, then $e(Q; X \cap Y) > 0$.

(e) "Bézout's Theorem": If $X \cap Y = \{Q_1, \dots, Q_r\}$, then

$$\sum_{i=1}^{r} e(Q_i; X \cap Y) = \operatorname{degree} X \cdot \operatorname{degree} Y.$$

Remark 3.1.1. Here is the geometric motivation for vanishing, which comes from differential geometry. If $\dim X + \dim Y$ is too small, then it should be possible to perturb X and Y slightly to new varieties that do not intersect, so the measure of intersection should be 0. In other words, in this case the intersection is accidental.

The motivation for positivity is similar. If $\dim X + \dim Y$ is maximal, then every perturbation of X and Y should intersect, so the measure of intersection should not disappear. In other words, the intersection is essential in this case and should be positively measured.

3.2 First steps

Definition 3.2.1. Let R be a Noetherian ring, and M, N finitely generated R-modules with $\operatorname{length}_R(M \otimes_R N)$ finite. Assume that M or N has finite projective dimension. (By the symmetry of Tor, we assume $\operatorname{pdim}_R M < \infty$.) For each $i \geq 0$, the module $\operatorname{Tor}_i^R(M, N)$ has finite length. Furthermore, $\mathrm{Tor}_i^R(M,N) = 0$ for each $i > \mathrm{pdim}_R M$. Thus, the *intersection multiplicity* of M and N

$$\chi_R(M,N) = \sum_{i \geq 0} (-1)^i \operatorname{length}_R(\mathrm{Tor}_i^R(M,N))$$

is a well-defined integer.

Exercise 3.2.2. Assume that R is local with maximal ideal m, and let $\mathbf{x} = x_1, \dots x_n \in m$ be an R-sequence. Let N be a finitely generated R-module such that $N/(\mathbf{x})N$ has finite length, i.e., $\operatorname{length}_R(R/(\mathbf{x}) \otimes_R N) < \infty$.

(a) The "dimension theorem" yields

$$\dim N = \inf\{l \in \mathbb{N} \mid \exists\, \mathbf{y} = y_1, \dots y_l \in m \text{ such that } \operatorname{length}_R(N/(\mathbf{y})N) < \infty\} \leq n.$$

(b) Since $\mathbf{x}$ is R-regular, one has $\dim(R/(\mathbf{x})) = \dim R - n$. This yields (in)equalities

$$\dim(R/(\mathbf{x})) + \dim N \leq \dim R - n + n = \dim R.$$

Furthermore,

$$\begin{aligned}
\chi_R(R/(\mathbf{x}), N) &= \sum_i (-1)^i \operatorname{length}_R(\mathrm{Tor}_i^R(R/(\mathbf{x}), N)) \\
&= \sum_i (-1)^i \operatorname{length}_R(\mathrm{H}_i(\mathbf{x}; N)) \\
&= e_{(\mathbf{x})}(N, n)
\end{aligned}$$

and thus,

$$\chi_R(R/(\mathbf{x}), N) \text{ is } \begin{cases} 0 & \text{if } \dim N < n, \text{ i.e., if } \dim(R/(\mathbf{x})) + \dim N < \dim R \\ > 0 & \text{if } \dim N = n, \text{ i.e., if } \dim(R/(\mathbf{x})) + \dim N = \dim R. \end{cases}$$

Exercise 3.2.3. Let $R = k[X_1, \dots, X_d]$ and let $\mathbf{x}$ be an R-sequence and N a finitely generated R-module such that $N/(\mathbf{x})N \neq 0$ has finite length. Assume that each $Q \in \mathrm{Min}_R(N)$ satisfies $\dim(R/Q) = \dim N$. (This assumption is needed for parts (a) and (c), but not for part (b).)

(a) $\dim(R/(\mathbf{x})) + \dim N \leq \dim R$.

(b) $\chi_R(R/(\mathbf{x}), N) = 0$ if $\dim(R/(\mathbf{x})) + \dim N < \dim R$.

(c) $\chi_R(R/(\mathbf{x}), N) > 0$ if $\dim(R/(\mathbf{x})) + \dim N = \dim R$.

(Hint: Pass to the localizations at maximal ideals in the support of $N/(\mathbf{x})N$.)

Conjecture 3.2.4. Assume that R is a regular local ring and M and N are finitely generated R-modules such that $\operatorname{length}_R(M \otimes_R N) < \infty$.

(a) "dimension inequality": $\dim M + \dim N \leq \dim R$.

(b) "nonnegativity": $\chi_R(M,N) \geq 0$.

(c) "vanishing": If $\dim M + \dim N < \dim R$, then $\chi_R(M,N) = 0$.

(d) "positivity": If $\dim M + \dim N = \dim R$, then $\chi_R(M,N) > 0$.

Remark 3.2.5. The hypothesis $\mathrm{length}_R(M \otimes_R N) < \infty$ in Conjecture 3.2.4 is crucial, as the dimension inequality fails easily without it, and $\chi_R(M,N)$ is only defined for pairs M, N whose tensor product has finite length.

The regularity hypothesis in the conjecture is trickier. If this is dropped, then we need to assume that at least one of the modules M or N has finite projective dimension in order for $\chi_R(M,N)$ to be defined. That this is not enough is shown by an example of Dutta, Hochster, and McLaughlin [3]: With $R = k[\![X,Y,Z,W]\!]/(XY-ZW)$ and $P = (X,Z)R$, there exists a module M of finite length and finite projective dimension such that $\chi_R(M,R/P) < 0$. Thus, such generalizations of nonnegativity and vanishing both fail over this ring. It is not known if the dimension inequality will fail in this context. Furthermore, it is not known, in general, whether nonnegativity or vanishing holds if the regularity hypothesis on R is replaced by the assumption that both M and N have finite projective dimension. See Theorem 3.4.6 for a "special case" that is known.

3.3 The affine case

Here's what happens in $k[X_1, \dots X_d]$. It suggests how to deal with the case when R is a regular local ring containing a field. The technique is called "reduction to the diagonal". We note that every finitely generated module over this ring has finite projective dimension by Hilbert's Syzygy Theorem.

Proposition 3.3.1. Let k be a field and $R = k[X_1, \dots, X_d]$ a polynomial ring. Fix prime ideals $P, Q \subset R$ such that $0 < \mathrm{length}_R(R/P \otimes_R R/Q) < \infty$.

(a) $\dim(R/P) + \dim(R/Q) \leq \dim R$

(b) $\chi_R(R/P, R/Q) \geq 0$

(c) $\chi_R(R/P, R/Q) = 0$ if and only if $\dim(R/P) + \dim(R/Q) < \dim R$.

Proof. We begin with the geometric motivation for the proof. The ideals P, Q correspond to affine subvarieties $V, W \subseteq \mathbb{A}^d$. Inside the product $\mathbb{A}^d \times \mathbb{A}^d \cong \mathbb{A}^{2d}$ there are two subvarieties of note: the diagonal Δ which is isomorphic to $\mathbb{A}^d$, and the product $V \times W$. The classical descriptions of these are:

$$\Delta = \{(a,a) \in \mathbb{A}^d \times \mathbb{A}^d \mid a \in \mathbb{A}^d\}$$
$$V \times W = \{(v,w) \in \mathbb{A}^d \times \mathbb{A}^d \mid v \in V \text{ and } w \in W\}.$$

There is an isomorphism
$$V \cap W \cong (V \times W) \cap \Delta$$
and the idea of this proof is to replace the triple $(V, W, \mathbb{A}^d)$ with the triple $(V \times W, \Delta, \mathbb{A}^{2d})$. (Hence, the phrase "reduction to the diagonal".) One hopes to make this replacement

because Δ is so nice (it is a linear subvariety) and so the result should be (and is) easier to prove for this second triple. The rest of the proof consists of the algebraic realization of this process.

We begin by setting

$$S = R \otimes_k R \cong k[Y_1, \dots, Y_d, Z_1, \dots, Z_d]$$

where the isomorphism is given by mapping $X_i \otimes 1 \mapsto Y_i$ and $1 \otimes X_i \mapsto Z_i$, and letting N be the S-module

$$N = R/P \otimes_k R/Q \cong_S S/(P \otimes_k R + R \otimes_k Q)$$

where the denominator of the right hand side makes sense because $P \otimes_k R$ and $R \otimes_k Q$ inject into $R \otimes_k R = S$ by the flatness of R over k. (Geometrically, S corresponds to $\mathbb{A}^d \times \mathbb{A}^d$ and N corresponds to the product $V \times W \subseteq \mathbb{A}^d \times \mathbb{A}^d$.)

The k-algebra homomorphism $S \cong k[Y_1, \dots, Y_d, Z_1, \dots, Z_d] \twoheadrightarrow k[X_1, \dots, X_d] = R$ given by $Y_i \mapsto X_i$ and $Z_i \mapsto X_i$ is surjective with kernel $\Delta = (Y_1 - Z_1, \dots, Y_d - Z_d)S$. Thus, R is identified with the S-algebra $R \cong S/\Delta$. (Geometrically, this "copy" of R corresponds to the diagonal in $\mathbb{A}^d \times \mathbb{A}^d$.

Claim. Every $\mathcal{Q} \in \operatorname{Min}_S(N)$ satisfies $\dim(S/\mathcal{Q}) = \dim_S N = \dim_R(R/P) + \dim_R(R/Q)$.

(Sketch of proof.) Let $p = \dim(R/P)$ and $q = \dim(R/Q)$. The Noether Normalization Lemma guarantees the existence of polynomial subrings

$$A = k[U_1, \dots, U_p] \hookrightarrow R/P \qquad B = k[V_1, \dots, V_q] \hookrightarrow R/Q$$

with each embedding module-finite. Tensoring over k yields a module-finite k-algebra embedding

$$A \otimes_k B \hookrightarrow R/P \otimes_k R/Q$$

and therefore the going-up theorem provides equalities

$$\dim N = \dim(R/P \otimes_k R/Q) = \dim(A \otimes_k B) = p + q = \dim(R/P) + \dim(R/Q).$$

Furthermore, each minimal prime $\mathcal{P} \subset R/P \otimes_k R/Q$ has $\mathcal{P} \cap (A \otimes B) = 0$ and, thus, gives rise to a module-finite injection

$$A \otimes_k B \hookrightarrow (R/P \otimes_k R/Q)/\mathcal{P}.$$

Another application of going-up yields

$$\dim((R/P \otimes_k R/Q)/\mathcal{P}) = \dim(R/P) + \dim(R/Q). \tag{3.1}$$

The minimal primes $\mathcal{P} \subset R/P \otimes_k R/Q$ are in bijection with the primes $\mathcal{Q} \in \operatorname{Min}_S(N)$ via the surjection $S \twoheadrightarrow R/P \otimes_k R/Q$. For any such $\mathcal{Q}$ with its partner $\mathcal{P}$, the isomorphism

$$S/\mathcal{Q} \cong (R/P \otimes R/Q)/\mathcal{P}$$

gives the equality

$$\dim(S/\mathcal{Q}) = \dim((R/P \otimes R/Q)/\mathcal{P})$$

which, coupled with equation (3.1), completes the proof of the claim.

(Geometrically, the claim is saying that, even though $V \times W$ is not a variety (i.e., reduced and irreducible) it is equidimensional of dimension $\dim V + \dim W$. Algebraically, the claim is allowing us to apply Exercise 3.2.3 to the S-modules S/Δ and N.)

To complete the proof, it is straightforward to check the R-isomorphism

$$S/\Delta \otimes_S N \cong R/P \otimes_R R/Q.$$

(Geometrically, this corresponds to the isomorphism $\Delta \cap (V \times W) \cong V \cap W$.) In particular, the tensor product $S/\Delta \otimes_S N$ has finite length over S. Thus, Exercise 3.2.3 gives the inequality in the following sequence

$$\begin{aligned} \dim(R/P) + \dim(R/Q) &= \dim N \\ &= (\dim N + d) - d \\ &= (\dim N + \dim(S/\Delta)) - d \\ &\leq \dim S - d \\ &= d. \end{aligned}$$

For each $i \geq 0$, we have an R-isomorphism

$$\operatorname{Tor}_i^S(S/\Delta, N) \cong \operatorname{Tor}_i^R(R/P, R/Q)$$

and this yields the equality in the following sequence

$$\chi_R(R/P, R/Q) = \chi_S(S/\Delta, N) \geq 0$$

where the inequality is by Exercise 3.2.3. Furthermore, equality holds if and only if

$$\begin{aligned} \dim(S/\Delta) + \dim N < \dim S &\iff d + (\dim(R/P) + \dim(R/Q)) < 2d \\ &\iff \dim(R/P) + \dim(R/Q) < d \end{aligned}$$

which provides the final desired conclusion. $\square$

3.4 The local case

Serre verified the dimension inequality part of Conjecture 3.2.4 in general, and the other parts of the conjecture in the equal characteristic and unramified cases.

Theorem 3.4.1. Assume that R is a regular local ring and M and N are finitely generated R-modules such that $\operatorname{length}_R(M \otimes_R N) < \infty$.

(a) $\dim M + \dim N \leq \dim R$.

(b) Assume that R has equal characteristic or is unramified. Then $\chi_R(M, N) \geq 0$ with equality if and only if $\dim M + \dim N < \dim R$.

Remark 3.4.2. This theorem shows that we can use χ to give geometric intersection multiplicities as described in Section 3.1. Indeed, let $X, Y \in \mathbb{P}^d_k$ be subvarieties such that $X \cap Y$ is a finite set of (closed) points, and let $\mathcal{O} = \mathcal{O}_{\mathbb{P}^d_k}$ be the structure sheaf and $\mathcal{I} = \mathcal{I}_X$ and $\mathcal{J} = \mathcal{I}_Y$ be the appropriate sheaves of ideals. For each $Q \in X \cap Y$, set

$$e(Q; X \cap Y) = \chi(\mathcal{O}_Q/\mathcal{I}_Q, \mathcal{O}_Q/\mathcal{J}_Q).$$

That this definition satisfies the desired properties (a)–(d) follows from Theorem 3.4.1. Bézout's Theorem is also satisfied, but we will not prove that here.

To prove Theorem 3.4.1, say when R contains a field, one might try to use reduction to the diagonal like in Proposition 3.3.1. However, there are inherent difficulties as the tensor product of two k-algebras may fail to be Noetherian.

Exercise 3.4.3. If k is a field, then the ring $k[\![X]\!] \otimes_k k[\![X]\!]$ is not Noetherian. (Hint: Look at a field extension $k(X) \hookrightarrow k((X))$ which has infinite transcendence degree.)

If one is clever like Serre, though, one uses completed tensor products to get around this problem.

Definition 3.4.4. Let $R = k[\![X_1, \ldots, X_d]\!]$ with k a field and let M and N be finitely generated R-modules. The *completed tensor product* of M and N over k is

$$M \widehat{\otimes}_k N = \varprojlim_{(p,q)} M/m^p M \otimes_k N/m^q N.$$

For a thorough account of the properties of the completed tensor product and its companions the completed Tor's, see [5, (0.7.2)]. Here are some basic facts that show one how to prove Theorem 3.4.1 for the ring $R = k[\![X_1, \ldots, X_d]\!]$. The general case of equal characteristic reduces to this one by passing to the completion.

Fact 3.4.5. Let $R = k[\![X_1, \ldots, X_d]\!]$ with k a field and let M and N be finitely generated R-modules.

(a) The ring $S = R \widehat{\otimes}_k R$ is isomorphic to $k[\![Y_1, \ldots, Y_d, Z_1, \ldots, Z_d]\!]$.

(b) $M \widehat{\otimes}_k N$ is a finitely generated S-module.

(c) The ring homomorphism $S \twoheadrightarrow R$ given by $Y_i, Z_i \mapsto X_i$ is well-defined and surjective with kernel $\Delta = (Y_1 - Z_1, \ldots, Y_d - Z_d)S$.

(d) There is an equality $\dim_S M \widehat{\otimes}_k N = \dim_R M + \dim_R N$.

(e) For each $i \geq 1$, there is a natural isomorphism $\operatorname{Tor}_i^S(S/\Delta, M \widehat{\otimes}_k N) \cong \operatorname{Tor}_i^R(M, N)$.

The proof of Theorem 3.4.1 in the unramified case is slightly more delicate, but similar. Finally, the dimension inequality in the ramified case follows from the multiplicity results in the unramified case, as a complete ramified regular local ring can be written as a quotient of an unramified regular local ring by a regular element. See [14] for more details.

Roberts [12] and Gillet-Soulé [4] have verified the Vanishing Conjecture in the ramified case. Robert's proof uses techniques from intersection theory, primarily, the formal properties of localized Chern characters. Gillet-Soulé's proof is K-theoretic, focusing on the Adams operations.

Theorem 3.4.6. Assume that R is a local complete intersection ring and M and N are finitely generated R-modules such that $\operatorname{pdim}_R M$, $\operatorname{pdim}_R N$, and $\operatorname{length}_R(M \otimes_R N)$ are all finite. If $\dim M + \dim N < \dim R$, then $\chi_R(M, N) = 0$. $\square$

Gabber verified the Nonnegativity Conjecture in the ramified case; see [1, 7, 13]. The proof depends upon de Jong's theory of regular alterations.

Theorem 3.4.7. If R is a regular local ring and M and N are finitely generated R-modules such that $\operatorname{length}_R(M \otimes_R N) < \infty$, then $\chi_R(M, N) \geq 0$.

Remark 3.4.8. The Positivity Conjecture is still open in the ramified case.

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