Some Variational Problems in Geometry II

Andrejs Treibergs
University of Utah

Abstract. We describe Sacks and Uhlenbeck's proof that smooth manifolds satisfying some topological condition contain nonconstant minimal immersions of two-spheres. This is an example of a geometric variational problem in which a special minimizing seqence is chosen that exhbits concentration, yet with sufficient regularity properties to show convergence to a smooth solution away from finitely many singular points. By conformally blowing up near the singular points, another minimizing sequence is constructed that converges to another solution, a "bubble." As this lecture is aimed at nonspecialists in geometry, we provide some background materials about minimal and harmonic maps.

This is a continuation of the lecture [T]. We describe Sacks and Uhlenbeck's proof [SaU1] that smooth, compact, n-dimensional manifold N satisfying some topological condition (*e.g.*, that $\pi_2(N) \neq 0$) contains nonconstant minimal two-spheres. This is an example of a geometric variational problem in which a special minimizing seqence is chosen (by solving a related variational problem) that exhbits concentration, yet with sufficient regularity properties to show convergence to a smooth solution away from finitely many singular points. By conformally blowing up near the singular points, another minimizing sequence is constructed that converges to another solution, a "bubble." Another source for the material is [SY].

In what follows M^2 and N^n will be a smooth compact oriented boundaryless Riemannian manifolds of dimension 2 and $n \geq 3$, where usually, $M = \mathbf{S}^2$, the two-sphere. We will be interested in finding a map $f \in C(M,N)$ that is minimal but is not a map to a point. Let us denote the constant or point maps by

$$\mathcal{K}_0 = \{f : M \to N : \text{there is } y \in N \text{ so that } f(x) = y \text{ for all } x \in N.\}$$

We shall say that f is nontrivial if it is not freely homotopic to a map in $\mathcal{K}_0$ which means it cannot be continuously deformed to a map in $\mathcal{K}_0$. For example, every map $f \in C(\mathbf{S}^2, \mathbf{S}^3)$ is trivial whereas $f \in C(\mathbf{S}^2, \mathbf{S}^1 \times \mathbf{S}^2)$ given by $x \mapsto ((1,0),x)$ is nontrivial. The assumption $\pi_2(N) \neq 0$ simply says that there is at least some nontrivial $f_0 \in C(\mathbf{S}^2, N)$, which we may assume to be in $C^\infty(M,N)$ by smoothing. A minimal surface is a map that is critical for the area functional $\text{Area}_N(f(M))$.

In §1 We show that finding minimal maps is equivalent to finding harmonic maps, which are critical for the energy functional. There we formulate the variational problems and establish the equivalence. We present Sacks and Uhlembeck's proof in §2. We mention some open problems in §3.

1. Conformal, minimal, and harmonic mappings of surfaces.

This material is standard from an introduction to differntial geometry, *e.g.* [KRST], [T]. By the Nash Embedding Theorem [G] we may assume that the manifold is smoothly isometrically embedded into some Euclidean space of sufficiently high dimension $N \subset \mathbf{R}^k$. This means that the metric ds_N^2 is the submanifold metric inherited from the ambient $\mathbf{R}^k$. The Sobolev space of maps is

$$L^{1,2}(M,N) = \{L^{1,2}(M,\mathbf{R}^k) : f \in N \ a.e.\}$$

the restriction of vector valued maps whose first derivatives are in L^2. Since we have assumed that N is compact, $L^{1,2}(M, N) \subset L^\infty(M, \mathbf{R}^k)$. In this formulation, the *energy* of a map $f \in L^{1,2}(M, N)$ is just the ordinary Dirichlet Integral

$$E(f) = \tfrac{1}{2} \int_M |df|_M^2 \, d\operatorname{Area}_M .$$

The harmonic map problem is to find minimizers of E among nontrivial maps.

The definition of Sobolev spaces of maps is subtle [SY]. For example $C^\infty(M, N)$ may not be dense in $L^{1,2}(M, N)$ for general m. An example of this is given by Schoen and Uhlenbeck [SU2]. Furthermore $\overline{C^\infty(M, N)}$ may not be weakly closed, so it may not suffice to consider only limits of sequences of smooth or continuous maps for the harmonic map problem [B]. $L^{1,2}(M, N)$ is not a Hilbert manifold. However, if the dimension of M is $m = 2$ then Schoen and Uhlenbeck[SU2] showed that $C^\infty(M, N)$ is dense in $L^{1,2}(M, N)$.

Another difficulty for geometric problems is that an $f \in L^{1,2}(M, N)$ is not continuous, since dimension two is critical for the Sobolev inequality. Thus topological conditions, such as nontriviality of the map cannot be easily passed to the limit. However, some continuity properties persist, such as the fact that the map on almost every one-dimensional coordinate curve is continuous [M, Theorem 3.1.8]. These properties were exploited by Lemaire [L] who also proved the existence of minimal spheres by looking for minimizing sequences directly in $L^{1,2}(M, N)$.

Harmonic and minimal maps defined. To define the maps we must consider two types of variations of $f \in L^{1,2}(M, N)$. The first type, which I call the *range variations* may be constructed as follows. Consider any $\eta \in C^\infty(M, \mathbf{R}^k)$. As $f + t\eta$ may not be in N, we project it to N by the nearest point projection $\Pi : U \to N$ where $U \in \mathbf{R}^k$ is some small enough neighborhood of $f(M)$ such that Π is smooth. The variation is $f_t = \Pi(f + t\eta)$ for $|t|$ small. If f is a critical point for E with respect to range variations, then f satifies the harmonic map equation weakly

(1) $$\Delta_M f + A(f)(df, df) = 0$$

where $A(f(x))(\cdot, \cdot)$ is the second fundamental form of N at $f(x)$. A is by definition [Le], the normal part (perpendicular to the tangent space $T_{f(x)}N$) of the Laplacian applied to $f \in \mathbf{R}^k$.

The other type of variation is the *domain variation* which may be described as follows. Let $\psi_t : M \to M$ be a smooth 1-parameter family of diffeomorphisms such that ψ_0 is the identity. The domain variation is $f_t = f \circ \psi_t$.

We call a map *harmonic* is it is a critical point of E under both domain and range variations. Morrey [M] proved that if $u : M \to N$ is harmonic then $u \in C^\infty(M, N)$.

Fact 1. *Every smooth two dimensional manifold can be locally parameterized by isothermal coordinates. This means that around every point of M there is a curvilinear coordinate system (x_1, x_2) so that the metric takes the form*

$$ds_M^2 = \lambda(x_1, x_2) \left(dx_1{}^2 + dx_2{}^2 \right)$$

where $\lambda > 0$ is a positive smooth function.

See, it e.g., [T]. For example, if $\sigma : \mathbf{S}^2 \to \mathbf{R}^2$ is stereographic projection, then in the standard stereographic coordinates $\sigma(P) = (x_1, x_2) \in \mathbf{R}^2$, the metric of the two sphere, $\sigma^* g_M$, becomes

$$ds_{\mathbf{S}^2}^2 = \frac{4 \left(dx_1{}^2 + dx_2{}^2 \right)}{\left(1 + x_1^2 + x_2^2 \right)^2}.$$

In the jargon, every surface is locally conformal to the Euclidean plane. A *conformal* map between surfaces $\varphi : M \to M'$ is a map that preserves angles and infinitessimal circles. Thus, in terms of the metric, the pulled back metric becomes just a multiple (by a positive function) of the original metric

$$\tau g = \varphi^* g'.$$

For example, if $R = \begin{pmatrix} a & -b \\ b & a \end{pmatrix}$ is the conformal matrix for any $a, b \in \mathbf{R}$ then the map from $\mathbf{R}^2 \to \mathbf{R}^2$ given multiplication by R is conformal.

Expressing M in isothermal coordinates, how do we write the pullback metric from N? Let $g_{ij} = \dfrac{\partial f}{\partial x_i} \cdot \dfrac{\partial f}{\partial x_j}$ be the metric of the surface $f(M) \subset \mathbf{R}^k$. Then the metric pulled back to M becomes

$$\tau(x_1, x_2)\, \lambda(x_1, x_2)\, \left(dx_1{}^2 + dx_2{}^2\right) = \sum_{i,j=1}^{2} g_{ij}\, dx_i\, dx_j = f^* ds^2_{\mathbf{R}^k}.$$

Thus $f : M \to N$ is conformal in these coordinates if and only if

$$g_{11} = \tau\,\lambda = g_{22}, \qquad g_{12} = g_{21} = 0$$

for some positive function τ. If $\tau \geq 0$ but is allowed to vanish, then f is said to be *weakly conformal*.

Fact 2. *Let $\varphi : M \to M'$ be a conformal map between surfaces. Then the energy is preserved: for all $f \in L^{1,2}(M', \mathbf{R}^k)$,*

$$E_M(\varphi^* f) = E_{M'}(f).$$

This follows from how the norm of the gradient and area form change under conformal reparameterizaion. The transformation formulæ are

$$\frac{1}{\tau}|d(\varphi^* f)|^2_M = |df|^2_{M'}, \qquad \tau\, d\,\mathrm{Area}_M = \varphi^* d\,\mathrm{Area}_{M'}.$$

Thus, by the change of variables formula,

$$E_{M'}(f) = \tfrac{1}{2} \int_{M'} |df|^2_{M'}\, d\,\mathrm{Area}_{M'} = \tfrac{1}{2} \int_M |d(f \circ \varphi)|^2_M\, d\,\mathrm{Area}_M = E_M(\varphi^* f).$$

In particular, for $U \subset \mathbf{R}^2$ an isothermal coordinte chart for M, then the identity mapping is a conformal map from $\left(U, dx_1{}^2 + dx_2{}^2\right) \to \left(U, \lambda(x_1, x_2)(dx_1{}^2 + dx_2{}^2)\right)$ so we have

$$E_U(f) = \tfrac{1}{2} \int_U |df|^2\, dx_1\, dx_2$$

is the ordinary $\mathbf{R}^2$ Dirichlet integral.

The area of the image of an isothermal chart $U \in \mathbf{R}^2$ is the usual induced area formula

$$\mathrm{Area}_U(f) = \int_U \sqrt{\det(g_{ij})}\, dx_1\, dx_2.$$

A surface $f : M \to N$ is *minimal* if it is a critical point for A under range variations. Domain variations preserve the area.

Lemma 3. [ES] *Suppose that $f_0 : M \to N$ is a weakly conformal immersion and $n \geq 3$. Then f_0 is harmonic if and only if f_0 is a minimal branched immersion.*

Proof. Let $f : M \to N$. Then in a local isothermal chart $(x_1, x_2) \in U$ for M,

$$E_U(f) = \tfrac{1}{2} \int_U \mathrm{tr}(g_{ij})\, dx_1\, dx_2 \geq \int_U \sqrt{\det(g_{ij})}\, dx_1\, dx_2 = \mathrm{Area}_U(f)$$

by the arithmetic-geometric inequality. Equality holds if and only if the eigenvalues of g_{ij} are equal, in other words when f is weakly conformal.

Now suppose that f_t is a variation for $t \in (-\varepsilon, \varepsilon)$ such that f_0 is weakly conformal. Then

$$E(f_t) - F(f_0) \geq A(f_t) - A(f_0).$$

Dividing by t and taking limits as $t \to 0\pm$ shows that at f_0, $\delta E(f_0) = \delta A(f_0)$. It follows that f_0 is minimal if and only if f_0 is harmonic.

If f_0 is minimal, it satisfies the equation (1) weakly. Thus by a theorem of Hartman and Wintner [HW], or Aronszajn [A], the singularities of f (where $\det(g_{ij}) = 0$) are isolated and their vanishing can't be of infinite order so are like $z \mapsto z^k$, *i.e.*, branch points [GOR].

The two-spheres are special in the sense that they have only one conformal type. This means that any metric on $\mathbf{S}^2$ is the pullback of any other metric by a conformal diffeomorphism.

Lemma 4. *If $f : \mathbf{S}^2 \to M$ is harmonic, then it is weakly conformal.*

Proof. Following [SaU1], using the fact that the second fundamental $A(f)(df, df)$ form of N is normal to N whereas $f_i = \dfrac{\partial f}{\partial x_i}$ are tangent to N, and $(x_1, x_2) \in \mathbf{R}^2$ are stereographic coordinates for the two-sphere, by taking inner products with (1),

$$0 = f_1 \cdot \Delta f = \tfrac{1}{2} \frac{\partial}{\partial x_1} \left(g_{11} - g_{22} \right) + \frac{\partial}{\partial x_2} g_{12}$$

$$0 = f_2 \cdot \Delta f = \tfrac{1}{2} \frac{\partial}{\partial x_2} \left(g_{22} - g_{11} \right) + \frac{\partial}{\partial x_1} g_{12}$$

which are Cauchy Riemann equations on $\mathbf{R}^2$. Hence

$$w = \tfrac{1}{2} \left(g_{11} - g_{22} \right) + i g_{12}$$

is a holomorphic function on $\mathbf{R}^2$. However, by the Schwarz inequality, $g_{12}^2 \leq g_{11} g_{22}$ so that

$$\int_{\mathbf{R}^2} |w| \, dx_1 \, dx_2 = \int_{\mathbf{R}^2} \sqrt{\tfrac{1}{4} \left(g_{11} - g_{22} \right)^2 + g_{12}{}^2} \, dx_1 \, dx_2 \leq E(f) < \infty.$$

It follows that $w \equiv 0$ vanishes identically on $\mathbf{R}^2$, thus f is weakly harmonic.

By Lemma 3 and Lemma 4, if one finds a nonconstant harmonic map $f : \mathbf{S}^2 \to N$ then it is a branched minimal immersion. A separate argument has to be made to show that f is nontrivial.

The minimizing procedure of Sacks and Uhlenbeck. Let $M = \mathbf{S}^2$ with a metric scaled so that $\mathrm{Area}(M) = 1$. Sacks and Uhlenbeck consider the perturbed energy functional for any $\alpha \geq 1$ given by

$$E_\alpha(f) = \int_M \left(1 + |df|^2 \right)^\alpha d\,\mathrm{Area}.$$

Then $1 + 2E(f) = E_1(f) \leq E_\alpha(f)$. Choose a nontrivial $f_0 \in C(M, N)$, which by approximation we may suppose is C^∞. Let $B = (1 + \sup_M |df_0|^2)^2$.

The idea of Sacks and Uhlenbeck is to establish that there is $u_\alpha \in C^\infty(M, N)$ which is homotopic to f_0 and which minimizes the preturbed energy among all maps homotopic to f_0

$$E_\alpha(u_\alpha) = \inf \left\{ E_\alpha(f) : f \in L^{1,2\alpha}(M, N) \text{ and } f \simeq f_0 \right\}.$$

Then $E_\alpha(u_\alpha) \leq B$ for all $1 \leq \alpha \leq 2$. They show that u_α is a close to a minimizing sequence for E; it converges to an E-critical function u weakly in $L^{1,2}(M, \mathbf{R}^k)$, almost everywhere so that $u \in L^{1,2}(M, N)$, and strongly in $L^2(M, N)$. The fact that u_α satisfy the E_α-Euler equations which admit some uniform apriori estimates allows them to conclude more about the nature of the limit. They show that the minimizing sequence concentrates at most at finitely many points. Away from these points the convergence is smooth. Near these points, either the gradient remains bounded uniformly, in which case the point is removeable, or the gradient is unbounded. By a conformal blowup reparameterization near the point, they get another minimizing sequence $\tilde{u}_\alpha$ that in fact converges to another limiting solution $\tilde{u}$, called a *bubble*, which provides a nonconstant map $\tilde{u} : \mathbf{S}^2 \to N$. The limiting function $\tilde{u}$ of the sequence $\tilde{u}_\alpha$ may not be homotopic to f_0 and it may not even be nontrivial, nonetheless, it provides the desired nonconstant map.

Fact 5. *Let $\alpha > 1$ and $f_0 \in C(M, N)$ be a nontrivial map. Then there is $u_\alpha \in C^\infty(M, N)$ which is homotopic to f_0 and which minimizes the α-energy amongst maps $f \in L^{1,2\alpha}(M, N)$ which are homotopic to f_0.*

Idea of the proof. First observe that since $\alpha > 1$ the maps $f \in L^{1,2\alpha}(M, N) \subset C^{0,1-1/\alpha}(M, N)$ are Hölder continuous by the Sobolev embedding theorem. Moreover, $L^{1,2\alpha}(M, N)$ is a C^2 separable Banach manifold

and the energy functional E_α satisfies the Palais-Smale condition (C) and the Ljusternik-Scnirelmann theory works for this functional. Hence E_α takes on its minimum in every connected component of $L^{1,2\alpha}(M, N)$, so in particular there is an energy minimizer in the component of maps homotopic to f_0 [P].

The critical maps of E_α in $L^{1,2\alpha}(M, N)$ are smooth if $\alpha > 1$. The Euler Lagrange equations $\delta E_\alpha = 0$ reduce to

$$\Delta_\alpha u + A(u)(du, du) = \Delta u + 2(\alpha - 1)\frac{(d^2 u, du)}{1 + |du|^2} + A(u)(du, du) = 0.$$

For some small $1 < \alpha_0 < 2$ we have for $1 < \alpha \le \alpha_0$ that $L_\alpha : L^{2,4}(M, N) \to L^4(M, N)$ is boundedly invertible. As $L^{4,2}(M, N) \subset C^{1,1/5}(M, N)$, the smoothness of critical values follows from Schauder Theory. As the spaces of maps $C(M, N)$ through $L^{1,2\alpha}(M, N)$ through $C^\infty(M, N)$ have the same homotopy type, a C^∞ minimizer in the free homotopy class of f_0 in $L^{1,2\alpha}(M, N)$ also minimizes E_α in its connected component of $C^\infty(M, N)$. But since $[f_0] \ne 0$ in $\pi_2(N)$ by hypothesis, u_α is not in the component of the point maps $\mathcal{K}_0$ [P].

Sacks and Uhlenbeck also consider other topological hypotheses, such as the case when $\pi_2(N) = 0$ but the universal cover of N is not contractible. Then the argument that there is a u_α which is nonconstant is a little more involved. The idea is that if here were no critical points other than $\mathcal{K}_0$ then there would be a energy decreasing deformation retraction of $E_\alpha^{-1}[1, B)$ to $\mathcal{K}_0$, which would collapse all the topology of N.

Then Sacks and Uhlenbeck show that the sequence u_α satisfies a weak lower-semicontinuity statement.

Fact 6. *Suppose that* $\{u_\alpha\}_{1 < \alpha \le \alpha_0} \subset L^{1,2}(M, N)$ *is a sequence of critical maps for* E_α*, and that there is* $B < \infty$ *so that* $E_\alpha(u_\alpha) < B$ *for all* $1 < \alpha \le \alpha_0$*. Then there is a subsequence* $u_{\alpha'}$ *that converges weakly* $u_{\alpha'} \rightharpoonup u$ *in* $L^{1,2}(M, \mathbf{R}^k)$*, that the convergence* $u_{\alpha'} \to u$ *is a.e. in* M *(so that* $u \in L^{1,2}(M, N)$*) and strongly in* L^2*. Moreover*

$$\lim_{\alpha' \to 1} E\left(u_{\alpha'}\right) \ge E(u).$$

The u_α are not assumed to be minimizing so it is not asserted for u. In Fact 10, the harmonicity of u is established. It is not known if u is continuous. And it can happen that $\lim_{\alpha' \to 1} E\left(u_{\alpha'}\right) > E(u)$ or that $u \in \mathcal{K}_0$ is a point (constant) map.
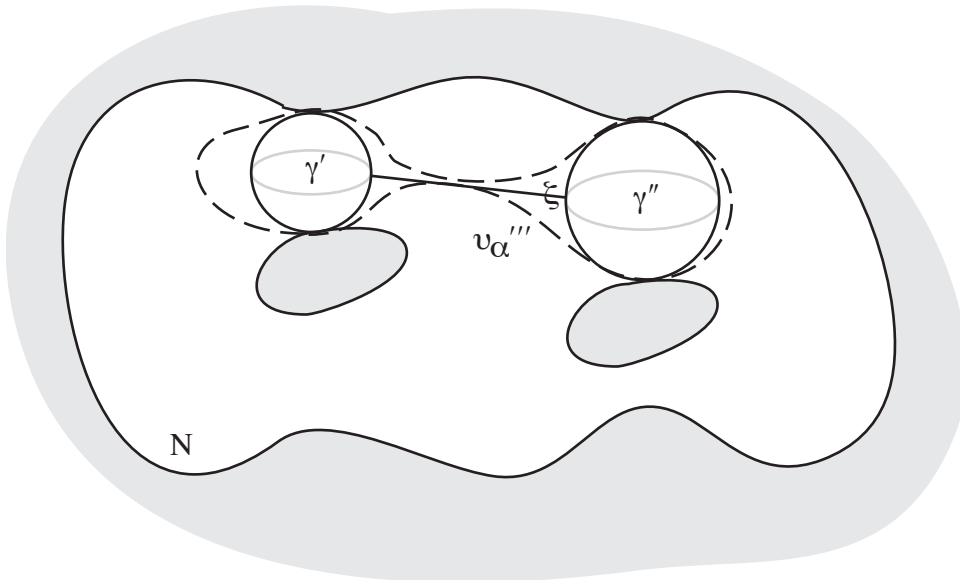

Fig 1. Minimizing sequence tending to the sum of two spheres may develop a degenerate neck.

The rest of the argument is to show that the Sacks and Uhlenbeck sequence $u_{\alpha'}$ converges smoothly away from finitely many points and that the limiting map if it exists, or the limiting map of the rescaled sequence (the bubble) is a nonconstant harmonic map.

Digression on bubbles: why the minimizing sequence shouldn't be converging smoothly. For the sake of illustration, suppose that there are two disjoint spheres in the manifold which are E-minimizers in their homotopy classes. Say they are given by smooth maps $\gamma', \gamma'' : \mathbf{S}^2 \to N$ which correspond to two different generators of $\pi_2(M)$. It may happen (see Figure 1.) that the minimizers u'_α and u''_α of E_α in the two free homotopy classes converge smoothly to γ' and γ'', resp. Let ζ be a length-minimizing path from $\gamma'(\mathbf{S}^2)$ to $\gamma''(\mathbf{S}^2)$, and let $\gamma''' : \mathbf{S}^2 \to N$ be such that upper part or the sphere is mapped to γ', the equatorial belt is mapped to a small tube following ζ and the lower part is mapped to γ''. The E_α minimizers, v_α''', homotopic to γ''' could then be expected to approximate γ' and γ'' but also collapse to the degenerate line ζ as $\alpha \to 1$. In other words, the minimizing sequence for γ''' will develop singularities, and not tend (without taking all bubbles into account) to a minimizer in the γ''' component. You would do better energetically if you could cut off part of the map and consider the sequence from half of the sphere. This is what will happen in effect. One end of the sphere will be rescaled by conformal deformation to confine the gradient, and will limit to the bubble; the other end will be collapsed to a point, taking some energy with it, as it is excluded from the rescaled limit. Thus, although the minimizing sequence will spawn rescaled related sequences that converge to some solution, it will not be possible to find minimizers in every homotopy class, such as for γ'''. It may happen that the minimizing sequences may have more than one bubble, or may form bubbles on bubbles. A method of "bubble trees" to keep track of all of the bubbles rather than discarding everything except the last bubble was developed by Parker and Wolfson [PW] for a very similar problem.

Apriori estimates. The regularity is a local property. Thus, we may consider topological disks $D \subset M$ of various sizes that may identify with closed coordinate disks $D(z, R) \subset \mathbf{R}^2$ of center z and radius R in an isothermal coordinate chart. By dilating the disk to unit radius, the Euler equation transforms on $D(z, 1)$ to

$$\Delta u + 2(\alpha - 1)\frac{(d^2 u, du)}{R^2 + |du|^2} + A(u)(du, du) = 0.$$

Thus Sacks and Uhlenbeck are able to give apriori estimates for this equation that are uniform for $1 < \alpha \leq \alpha_0$ and $0 < R$. We mention only a couple of the estimates.

The first is a removeable singularities theorem for harmonic maps.

Fact 7. *Let $0 < R < \infty$ and suppose $f : D(R) - \{0\} \to N$ is a smooth harmonic map. Suppose that the energy is finite*

$$\int_{D(R) - \{0\}} |df|^2 \, dx_1 \, dx_2 < \infty.$$

Then f extends to a smooth harmonic map $f : D(R) \to N$.

By inversion, $D(1) - \{0\}$ is conformally diffeomorphic to $\mathbf{R}^2 - D(1)^\circ$. This says that a harmonic map can be extended across the point at infinity.

The second is a global estimate for small energy E_α-critical maps. It says uniformly that if the E_α-energy is small, then the map must be a point map.

Fact 8. *Given such M and N, there exists an $\alpha_0 > 1$ and $\delta > 0$ so that if v is an E_α-critical map such that $E_\alpha(v) < \delta$ for some $1 \leq \alpha \leq \alpha_0$, then v is a constant map $v \in \mathcal{K}_0$ and $E_\alpha(v) = 0$.*

The partial regularity result, first given by Morrey [M], is that for the sequences being considered, if the energy remains uniformly small then the convergence is smooth.

Fact 9. *Let $0 < R < \infty$ and $B < \infty$ be given. Suppose that $u_\alpha : D(R) \to N$ be critical for E_α, that $E_\alpha(u_\alpha) \leq B$ and that $u_\alpha \rightharpoonup u$ weakly in $L^{2,1}(D(R), \mathbf{R}^k)$ as $\alpha \to 1$. Then there is an $\varepsilon > 0$ so that if $E(u_\alpha) \leq \varepsilon$ then $u_\alpha \to u$ in $C^1\left(D(\frac{R}{2}), N\right)$ where $u : D(\frac{R}{2}) \to N$ is a smooth harmonic map.*

As E_α is almost conformally invariant, the magnitude ε is independent of R.

As an application, if $u_\alpha : D(R) \to N$ are E_α-critical and $\limsup_{\alpha \to 1} \sup_{D(R)} |du_\alpha| < \infty$ then there is a $\delta > 0$ so that $u_\alpha \to u$ converges in $C^1\left(D(\delta), N\right)$ to a smooth harmonic map $u : D(\delta) \to N$.

Fact 9 is now used to show that u_α converges smoothly except at finitely many points. The idea is to cover M by disks of small radius. Then, using Fact 9, most of the disks will have small energy and thus u_α

will converge in such disks. There is a number $h \in \mathbf{N}$ so that the following is true. For every $m \in \mathbf{N}$ an integer, let $R = 2^{-m}$ be the radius of the disks and let $U_R = \{x \in U : D(R, x) \subset U\}$. There are finitely many $\ell = \ell(m)$ center points $\mathcal{Z}_m := \{z_{1,m}, \dots, z_{\ell,m}\}$ so that $\{D(\frac{R}{2}, z_{i,m})\}_{i=1,\dots,\ell} \subset U$ cover U_R and so that every point $x \in U_R$ is contained in at most h balls $\{D(R, z_{i,m})\}_{i=1,\dots,\ell}$. The key thing to notice is that the number of disks ℓ whose energy exceeds ε is at most $\frac{hB}{\varepsilon}$, which is a number independent of m. Thus by Fact 9, the convergence as $\alpha \to 1$ is uniform except in the vicinity of ∂U or near the points $\mathcal{Z}_m$. Letting $m \to \infty$ yields the following partial regularity statement.

Fact 10. *Let $U \subset M$ be open and $u_\alpha : U \to N$ be critical for E_α. Suppose that $E_\alpha(U_\alpha) \leq B$ uniformly for some $B < \infty$ and that $u_\alpha \rightharpoonup u$ weakly in $L^{2,1}(U, M)$. Then there is a subsequence $\alpha' \to 1$ and finitely many points $z_1, \dots, z_\ell \in U$ so that $u_{\alpha'} \to u$ in $C^1(U - \{z_1, \dots, z_\ell\}, N)$ where $u : U - \{z_1, \dots, z_\ell\} \to N$ is a smooth hamonic map.*

Either du_α remains bounded near the singular points in which case the convergence can be smoothly extended across the singular point, or the gradient blows up and sequence can be rescaled to form one which converges to a new harmonic map, a *bubble*.

Theorem 11. *Suppose $u_\alpha : M \to N$ be the sequence of smooth E_α-critical maps constructed in the argument of Fact 10 such that $u_\alpha \to u$ in $C^1(M - \{x_1, \dots, x_\ell\}, N)$ but not in $C^1(M - \{x_2, \dots, x_\ell\}, N)$. Then there is a nonconstant harmonic map $\tilde{u} : \mathbf{S}^2 \to N$. Moreover*

$$(2) \qquad E(\tilde{u}) + E(u) \leq \limsup_{\alpha \to 1} E(u_\alpha).$$

Sketch of the proof. We show that by conformal rescaling, there is a sequence $\tilde{u}_\alpha$ of E_α critical maps converging to $\tilde{u}$.

Consider a single isothermal chart near x_1. Let $m \in \mathbf{N}$ be fixed but large enough so $D(x_1, 2^{-m})$ is in the chart. Let

$$b_\alpha = \sup_{x \in D(x_1, 2^{-m})} |du_\alpha(x)| = |du_\alpha(x_\alpha)|$$

be the sup of the gradient near x_1, and $x_\alpha \in D(x_1, 2^{-m})$ the point that realizes the max. We must have b_α unbounded or else, as we remarked after Fact 9, the C^1 convergence can be extended to a δ-disk about x_1. A subsequence, which we also call b_α tends to infinity as $\alpha \to 1$.

Now we conformally reparameterize the sphere. Observe that if $\sigma : \mathbf{S}^2 \to \mathbf{R}^2$ is stereographic projection from any point being the north pole (other than $-x_1$ which is used later), then a translation and dilation of $\mathbf{R}^2$ amounts to a conformal diffeomorphism of the sphere. In other words if $T : \mathbf{R}^2 \to \mathbf{R}^2$ is given by a dilation followed by translation $z \mapsto z_1 + cz$, then $\sigma^{-1} \circ T \circ \sigma : \mathbf{S}^2 \to \mathbf{S}^2$ is a conformal diffeomorphism. Thus we consider the dilated map

$$\tilde{u}_\alpha(z) = u_\alpha\left(x_\alpha + \frac{1}{b_\alpha}z\right) : D\left(x_1, b_\alpha 2^{-m}\right) \to N.$$

Observe that $|d\tilde{u}_\alpha(0)| = 1$ and $|d\tilde{u}_\alpha(z)| \leq 1$ for all $z \in D(x_1, b_\alpha 2^{-m})$. This sequence converges (in C^1 on compacta) to a smooth harmonic function $\tilde{u} : \mathbf{R}^2 \to N$. Furthermore, because the energy can be decomposed as

$$\frac{1}{2}\int_{D(x_1, 2^{-m}b_\alpha)} |d\tilde{u}_\alpha|^2 + \frac{1}{2}\int_{\mathbf{S}^2 - D(x_1, 2^{-m})} |du_\alpha|^2 = E(u_\alpha) \leq B,$$

for a subsequence $\alpha' \to 1$ this converges to

$$E(\tilde{u}) + \frac{1}{2}\int_{\mathbf{S}^2 - D(x_1, 2^{-m})} |du|^2 = \limsup_{\alpha' \to 1} E(u_{\alpha'}).$$

Letting $m \to \infty$ gives (2). But since $E(\tilde{u}) < \infty$ on the plane, then the removeable singularities theorem, Fact 7, implies that the singularity at infinity is removeable and that $\tilde{u} : \mathbf{S}^2 \to N$ is a smooth harmonic

map. Furthermore, since the convergence is in C^1, we must also have $|d\tilde{u}(0)| = 1$, thus $\tilde{u}$ cannot be a map to a point.

If bubbles form we get the desired harmonic map. The remaining possibility is that there are no bubbles so $u_\alpha \to u$ in $C^1(M, N)$. Since we have chosen u_α to be nontrivial, it follows that $E_\alpha(u_\alpha) \geq \delta > 0$ for all $1 \leq \alpha \leq \alpha_0$ by Fact 8. In this case $E(u) = \lim_{\alpha \to 1} E_\alpha(u_\alpha) \geq \delta$, so that u is not constant, completing Sacks and Uhlenbeck's argument.

The statements in [SaU1] and [SaU2] are far sharper. For one, if $\pi_2(N) \neq 0$ then they show that there exists a generating set for $\pi_2(N)$ consisting of immersed branched minimal spheres.

3. Some open problems for minimal spheres.

Finding an extremal mapping $f : M \to N$ has the drawback that the resulting surface may not be an embedding. Pitts [Pi] showed that every smooth manifold N^n ($3 \leq n \leq 6$) has an embedded minimal hypersurface M^{n-1} using geometric measure theory instead of mapping. Schoen and Simon [SS] extended this to $n = 7$ and identified the singularities for $n \geq 8$. However, this method does not say anything about the topology of the minimizer. Meeks and Yau [MY] showed that if a minimal sphere in a compact three manifold minimizes area in its homotopy class, then it must be embedded. Meeks, Simon and Yau [MSY] proved that an area minimizing sequence of surfaces isotopic to a given surface in a three manifold converges as a measure to a sum of embedded minimal spheres. Smith [Sm] proved using a minimax argument that there are embedded minimal $\mathbf{S}^2$'s in $(\mathbf{S}^3, g)$, where g is an arbitrary smooth metric. Further generalizations and extensions are listed in Simon's survey [Si].

The question of existence of embedded minimal spheres in three manifolds has application to questions of three dimensional topology, such as the spherical spaceform problem, which asks if any free action of a finite group on the three-sphere is topologically conjugate to an orthogonal action.

Problem 1. (Pitts-Rubenstein, or Problem 29 of Yau [Y], [CM]) Let N^3 be a closed simply-connected three manifold. Does there exist a bound on the Morse Index of all closed embedded minimal surfaces of fixed genus?

We end by mentioning a couple of problems posed by J. Wolfson [W]. They are area minimization questions for surfaces satisfying an additional pointwise constraint. Let (N, ω) denote a smooth $2n$-dimensional symplectic manifold. The everywhere-given two form ω is closed ($d\omega = 0$) and everywhere nondegenerate ($\omega \wedge \omega \wedge \cdots \wedge \omega = c\, d$ vol where $c \neq 0$). An example is $\mathbf{R}^{2n}$ with $\omega = \sum_{i=1}^n dx_i \wedge dy_i$. A submanifold $f : M \to N$ is called *Lagrangian* if $f^*\omega = 0$. For example, if $\varphi : \mathbf{R}^n \to \mathbf{R}$ then the n-dimensional submanifold given by the graph $x \mapsto (x, \nabla\varphi(x))$ is Lagrangian. An almost complex structure J (like multiplication by $\sqrt{-1}$) is said to be compatible with ω if $\omega(JX, JY) = \omega(X, Y)$ for all vectors X, Y. Given a symplectic form and compatible almost-complex structure, then the compatible metric is defined by $g(X, Y) = \omega(JX, Y)$. Schoen and Wolfson [SW] have proved that for any Lagrangian homotopy class $[\theta] \in \pi_2(N)$ in a smooth Lagrangian 4-manifold N, there is a finite collection of Lagrangian, compatible-metric area minimizing spheres $f_i \in L^{1,2}(\mathbf{S}^2, N)$ such that $\sum_i [f_i] = [\theta]$.

The general problem has application in mirror manifold theory. Strominger, Yau and Zaslow propose to construct the mirror manifold using the moduli space of special Lagrangian real 3-tori of a Calabi-Yau three-fold.

Problem 2. Given a Lagrangian homology class θ in a smooth symplectic manifold, among the cycles representing the class, find a canonical one with minimal compatible volume.

Wolfson [W] mentions a related problem from nonlinear elasticity theory. Let D denote the unit disk in the plane and consider the set of maps $f : D \to D$ with prescribed Dirichlet boundary data and with Jacobean identically equal to one.

Problem 3. Find an energy minimizer in this class of maps.

The existence of the minimizer has been established but nothing is known about its regularity. If d Area is the area form on D, then $\omega = p_1^*(d\,\text{Area}) - p_2^*(d\,\text{Area})$ is a symplectic form on $D \times D$, where p_i are the projections $D \times D \to D$. The condition $\text{Jac}(f) = 1$ is equivalent to $\text{Graph}(f)$ is Lagrangian.

References

[A]   N. Aronszajn, *A unique continuation theorem for solutions of elliptic partial differential equations or inequalities of second order*, Pacific J. Math. **9** (1959), 405–448.

[B]   F. Bethuel, *A characterization of maps in $H^1(B^3, S^2)$ which can be approximated by smooth maps*, Ann. Inst. Poincaré Anal. Non Lin. **7** (1990), 269–286.

[CM]   T. Colding & W. Minicozzi, *Minimal Surfaces*, Lecture Notes **4**, Courant Inst. Math. Sci., 1999.

[ES]   J. Eells & J. Sampson, *Harmonic mappings of Riemannian manifolds*, Amer. J. Math. **86** (1964).

[G]   M. Gromov, *Partial Differential Relatons*, Ergebnisse der Math. Bd. 3, **9**, Springer Verlag, Berlin, 1986, p. 223.

[GOR]   R. Gulliver, R. Osserman & H. Royden, *A theory of branched immersions of surfaces*, Amer. J. Math. **95** (1973), 750–812.

[HW]   P. Hartman and A. Wintner, *On the local behavior of solutions of nonparabolic equations*, Amer. J. Math. **75** (1953), 449–476.

[KRST] N. Korevaar, J. Ratzkin, N. Smale & A. Treibergs, *A Connoiseur's Guide to Constant Mean Curvature Surfaces in $\mathbb{R}^3$*, VIGRE Minicourse on Variational Methods and Nonlinear PDE, University of Utah, Lecture Notes (2002), http://www.math.utah.edu/~rudd/minicourse/ratzkin.pdf.

[Le]   J. Lee, *Riemannian Manifolds: An Introduction to Curvature*, Graduate Texts in Mathematics, **176**, Springer, New York, 1991.

[L]   L. Lemaire, *Applications harmoniques des surfaces Riemanniennes*, Jour. Differential Geometry **13** (1978), 51–87.

[MSY]   W. Meeks III, L. Simon & S. T. Yau, *Embedded minimal surfaces, exotic spheres, and manifolds with positive Ricci curvature*, Annals Math. **116** (1982), 621–659.

[MY]   W. Meeks III & S. T. Yau, *Topology of three manifolds and the embedding problems in minimal surface theory*, Annals Math. **112** (1980), 441–485.

[M]   C. Morrey, *Multiple Integrals in the Calculus of Variations*, Grundlehren der mathematischen Wissenschaften, **130**, Springer Verlag, New York, 1966.

[P]   R. Palais, *Foundations of Global Nonlinear Analysis*, Benjamin, New York, 1968.

[PW]   T. Parker & J. Wolfson, *Pseudo-Holomorphic maps and Bubble Trees*, J. Geom. Analysis **3** (1993), 63–98.

[Pi]   J. Pitts, *Existence and Regularity of Minimal Surfaces on Riemannian Manifolds*, Mathenmatical Notes **27**, Princeton U. Press, Princeton, 1981.

[SaU1] J. Sacks & K. Uhlenbeck, *The existence of minimal immersions of 2-spheres*, Annals of Math. **113** (1981), 1–24.

[SaU2] J. Sacks & K. Uhlenbeck, *Minimal immersions of closed Riemann Surfaces*, Trans. Amer. Math. Soc. **113** (1982), 639–652.

[SS]   R. Schoen & L. Simon, *Regularity of stable minimal hypersurfaces*, Commun. Pure Appl. Math. **34** (1981), 741–797.

[SU1]   R. Schoen and K. Uhlenbeck, *A regularity theory for harmonic maps*, J. Diff. Geom. **17** (1982), 307–335.

[SU2]   R. Schoen and K. Uhlenbeck, *Boundary regularity and the Dirichlet problem for harmonic maps*, J. Diff. Geom. **18** (1983), 253–268.

[SW]   R. Schoen & J. Wolfson, *Minimizing area among Lagrangian surfaces: The mapping problem*, J. Diff. Geom. **58** (2001), 1–86.

[SY]   R. Schoen & S.-T. Yau, *Lectures on Harmonic Maps*, Conference Proceedings and Lecture Notes in Geometry and Topology, **2**, International Press Inc., Boston, 1997.

[Si]   L. Simon, *Survey lectures on minimal submanifolds*, Seminar on Minimal Submanifolds, (E. Bombieri, ed.), Princeton U. Press, Princeton, 1983, pp. 3–52.

[Sm]   F. Smith, *On the Existence of Embedded Minimal 2-Spheres in the 3-Sphere, Endowed with an Arbitrary Metric*, thesis, Univesity of Melbourne, Melbourne, 1986.

[T]   A. Treibergs, *Some variational problems in geometry I*, VIGRE Minicourse on Nonconvex Variational Problems & Applications, University of Utah, Lecture Notes (2005), http://www.math.utah.edu/vigre/minicourses/nvpa/ATmain.pdf

[W]   J. Wolfson, *Lagrangian cycles and volume*, Conformal, Riemannian and Lagrangian Geometry: The 2000 Barrett Lectures, University Lecture Series **27**, (A. Freire, ed.), Amer. Math. Soc., Providence, 2002.

[Y]   S. T. Yau, *Open problems in geometry*, Proc. Symp. Pure Math., vol. 54.I, Amer. Math. Soc., Providence, 1993.

UNIVERSITY OF UTAH,
DEPARTMENT OF MATHEMATICS
155 SOUTH 1400 EAST, RM 233
SALT LAKE CITY, UTAH 84112-0090
*E-mail address:*treiberg@math.utah.edu