

An example

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1. Harmonic functions

Let $G = \mathrm{SL}_2(\mathbb{R})$.

The group G acts on \mathcal{H} by Möbius transformations and commutes with the non-Euclidean Laplacian

$$\Delta_{\mathcal{H}} = y^2 \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \right)$$

The space of functions f defined everywhere on \mathcal{H} satisfying

$$\Delta_{\mathcal{H}} f = \lambda f$$

(eigenfunctions of $\Delta_{\mathcal{H}}$) will therefore be stable under G . Such functions are necessarily real analytic in \mathbb{D} .

This illustrates one of the two ways in which representations of G arise naturally. The other is illustrated by the boundary values of these eigenfunctions on $\mathbb{P}^1(\mathbb{R})$.

If we write this Laplacian in non-Euclidean radial coordinates it becomes

$$\frac{\partial^2}{\partial r^2} + \frac{1}{\tanh r} \frac{\partial}{\partial r} + \frac{1}{\sinh^2 r} \frac{\partial^2}{\partial \theta^2}$$

In small regions Euclidean and non-Euclidean space are asymptotically alike, so it is reassuring to compare this operator with the Euclidean Laplacian in radial coordinates

$$\frac{\partial^2}{\partial r^2} + \frac{1}{r} \frac{\partial}{\partial r} + \frac{1}{r^2} \frac{\partial^2}{\partial \theta^2}$$

which looks the same around $r = 0$ since $\tanh r \sim \sinh r \sim r$ for small r .

The Laplacian commutes with rotations. If f is any solution of $\Delta_{\mathcal{H}} f = \lambda f$ we can write

$$f = \sum e^{in\theta} f_n(r)$$

where each term is also an eigenfunction. We may first look for solutions of the form $e^{in\theta} f(r)$ —in effect separating variables in polar coordinates. Since $\Delta_{\mathcal{H}}$ in radial coordinates is

$$\frac{\partial^2}{\partial r^2} + \frac{1}{\tanh r} \frac{\partial}{\partial r} + \frac{1}{\sinh^2 r} \frac{\partial^2}{\partial \theta^2}$$

we get the equation

$$f''(r) + \frac{1}{\tanh r} f'(r) - \frac{n^2}{\sinh^2 r} f(r) = \lambda f(r)$$

Like the Euclidean one, this has an **irregular singularity** at ∞ , like for example the equation $f'' - f = 0$ with exponentially increasing and decreasing solutions $e^{\pm r}$.

On \mathcal{H} the non-Euclidean distance r from i (our non-Euclidean origin) to iy with $y \geq 1$ is

$$\int_1^y \frac{ds}{s} = \log y$$

so that $y = e^r$. If we make a change of variables $y = e^r$ in our differential equation we get

$$y^2 \frac{d^2 f}{dy^2} + \frac{2y^3}{y^2 - 1} \frac{df}{dy} - \frac{4n^2 y^2}{(y^2 - 1)^2} f = \lambda f$$

which has a **regular singularity** at $y = 1$ (and other places).

This equation and other similar equations will be examined in much detail later, but for now I'll look only at the case $\lambda = 0$.

So now we want to solve

$$y^2 \frac{d^2 f}{dy^2} + \frac{2y^3}{x^2 - 1} \frac{df}{dy} - \frac{4n^2 y^2}{(y^2 - 1)^2} f = 0$$

This has a regular singularity at $y = 1$, and the general theory of such equations tells us that there is a basis of solutions

$$(y - 1)^n (a_0 + a_1(y - 1) + \cdots),$$
$$(y - 1)^{-n} (a_0 + b_1(y - 1) + \cdots)$$

if $n \neq 0$ and a basis

$$a_0 + a_1(y - 1) + \cdots,$$
$$(b_0 + b_1(y - 1) + \cdots) \log |y - 1|$$

if $n = 0$.

We are interested for the moment only in well-behaved solutions, so we discard half of these. What remains are solutions whose Taylor series start out

$$(y - 1)^{|n|} (c_0 + c_1(y - 1) + \dots)$$

We could have predicted this!

We may as well replace \mathcal{H} by \mathbb{D} . This makes rotational symmetry manifest.

Let's change from \mathcal{H} to \mathbb{D} . The point iy goes to $t = (y - 1)/(y + 1)$ on the x -axis. The Laplacian becomes

$$(1 - r^2)^2 \Delta$$

so we are looking for functions harmonic in the usual sense.

The equation $\Delta f = 0$ in the unit disk—indeed, even in the whole complex plane—has as solutions the constants together with all functions z^n and \bar{z}^n for $n > 0$.

What we have proved, together with the theory of Fourier series, is that these are all we get:

Theorem. *Every solution of $\Delta = 0$ which is finite in a neighbourhood of 0 is a sum (possibly infinite) of the form*

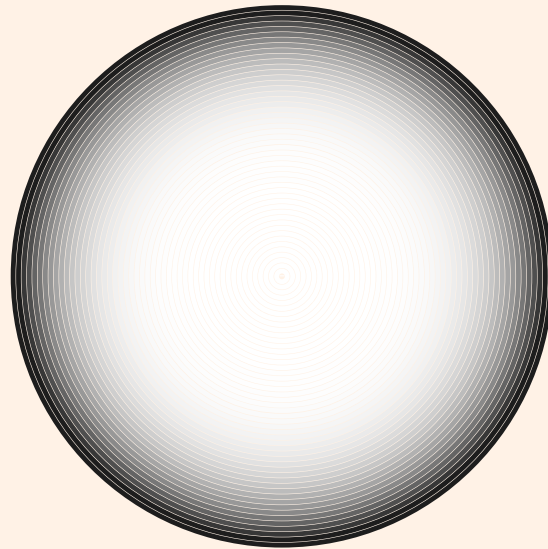
$$f(z) = c_0 + \sum_{n>0} c_n z^n + \sum_{n<0} c_n \bar{z}^{|n|} .$$

This is not quite obvious. Rotationally symmetric eigenfunctions whose eigenvalue is not 0 will usually, as we'll see later on, be infinite sums $\sum_0^\infty c_k r^{2k}$ instead of the simple constants.

2. Group and Lie algebra

We demand of our functions that they be well defined throughout \mathbb{D} , the entire unit disk, not just around 0, since G acts transitively on \mathbb{D} . This imposes growth conditions on terms in the infinite series $c_0 + \sum_{n>0} c_n z^n + \sum_{n<0} c_n \bar{z}^{|n|}$.

But this raises some interesting questions—there are many different conditions we can impose on the functions concerning their behaviour near the boundary of \mathbb{D} , the unit circle $|z| = 1$, and still retain stability under G .



Various G -stable conditions that can be imposed on spaces of harmonic functions (increasing in strictness):

- they must be real analytic throughout \mathbb{D} ;
- they must be of moderate growth near the unit circle.
- they must have smooth boundary values on the unit circle;

series

- they must be real analytic in an open neighbourhood of the closed unit disk;

All of these conditions are of some importance, but at the same time all these representations are in some sense the same—they all contain the polynomials in z and \bar{z} .

All these representations of G contain exactly the same eigenfunctions of K .

The space spanned by the polynomials in z and \bar{z} is not stable under G , but

- it is stable under K ;
- it is also stable under the Lie algebra \mathfrak{g} of G .

What this last means in concrete terms is that it is stable under the differential operators associated to the vector fields derived from \mathfrak{g} —that is to say, generated by one-parameter flows on G :

$$A: \alpha = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$

$$N: \nu_+ = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$$

$$K: \kappa = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$$

These vector fields these give rise to are

$$\alpha: 1 - z^2$$

$$\nu_+: (1/2i)(z - 1)^2$$

$$\kappa: -2iz$$

where the holomorphic function $f(z) = p + iq$ identified as a real vector field becomes

$$\begin{aligned} p \frac{\partial}{\partial x} + q \frac{\partial}{\partial y} &= (p + iq) \left(\frac{\partial}{\partial x} - i \frac{\partial}{\partial y} \right) \\ &\quad + (p - iq) \left(\frac{\partial}{\partial x} + i \frac{\partial}{\partial y} \right) \\ &= f(z) \frac{\partial}{\partial z} + \overline{f(z)} \frac{\partial}{\partial \bar{z}} \end{aligned}$$

Certain complex vector fields are useful because they raise and lower K -weights. Recall that

$$x_{\pm} = \alpha \mp i(\kappa + 2\nu_+).$$

Then

$$\kappa + 2\nu_+ = -i(1 + z^2)$$

$$\rightsquigarrow 2xy \frac{\partial}{\partial x} - (1 + x^2 - y^2) \frac{\partial}{\partial y}$$

$$i(\kappa + \nu_+) \rightsquigarrow 2xy i \frac{\partial}{\partial x} - (1 + x^2 - y^2) i \frac{\partial}{\partial y}$$

$$\alpha \rightsquigarrow (1 - x^2 + y^2) \frac{\partial}{\partial x} - 2xy \frac{\partial}{\partial y}$$

$$x_+ = \alpha - i(\kappa + 2\nu_+) \rightsquigarrow -z^2 \frac{\partial}{\partial z} + \frac{\partial}{\partial \bar{z}}$$

$$x_- = \alpha + i(\kappa + 2\nu_+) \rightsquigarrow \frac{\partial}{\partial z} - \bar{z}^2 \frac{\partial}{\partial \bar{z}}$$

This leads to the idea of considering certain representations of the pair (\mathfrak{g}, K) that satisfy various modest compatibility conditions—instead of representations of G itself.

The advantage of this reduction is simplification. **In any smooth representation of G the elements of the Lie algebra act as linear operators.** Lie algebras are very tightly structured, so the representation of the Lie algebra is often also structured. If

$$\varphi_n = \begin{cases} 1 & \text{if } n = 0 \\ z^n & \text{if } n > 0 \\ \bar{z}^n & \text{if } n < 0 \end{cases}$$

the messy formulas on the previous page boil down to

$$\kappa \varphi_n = -2ni \varphi_n$$

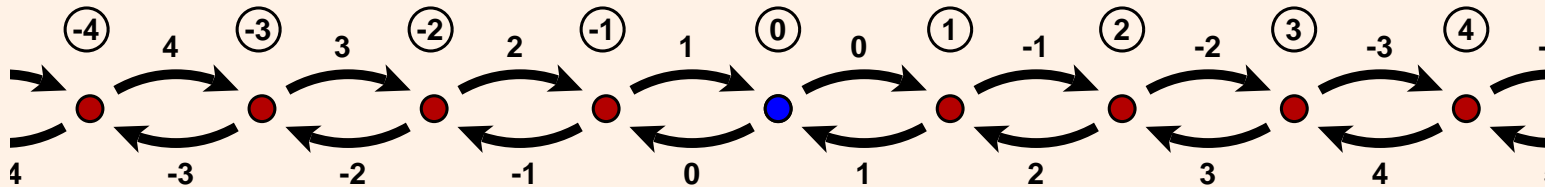
$$x_+ \varphi_n = -n \varphi_{n+1}$$

$$x_- \varphi_n = n \varphi_{n-1}$$

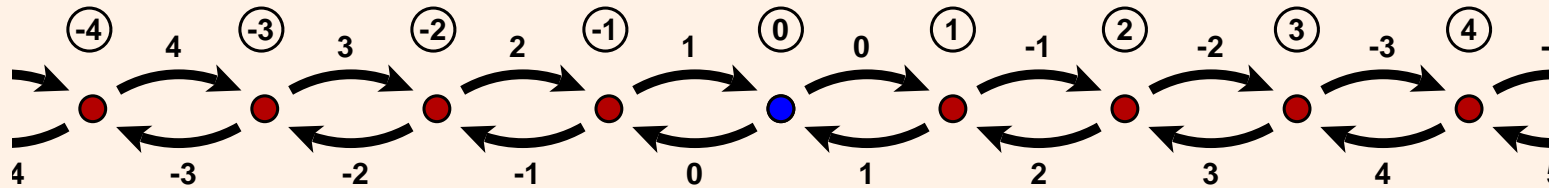
$$\kappa \varphi_n = -2ni \varphi_n$$

$$x_+ \varphi_n = -n \varphi_{n+1}$$

$$x_- \varphi_n = n \varphi_{n-1}$$



In some sense we understand this space quite well. We can see that the constants form a subspace stable under the Lie algebra, and of course the group as well. No surprise.



But we can also see that the quotient by the constants is the direct sum of two components, holomorphic and anti-holomorphic. And that each of these is irreducible. Each of these corresponds to a G -stable component, too.

We can summarize the situation in a long exact sequence (where H is the space of harmonic polynomials):

$$0 \longrightarrow \mathbb{C} \longrightarrow H \longrightarrow D_+ \oplus D_- \longrightarrow 0$$

There are analogous sequences for every irreducible finite-dimensional representation of $SL_2(\mathbb{R})$. The representations D_{\pm} occurring here turn out to be unitary, and strongly so.

3. Restriction to P

There are almost always two important ways to look at a representation (of G or its Lie algebra). One is to describe its restriction to a maximal compact subgroup, in this case $K = \mathrm{SO}_2$. The other is to describe its restriction to a minimal parabolic subgroup, in this case P .

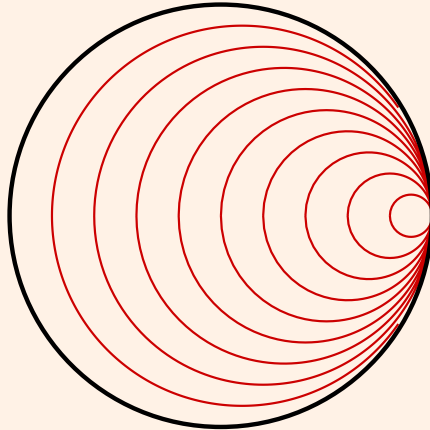
We know what the restriction of our representation to K is—the direct sum of characters ε^{2n} , where

$$\varepsilon: \begin{bmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{bmatrix} \longmapsto e^{i\theta}$$

What can we say about its restriction to P ?

Why is this an interesting question?

I am not going to describe completely its restriction to P , but as I did with K , I am going to look at it in the neighbourhood of the point fixed by P . For \mathcal{H} this is ∞ , but for \mathbb{D} it is 1.



I am going to look at the G -space of functions that take smooth boundary values on the unit circle. Such functions have a Taylor series at 1 of the form

$$c_0 + \sum_{n>0} c_n (z - 1)^n + \sum_{n<0} c_n (\bar{z} - 1)^{|n|}$$

How does the Lie algebra act on this space of Taylor series?

$$\begin{aligned}
\kappa &= -2iz \\
&= -2i(1 + (z - 1)) \\
\alpha &= (1 - z^2) \\
&= -(z - 1)(2 + (z - 1)) \\
\nu_+ &= (1/2i)(z - 1)^2
\end{aligned}$$

leading to (for example)

$$\begin{aligned}
\kappa : (z - 1)^m &\longmapsto -2im(z - 1)^{m-1}(1 + (z - 1)) \\
\alpha : (z - 1)^m &\longmapsto -m(z - 1)^m(2 + (z - 1)) \\
\nu_+ : (z - 1)^m &\longmapsto (m/2i)(z - 1)^{m-1}
\end{aligned}$$

The operators α and ν_+ take the ideal \mathfrak{m} of functions vanishing at 1 to itself, since P fixes 1. They also take each \mathfrak{m}^n to itself. The operator ν_+ even takes \mathfrak{m}^n to \mathfrak{m}^{n+1} .

This is closely related to certain simple representations of the Lie algebra known as **Verma modules**. They are important in representation theory of the group because, as is the case here, they are related to the behaviour of certain functions at ∞ . Also to understanding the characters of a representation.

4. Boundary values

I have now at hand five spaces of interest—one a representation of the Lie algebra alone, the rest of the group itself (order of decreasing strictness):

- harmonic polynomials;
- functions real analytic in an open neighbourhood of the closed unit disk;
- functions possessing smooth boundary values on the unit circle;
- functions of moderate growth near the unit circle.
- functions real analytic throughout \mathbb{D} ;

These are all characterized by their boundary values:

- **trigonometric polynomials;**
- **functions real analytic on the unit circle;**
- **smooth functions on the unit circle;**
- **distributions on the unit circle;**
- **hyperfunctions.**

Again, all of these are in some way equivalent, although in other ways quite different. Furthermore, in this case the map from a function \mathbb{D} to its boundary value is an isomorphism with a simple space of analytical objects on \mathbb{S}^1 . The Cauchy integral and its anti-holomorphic analogue, or equivalently the Poisson kernel, give the map going back.

Let's look at just one of these spaces, that of harmonic functions with smooth boundary values. The boundary values are all of $C^\infty(P \setminus G)$, and behaviour of $f(z)$ as z approaches a point on $|z| = 1$ is **asymptotic**, described by a non-convergent Taylor series.

In general, representations of G (or, rather, of (\mathfrak{g}, K)) will have two different kinds of incarnations. One is by means real analytic functions on G that are eigenfunctions of certain differential operators like Δ —its **matrix coefficients**.

The other is as smooth sections of real analytic vector bundles on flag manifolds like $\mathbb{P}^1(\mathbb{R})$. These sections are boundary values of functions in the first type. The exact description is closely related to the structure of the representation when restricted to a parabolic subgroup such as P .

5. Harmonic forms

If $f(z) = c_0 + \sum_{n>0} c_n z^n + \sum_{n<0} c_n \bar{z}^{|n|}$ is a harmonic function on \mathbb{D} then

$$df = \frac{\partial f}{\partial z} dz + \frac{\partial f}{\partial \bar{z}} d\bar{z}$$

In this section I shall look at the holomorphic components of such forms:

$$\omega = \left(\sum c_n z^n \right) dz = f(z) dz$$

where $f(z)$ is holomorphic. The group acts on these (right action) via the transpose, taking $f(z) dz$ to

$$\frac{f(g(z)) dz}{J(g, z)^2}$$

If ω is a holomorphic form then $\omega \wedge \bar{\omega}$ is a 2-form on \mathbb{D} and the metric $\omega \wedge \bar{\omega}$ is both positive definite and G -invariant.

The representations D_{\pm} possess positive definite G -invariant inner products, and occur as discrete summands in the Hilbert space of all square-integrable 1-forms on \mathbb{D} .

$SL_2(\mathbb{R})$

The End