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Lecture 1

Passive Scalar in Turbulent Velocity Field

$$\frac{\partial\theta(t,x)}{\partial t} = -\mathbf{v}(t,x)\cdot\nabla\theta(t,x),\ t>0,\ x\in\mathbb{R}^d;$$
$$\theta(0,x) = \theta_0(x);$$

$\mathbf{v} = (v^1,\dots,v^d) \in \mathbb{R}^d$, $d \geq 2$.

If each v^i is Lipschitz continuous in x, then

$$\theta(t,x) = \theta_0(X^x_{t,0});$$

X is the flow of $\mathbf{v}$:

$$\frac{dX^x_{s,t}}{dt} = \mathbf{v}(t,X^x_{s,t}),\ t>s,\ X^x_{s,s} = x.$$

Turbulent Transport

What if $\mathbf{v}$ is not Lipschitz continuous in x?

- Example — Kolmogorov's theory: $\mathbf{v}$ is Hölder $\approx 1/3$.
- Difficulty — Existence but no uniqueness for the flow equation.

$$\frac{dX_{s,t}^x}{dt} = \mathbf{v}(t, X_{s,t}^x),\ t > s,\ X_{s,s}^x = x.$$

- How to find θ?

$$\theta(t,x) = \theta_0(X_{t,0}^x) \text{ is not true}$$

Regularization

- Introducing viscosity (κ-limit):

$$\frac{\partial \theta^\kappa(t,x)}{\partial t} = \kappa \Delta \theta^\kappa(t,x) - \mathbf{v}(t,x) \cdot \nabla \theta^\kappa(t,x),\ t > 0,\ x \in \mathbb{R}^d;$$

$$dX_{s,t}^{\kappa,x} = \mathbf{v}(t, X_{s,t}^{\kappa,x})dt + \sqrt{2\kappa}dw(t),\ t > s,\ X_{s,s}^{\kappa,x} = x.$$

- Smoothing out $\mathbf{v}$ (ε-limit):

$$\mathbf{v}^\varepsilon(t,x) = \frac{1}{\varepsilon^d} \int_{\mathbb{R}^d} \mathbf{v}(t,y) \psi \left(\frac{x-y}{\varepsilon} \right) dy$$

(Gawędzki and Vergassola (2000), E and Vanden Eijnden (2000))

Kraichnan's Model of Turbulence

Physical Model for v:

• $\mathbf{v}$ is a statistically homogeneous, isotropic, and stationary Gaussian vector field with zero mean and covariance

$$E(v^i(t,x)v^j(s,y)) = \delta(t-s)C^{ij}(x-y).$$

• For small x, $C^{ij}(x) \sim C^{ij}(0)(1-|x|^\gamma)$,
$0 < \gamma < 2$.

Mathematical Model for v (Le Jan and Raimond (2002), Baxendale, Harris (1988)):

• The matrix C is characterized by its Fourier transform:

$$\widehat{\mathbf{C}}(z) = \frac{A_0}{(1+|z|^2)^{(d+\gamma)/2}} \left(a\frac{zz^*}{|z|^2} + \frac{b}{d-1}\left(I - \frac{zz^*}{|z|^2}\right)\right),$$

• $a = 0 \Rightarrow \nabla \cdot \mathbf{v} = 0$;
• $b = 0 \Rightarrow \mathbf{v} = \nabla V$ for some scalar V.
• $\zeta = b/(a+b)$— degree of incompressibility.

- Representation of $\mathbf{v}$:

$$v^i(t,x) = \sum_{k\geq 1} \sigma_k^i(x)\dot{w}_k(t),$$

$\dot{w}_k(t)$ are independent standard Gaussian white noises; $\{\sigma_k,\ k \geq 1\}$ is a CONS in H_C.

- $H_C = H^{(d+\gamma)/2}(\mathbb{R}^d;\mathbb{R}^d)$, $a, b > 0$;
- σ_k^i is Hölder $\gamma/2$.

$$C^{ij}\left(x-y\right) = \sum_k \sigma_k^i(x)\sigma_k^j(y),$$

Thm *(Le Jan, Raimond, 2002)* For a suitable class of initial conditions $\theta_0, \theta\left(t,x\right)$

$$\theta(t,x) = \int \theta_0(y) P\left(X_{0,t}^x \in dy | \mathcal{F}_t^W\right) \tag{1}$$

where

$$X_{t,x}\left(s\right) = x + \int_s^t \sigma^k\left(X_{t,x}\left(r\right)\right) \circ \overleftarrow{dw_k}\left(r\right).$$

Le Jan and Raimond have also derived an equation for the measure $P\left(X_{0,t}^x \in dy|\mathcal{F}_t^W\right)$, similar to the Zakai equation of nonlinear filtering.

- *Statistical ("weak" in probabilistic sense) solution of the flow equation.*

- *Still very little info about θ (Transport equation is solved in the space of measures, no uniqueness was established)*

Transport Equation as an SPDE

- $\mathbb{F} = (\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq 0}, \mathbb{P})$, a stochastic basis with the usual assumptions.
- $(w_k(t), k \geq 1, t \geq 0)$, independent standard Wiener processes on $\mathbb{F}$.
- $\mathbf{v}$ divergence-free (incompressible flow$\Longrightarrow$ $div \sigma_k = 0$).

Since the Kraichnan velocity $\mathbf{V}$:

$$V^i(t,x) = \sum_{k\geq 1} \sigma_k^i(x) \dot{w}_k(t),$$

the transport equation is given by

$$d\theta(t,x) = -\sum_k \sigma_k(x) \cdot \nabla\theta(t,x) \circ dw_k(t).$$

or

$$\boxed{d\theta(t,x) = \tfrac{1}{2} C^{ij}(0) D_i D_j \theta(t,x) dt - \sigma_k^i(x) D_i \theta(t,x) dw_k(t)}$$

Notation: $D_i = \frac{\partial}{\partial x^i}$.

Summation convention: summation over a pair of repeating indices.

Stochastic PDEs with Multiplicative Noise;
Basic facts

Consider a stochastic evolution equation

$$du(t,x) = \mathcal{A}\,(t,x)\,u(t,x)dt + \sum_{k=1}^{\infty} \mathcal{M}^k\,(t,x)\,u(t,x)dw_k(t), u\,(0,x) = u_0\,(x)$$

where $\mathcal{A}$ and $\mathcal{M}$ are differential operators, and w_k are independent standard Wiener processes
.

$$(\mathbf{H}) : \mathcal{A} - \frac{1}{2}\mathcal{M}\mathcal{M}^* \text{ is elliptic}$$

If (H) does not hold, one coulld not guarntee that a solution of this equation is square integrable, i.e. $E\,\|u\,(t,\cdot)\|^2_{\mathbb{R}^d} < \infty$ *for all* t*;*

Examples:

1. $\mathcal{A} = \frac{1}{2}\Delta$, $\mathcal{M} = \varepsilon\nabla$, $\varepsilon < 1$−elliptic;
2. $\mathcal{A} = \frac{1}{2}\Delta$, $\mathcal{M} = \nabla$ −degenerate elliptic;
3. $\mathcal{A} = \frac{1}{2}\Delta$, $\mathcal{M} = \varepsilon\nabla$, $\varepsilon > 1$−non-elliptic;

The transport equation

$$d\theta(t,x) = \frac{1}{2}C^{ij}(0)D_iD_j\theta(t,x)dt - \sigma_k^i(x)D_i\theta(t,x)dw_k(t)$$

is degenerate elliptic!

(Krylov-R., P. Chow, J. Potthoff, B. Øksendal, etc. σ-smooth)

Wiener chaos:

$W(t) = (w_k(t),\ k \geq 1,\ 0 < t < T)$,

$\{m_i(s),\ i \geq 1\}$ — CONS in $L_2([0,T])$,

$\xi_i^k = \int_0^T m_i(s) dw_k(s)$.

$\mathcal{J} = \left(\alpha = (\alpha_i^k,\ i,k \geq 1) \ \middle|\ |\alpha| = \sum_{i,k} \alpha_i^k < \infty\right)$;

$\xi_\alpha = \prod_{i,k} \left(\frac{H_{\alpha_i^k}(\xi_{ik})}{\sqrt{\alpha_i^k!}} \right)$, where

$H_n(x) = (-1)^n \exp\left\{\frac{x^2}{2}\right\} \frac{d^n}{dx^n} \exp\left\{-\frac{x^2}{2}\right\}$.

Theorem. *(Cameron and Martin, 1947)*
The collection $\{\xi_\alpha,\ \alpha \in \mathcal{J}\}$ is an orthonormal basis in $L_2(\Omega, \mathcal{F}_T^W, \mathbb{P})$:
If $\eta \in L_2(\Omega, \mathcal{F}_T^W, \mathbb{P})$ and $\eta_\alpha = \mathbb{E}(\eta \xi_\alpha)$, then

$$\eta = \sum_{\alpha \in \mathcal{J}} \eta_\alpha \xi_\alpha$$

and

$$\mathbb{E}|\eta|^2 = \sum_{\alpha \in \mathcal{J}} \eta_\alpha^2.$$

The Propagator System

$d\theta(t,x) = \frac{1}{2}C^{ij}(0)D_iD_j\theta(t,x)dt - \sigma^i_k(x)D_i\theta(t,x)dw_k(t)$

If σ^i_k, θ_0 are smooth, then the STE has a nice square integrable solution, moreover $\theta(t,x) = \sum_{\alpha\in\mathcal{J}}\theta_\alpha(t,x)\xi_\alpha(t)$.

Define: $\xi_\alpha(t) = \mathbb{E}(\xi_\alpha|\mathcal{F}^W_t)$; $\xi_\alpha(0) = I(|\alpha| = 0)$.

Fact:

$$d\xi_\alpha(t) = \mathcal{D}\xi_\alpha(t)dW(t),$$

where

$\mathcal{D}\xi_\alpha(t) = m_i(t)\sqrt{\alpha^k_i}\xi_{\alpha^-(i,k)}(t)\ell_k$ is the Maliavin derivative, and $\alpha^-(i,k)$ is the multi-index with the components

$$(\alpha^-(i,k))^l_j = \begin{cases} \max(\alpha^k_i - 1, 0), & \text{if } i = j \text{ and } k = l, \\ \alpha^l_j, & \text{otherwise.} \end{cases}$$

By the Itô formula

$$\begin{aligned} \frac{\partial\theta_\alpha(t,x)}{\partial t} &= \frac{1}{2}C^{ij}(0)D_iD_j\theta_\alpha(t,x) \\ &- \sum_{i,k}\sqrt{\alpha^k_i}\sigma^j_k(x)D_j\theta_{\alpha^-(i,k)}(t,x)m_i(t); \\ \theta_\alpha(0,x) &= \theta_0(x)I(|\alpha| = 0) \end{aligned}$$

Solving the Pr opagator

Good news: σ_k^i do not have to be smooth or even continuous.
$|\alpha| = 0$:

$$\frac{\partial \theta_{(0)}(t,x)}{\partial t} = \frac{1}{2} C^{ij}(0) D_i D_j \theta_{(0)}(t,x)$$

$\theta_{(0)}(0,x) = \theta_0(x) \;\Rightarrow\; \theta_{(0)}(t,x) = \mathbb{T}_t \theta_0(x)$.

$\alpha = \delta_{ik}$:

$$\begin{aligned}\frac{\partial \theta_{(ik)}(t,x)}{\partial t} &= \frac{1}{2} C^{ij}(0) D_i D_j \theta_{(ik)}(t,x) \\ &- \sigma_k^j(x) D_j \theta_{(0)}(t,x) m_i(t);\; \theta_{(ik)}(0,x) = 0.\end{aligned}$$

$\theta_{ik}(t,x) = -\int_0^t m_i(s) \mathbb{T}_{t-s} \sigma_k^j D_j \mathbb{T}_s \theta_0(x) ds$.

In fact, with $\mathcal{M}_k = -\sigma_k^j D_j$,

$$\sum_{|\alpha|=N} |\theta_\alpha(t,x)|^2 = \sum_{k_1,\dots,k_N=1}^{\infty} \int_0^t \int_0^{s_N} \dots \int_0^{s_2} |\mathbb{T}_{t-s_N} \mathcal{M}_{k_N} \dots \mathbb{T}_{s_2-s_1} \mathcal{M}_{k_1} \mathbb{T}_{s_1} \theta_0(x)|^2 ds_1 \dots ds_N.$$

$\exists!$in L_2

$$d\theta(t,x) = \frac{1}{2}C^{ij}(0)D_iD_j\theta(t,x)dt - \sigma_k^i(x)D_i\theta(t,x)dw_k(t)$$

Thm (Lototsky and R., Russian Math. Surveys, 2003)
If $\theta_0 \in L_2(\mathbb{R}^d)$, then:

- For every $\varphi \in \mathbf{C}_0^\infty(\mathbb{R}^d)$, the random field $\theta(t,x) = \sum_{\alpha\in\mathcal{J}} \theta_\alpha(t,x)\xi_\alpha$ is a unique strong solution of the transport equation in that for any test-function φ,

$$\begin{aligned}(\theta,\varphi)(t) = (\theta_0,\varphi) + \frac{1}{2}\int_0^t C^{ij}(0)(\theta, D_iD_j\varphi)(s)ds \\ + \int_0^t (\theta, \sigma_k^i D_i\varphi)dw_k(s)\end{aligned}$$

- For $t > 0$,

$$\|\theta\,(t)\,\|^2_{L_2(\mathbb{R}^d)} = \sum_{\alpha\in\mathcal{J}} \|\theta_\alpha(t)\|^2_{L_2(\mathbb{R}^d)} < \|\theta_0\|^2_{L_2(\mathbb{R}^d)}.$$

- For

$$\begin{aligned}d_sX_t^{x,i}\,(s) &= -\sigma_k^i\,(X_t^x\,(s))\,\overleftarrow{dw_k}\,(s)\,, s\in[0,t),\\ X_t^x\,(t) &= x\end{aligned}$$

(martingale solution), and

$$\theta\,(t,x) = \mathbb{E}\left(\theta_0\,(X_t^x\,(0))\,|\mathcal{F}_t^W\right)$$

Re marks

- $\mathbb{E}\theta(t,x) = \theta_{\emptyset}(t,x)$. ($\emptyset$ is a multi-index with zero entries)

- $\mathbb{E}(\theta(t,x)\theta(s,y)) = \sum_{\alpha\in\mathcal{J}} \theta_\alpha(t,x)\theta_\alpha(s,y)$.

- By interpolation: $\mathbb{E}\|\theta\|^p_{L_p(\mathbb{R}^d)} < \|\theta_0\|^p_{L_p(\mathbb{R}^d)}$, $2 < p < \infty$. Weighted L_p (e.g. $\theta_0(x) = |x|$) are also OK.

- Conservation of energy, $\mathbb{E}\|\theta\,(t)\,\|^2_{L_2(\mathbb{R}^d)} = \|\theta_0\|^2_{L_2(\mathbb{R}^d)}$,

$\Updownarrow$

- Pathwise solution of the flow equation.

$d\theta(t,x) = \nu\Delta\theta(t,x)dt - \sigma_k^i(x)D_i\theta(t,x)dw_k(t),$

$\sigma_k,\ k \geq 1$ — CONS in $L_2(\mathbb{R}^d;\mathbb{R}^d) \Leftrightarrow \dot{W}$ is space-time white noise

Note: $\sum_{k\geq 1}\sigma_k^i(x)\sigma_k^j(x)$ diverges.

S-system:

$$\begin{aligned}\frac{\partial\theta_\alpha(t,x)}{\partial t} &= \nu\Delta\theta_\alpha(t,x)\\ &\quad - \sum_{i,k}\sqrt{\alpha_i^k}\sigma_k^j(x)D_j\theta_{\alpha^-(i,k)}(t,x)m_i(t);\end{aligned}$$

Still solvable, but now

$$\sum_{\alpha\in\mathcal{J}}\|\theta_\alpha\|^2_{L_2(\mathbb{R}^d)} = \infty$$

Weighted Wiener Chaos

Let $Q := \{q_1, q_2, ...\}$, $q_k > 0$, and $q^\alpha := \Pi_{\iota,k} q_k^{\alpha_i^k}$.

Definition. *The Q-weighted Wiener Chaos space* $L_{2,Q}(\mathcal{F}_T^W; L_2(\mathbb{R}^d))$ *is*

$$L_{2,Q}(\mathcal{F}_T^W; L_2(\mathbb{R}^d)) = \left\{ (u_\alpha) : \sum_{\alpha \in \mathcal{J}} q^{-2\alpha} \|u_\alpha\|^2_{L_2(\mathbb{R}^d)} < \infty \right\}.$$

Still write $u = \sum_{\alpha \in \mathcal{J}} u_\alpha \xi_\alpha$

Where this series converges?

Examples:

1. (Obvious) If $u(t) = 1 + \sum_{k \geq 1} \int_0^t u(s) dw_k(s)$, then $u \in L_{2,Q}(\mathcal{F}_T^W; \mathbb{R})$ for every $Q = (q_1, q_2, \ldots)$ so that $\sum_{k \geq 1} q_k^2 < \infty$.

2. (Nualart-R., JFA, 1997) If

$$du(t,x) = \Delta u(t,x)dt + u(t,x)dw(t,x), \qquad t > 0,\ x \in \mathbb{R}^d,\ \ d \geq 2,$$

then $u \in L_{2,Q}(\mathcal{F}_T^W; L_2(\mathbb{R}^d))$ for some Q.

Theorem (Lototsky-R., Annals of Prob. 2005)

Assume that $\theta_0 \in L_2(\mathbb{R}^d)$ and $|\sigma_k^i(x)| \leq C_k$. Let Q be a sequence with $q_k = \frac{\sqrt{\delta\nu}}{d 2^k C_k}$ for some $0 < \delta < 2$. If

$$\frac{\partial \theta_\alpha(t,x)}{\partial t} = \nu \Delta \theta_\alpha(t,x) - \sum_{i,k} \sqrt{\alpha_i^k} \sigma_k^j(x) D_j \theta_{\alpha^-(i,k)}(t,x) m_i(t);$$

then $\sum_{\alpha \in \mathcal{J}} q^{2\alpha} \|\theta_\alpha(t)\|^2_{L_2(\mathbb{R}^d)} < \infty$
and $\theta(t,x) = \sum_{\alpha \in \mathcal{J}} \theta_\alpha(t,x) \xi_\alpha$ satisfies

$$\theta \in L_{2,Q}\left(\mathcal{F}_T^W; \mathbf{C}((0,T); L_2(\mathbb{R}^d))\right).$$

This θ is called the *Wiener Chaos solution* of the totally turbulent transport equation $d\theta(t,x) = \nu \Delta \theta(t,x) dt - \sigma_k^i(x) D_i \theta(t,x) dw_k(t)$.

- Computable expressions for the solution and its moments from the S-system.
- New regularity results.
- Possibilities for generalization.

Further Directions

- Anticipating equations.
- Elliptic equations.
- Nonlinear equations.

The main results can be found in:

1. S. Lototsky, B. L. Rozovskii. Passive Scalar Equation in a Turbulent Incompressible Gaussian Velocity Field. *Russian Math. Surv.* 59 (2004), No.2,

2. S. Lototsky, B. L. Rozovskii. Wiener chaos solutions of linear stochastic evolution equations. *Ann. Probab.* (2006, to appear).