Variational Techniques for Sturm-Liouville Eigenvalue Problems

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Abstract

As part of a 2002 VIGRE mini-course at University of Utah, we present properties of eigenvalues and eigenfunctions for the second-order Sturm-Liouville boundary value problem. Most of our proofs are adapted from [1] and are given using variational methods. Examples are also discussed.

Introduction

We define the Sturm-Liouville operator L on a bounded interval $[a, b]$ via

$$Lu := -(pu')' + qu,$$

where $p \in C^1([a, b]), \quad q \in C([a, b]), \quad p \geq \nu > 0, \quad \nu \in \mathbb{R}, \quad$ and $q \geq 0$.
We are interested in the Sturm-Liouville Eigenvalue Problem (SLEP) with Dirichlet boundary conditions:

$$Lu = \lambda u, \quad \lambda \in \mathbb{R} \tag{1}$$

$$u(a) = u(b) = 0 \tag{2}$$

Definition 1. *Values of λ for which (1),(2) has a nontrivial solution are called* eigenvalues *and a nontrivial solution u corresponding to λ is called an* eigenfunction*. The pair (λ, u) is called an* eigenpair *for the SLEP (1), (2).*

First, we present an example of how a PDE can lead to a SLEP. This example can be found in [2].

Example 1. *In this example we use separation of variables to show how we can get solutions of the two dimensional Laplace's equation*

$$u_{xx} + u_{yy} = 0.$$

We look for solutions of the form

$$u(x,y) = X(x)Y(y).$$

¿From Laplace's equation we get

$$X''Y + XY'' = 0.$$

Separating variables and assuming $X(x) \neq 0$, $Y(y) \neq 0$ *we get*

$$\frac{X''}{-X} = \frac{Y''}{Y}.$$

Since the left hand side of this equation depends only on x *and the right hand side depends only on* y *we get that*

$$\frac{X''(x)}{-X(x)} = \frac{Y''(y)}{Y(y)} = \lambda,$$

where λ *is a constant. This leads to the Sturm-Liouville differential equations*

$$X'' = -\lambda X, \quad Y'' = \lambda Y. \tag{3}$$

It follows that if X *is a solution of the first differential equation in* (3) *and* Y *is a solution of the second equation in* (3)*, then*

$$u(x,y) = X(x)Y(y)$$

is a solution of Laplace's partial differential equation.

Existence of minimizers

Using standard variational methods of Lagrange multiplier type, we look for minimizers of the functional

$$F[u] = \int_a^b p(u')^2 + qu^2 \, dx,$$

whose corresponding Euler-Lagrange equation is given by the Sturm-Liouville equation, over

$$A = \{u \in H_0^1([a,b]) : \int_a^b u^2 \, dx = 1\}$$

so that a minimizer will yield the equation

$$\text{"}F'[u] = \lambda u\text{"},$$

$$\int_a^b pu'\phi' + qu\phi\,dx = \lambda \int_a^b u\phi$$

for all ϕ in $H_0^1([a,b])$. The minimization will yield some suprising and useful results about the eigenvectors of the operator.

For our purposes we consider the minimization problem in a closed subspace V of $H_0^1([a,b])$. First, to guarantee a minimum, we need $F[u]$ to be bounded below and thus require the assumption that $p(x) \geq \theta > 0$ and $q(x) > 0$. Then $F[u]$ has an infimum over A, which we call α, and a minimizing sequence u_k. To use our compactness results we would like the u_k to be bounded in $H_0^1([a,b])$, but this follows from the Poincire inequality and coercivity of the functional F, which we see in the chain of inequalities

$$M \geq F[u_k] = \int_a^b p(u_k')^2 + qu_k^2\,dx \geq \theta||u_k'||_{L^2} + 0 \geq C||u_k||_{H_0^1([a,b])}.$$

As $H_0^1([a,b])$ is a Hilbert space (i.e. a reflexive Banach space)

$$u_k \longrightarrow_{H_0^1([a,b])} u$$

weakly for some $u \in H_0^1([a,b])$. This u is our candidate for the minimizer of $F[u]$. All is left to show is that $F[u] \leq \liminf F[u_k]$ and $u \in A$, i.e. $||u||_{L^2} = 1$. These two conditions guarantee that u is indeed a minimizer in V.

WLSC

To develope this inequality, we introduce the (notation saving) symmetric, $H_0^1([a,b])$ coervive, continuous, bilinear form

$$a(u,v) = \int_a^b pu'v' + quv\,dx$$

associated to F.

$$F[u_k] \geq 2a(u,u_k) - a(u,u).$$

And so, in the limit,

$$\alpha \geq 2a(u,u) - a(u,u) = F[u].$$

u is contained in V and A

Since the u_k are bounded in $H_0^1([a,b])$, which is compactly embedded in L^2, we know of the existence of a $v \in L^2$ to which the u_k converge strongly in L^2. We don't yet, however, know that u and v are the same element. But for arbitrary $w \in L^2$, we have

$$(u - v, w)_{(L^2)} \leq |(u,w) - (u_k,w)| + |(v - u_k, w)|,$$

where the left term can be made arbitrarily small since weak convergence in $H_0^1([a,b])$ implies weak convergnce in L^2 and the second term can be made arbitrarily by the Cauchy inequality and strong convergence in L^2. Since L^2 is a Hilbert space, and now $(v-u,w)=0$ for any $w \in L^2$, we must conclude that $u=v$. But this proves that $u \in A$, because $u_k \longrightarrow_{L^2} v$ implies that $||v=u||_{L^2}=1$.

V is a closed subspace for which we may apply Mazur's theorem which asserts that closed, convex sets are weakly closed. It follows that $u \in V$ as well.

Conclusion

Combining the above two results we find that by WLSC

$$F[u] \leq \lim F[u_k] = \alpha$$

and because u is contained in A and V,

$$\alpha = \inf_{v \in A,V} F[v] \leq F[u].$$

We conclude that $F[u]=\alpha$ and that u the minimizes F over A and V. subsection*Eigenpairs As discussed, the inductive minimizing of the functional F over the subspaces V_i produces a chain of functions and their corresponding values under F,

$$u_1, u_2, ..., u_i, ... \qquad \lambda_1 \leq \lambda_2 \leq ... \leq \lambda_i = F[u_i] \leq ...$$

We will show that these u_i and λ_i are actually eigenpairs for the Sturm-Liouville operator. To avoid later confusion, we note that the homogeneity of the functional F allows us consider the following two problems as equivalent;

$$\min_{v \in A,V_i} F[v] = \lambda_i \quad \Longleftrightarrow \quad \min_{v \in V_i} \frac{F[v]}{||v||_{L^2}} = \lambda_i.$$

Thus, we will frequently see the inequality

$$F[u_i] \leq \lambda_i \int v^2 \, dx, \qquad v \in V_i,$$

without having to assume that $||v||_{L^2}=1$.

Obtaining Eigenpairs

To obtain our eigenpairs, we inductively define a sequence of closed subspaces of $H_0^1([a,b])$.
Define

$$A := \{u \in H_0^1([a,b]) \ : \ \|u\|_{L^2} = 1\}.$$

Let

$$V_1 := H_0^1([a,b]) \quad \text{and} \quad W_1 := V_1 \cap A.$$

Minimizing F on W_1, we get $u_1 \in W_1$ and

$$\lambda_1 = \inf\{F(u) \ : \ u \in W_1\} = F(u_1).$$

Next, let

$$V_2 :=< u_1 >^{\perp} \quad \text{and} \quad W_2 := V_2 \cap A.$$

Minimizing F on W_2, we get $u_2 \in W_2$ and

$$\lambda_2 = \inf\{F(u) \ : \ u \in W_2\} = F(u_2).$$

Continue this process with

$$V_n :=< u_1 >^{\perp} \cap < u_2 >^{\perp} \cap \ldots \cap < u_{n-1} >^{\perp} \quad \text{and} \quad W_n := V_n \cap A.$$

Minimize F on W_n to get the pair (λ_n, u_n).

Notice that

$$W_1 \supset W_2 \supset W_3 \supset \ldots \supset W_n \supset W_{n+1} \supset \ldots,$$

and hence,

$$\lambda_1 \leq \lambda_2 \leq \ldots \leq \lambda_n \leq \lambda_{n+1} \leq \ldots$$

Also, by construction, we get

$$\int_a^b u_n u_k \, \mathrm{d}x = \delta_{n,k} \quad \text{where} \quad \delta_{n,k} := \begin{cases} 1 & \text{if}\, n = k \\ 0 & \text{if}\, n \neq k \end{cases}$$

Proof of eigenpairhood

If $\phi \in V_i$ and $\epsilon \neq 0$, then $u_i + \epsilon\phi \in V_i$ and by definition of a minimizer,

$$a(u_i + \epsilon\phi, u_i + \epsilon\phi) \geq \lambda_i \int (u_i + \epsilon\phi)\, dx.$$

$$2\epsilon[a(u_i, \phi) - \lambda_i \int u_i\phi\, dx] + \epsilon^2[a(\phi, \phi) - \lambda_i \int \phi^2\, dx] \geq 0.$$

The second term is positive by definition of the minimum, and by taking ϵ negative and small enough, so that the first term dominates the second, we must conclude

$$a(u_i, \phi) - \lambda_i \int u_i\phi\, dx = 0.$$

Now, more generally, for arbitrary $\psi \in H_0^1([a, b]), \psi$ differs from a ϕ in V_i by a linear combination of u_k for $k = 1, ..., i - 1$. But since

$$a(u_i, u_k) = \int u_i u_k\, dx = \delta_i^k,$$

the before last equality holds for ψ in place of ϕ. This is exactly the weak formulation that (u_i, λ_i) form an eigenpair for the Sturm-Liouville operator.

Properties of the eigenpairs

Most obvious is that the eigenvalues are ordered as a result of the minimization process. More though is that they approach infinity. If this were not true, they would be bounded and thusly the u_i would be bounded. As a result, the u_i would L^2-converge strongly to an u with $\int u^2\,dx = 1$.

$$\int (u - u_i)^2\,dx \longrightarrow 0.$$

But

$$\int (u - u_i)^2\,dx = \lim_k \int (u_k - u_i)^2\,dx = 2$$

when $i \neq k$. This is a contradiction.

Also, these eigenpairs are complete, in that if (u, λ) is an eigen pair, then $u = u_i, \lambda = \lambda_i$ for some i and if $\lambda_i = \lambda_j$ then $i = j$. These facts follow from the following argument. Suppose that (u, λ) is an eigenpair. Then

$$\lambda \int u u_i\,dx = a(u, u_i) = \lambda_i \int u u_i\,dx,$$

$$(\lambda - \lambda_i) \int u u_i\,dx = 0$$

for all i, so that either $u = 0$ (the trivial case) or $\lambda = \lambda_i$ for some i and $u \perp u_j$ for $j \neq i$. Now suppose $u \notin V_i$ for the previously given i. Then $u \in V_j$ for some $j < i$ and thus we must conclude that u minimizes $F[u]$ over V_j since we have that

$$\lambda_j \leq \lambda_i.$$

But this means that there are two nontrivial minimizers of the function F over the same space, which, by the following calculation, is a contradiction.

Suppose u_1, u_2 are L^2 unit vectors both satisfing $F[u_1] = F[u_2] \leq F[v]$ for any $v \in V$. Then $\frac{u_1+u_2}{2}$ and $\frac{u_1-u_2}{2} \in V_2$, and so

$$F[u_1] \leq F[\frac{u_1 + u_2}{2}] \qquad F[u_2] \leq F[\frac{u_1 - u_2}{2}],$$

$$\int p(u_1')^2 + q u_1^2\,dx \leq \int \frac{p}{4}(u_1' + u_2')^2 + \frac{q}{4}(u_1 + u_2)^2\,dx,$$

$$\int p(u_2')^2 + q u_2^2\,dx \leq \int \frac{p}{4}(u_1' - u_2')^2 + \frac{q}{4}(u_1 - u_2)^2\,dx.$$

Adding the two inequalities together, we find that

$$\int p((u_1')^2 + (u_2')^2) + q(u_1^2 + u_2^2)\,dx \leq \int \frac{p}{2}((u_1')^2 + (u_2')^2) + \frac{q}{2}(u_1^2 + u_2^2)\,dx,$$

$$\int \frac{p}{2}((u_1')^2 + (u_2')^2) + \frac{q}{2}(u_1^2 + u_2^2)\,dx \leq 0.$$

This is a contradiction since we assume that eigenvalues are nontrivial.

Eigenvectors Span $\mathbf{H_0^1}([\mathbf{a},\mathbf{b}])$

Not only does the Sturm-Liouville operator have an infinite spectrum of eigenvalues but the eigenvectors actually form a basis for $H_0^1([a,b])$. For $v \in H_0^1([a,b])$, define

$$v_n = \sum_{i=1}^{n} (v, u_i)_{L^2} u_i.$$

We first show that v_n form a Cauchy sequence in $H_0^1([a,b])$. Notice that

$$0 \leq a(v - v_n, v - v_n) = a(v,v) - \sum_{i=1}^{n} \lambda_i c_i^2$$

$$\sum_{i=1}^{n} \lambda_i c_i^2 \leq a(v,v)$$

for all n. Thus the sum converges, making the tail sum approach zero. It follows from

$$C||v_n - v_m||_{H_0^1([a,b])} \leq a(v_n - v_m, v_n - v_m) = \sum_{i=m}^{n} \lambda_i c_i^2$$

that the v_n are a Cauchy sequence in $H_0^1([a,b])$. Now, it is not certain whether v_n actually converge to v or not, infact, all we know is that v_n converge to some w in $H_0^1([a,b])$, but we will show that the v_n converge to v in L^2, which gives the desired result. Notice, we used the coercivity of the form a, and know we use the fact (u_i, λ_i) are a minimizing pair in V_i.

Notice that $\int (v - v_n) u_i \, dx = 0$ for $i = 1, ..., n$ so that $v - v_n \in V_{n+1}$. Thus, if we pick n large enough so that $\lambda_n > 0$ we have

$$\int (v - v_n)^2 \, dx \leq \frac{1}{\lambda_n} a(v - v_n, v - v_n) \leq \frac{1}{\lambda_n} a(v, v).$$

Since, $\lambda_n \longrightarrow \infty$, the inequality shows that $v_n \longrightarrow v$ in L^2.

Eigenvalues are simple

Suppose we have two linearly independent eigenvectors u_1 and u_2 corresponding to some eigenvalue λ. Then, find real numbers c_1 and c_2 not both zero so that

$$c_1 u_1'(a) + c_2 u_2'(a) = 0.$$

Define $u(t) = c_1 u_1(t) + c_2 u_2(t)$. Then, u(t) is a nontrivial solution to SLEP and (λ, u) is an eigenpair due to linearity. From our boundary conditions, we see

$$u(a) = c_1 u_1(a) + c_2 u_2(a) = 0$$

and by choice of c_1, c_2, we have

$$u'(a) = c_1 u_1'(a) + c_2 u_2'(a) = 0.$$

Since 2nd order linear initial value problems have unique solutions, $u \equiv 0$. This contradicts the linear independence of u_1 and u_2. Hence, the eigenvalues are simple.

Obtaining simple eigenvalues depends on the Dirichlet Boundary conditions. This next example illustrates how periodic boundary conditions can yield eigenvalues that are not simple. The following example can be found in [2].

Example 2. *Consider the following SLEP*

$$u'' = -\lambda u, \tag{4}$$

$$u(-\pi) = u(\pi), \quad u'(-\pi) = u'(\pi). \tag{5}$$

We let $\lambda = \mu^2$ *where* $\mu > 0$. *A general solution of* (4) *in this case is*

$$u(\theta) = c_1 \cos(\mu\theta) + c_2 \sin(\mu\theta), \quad \theta \in [-\pi, \pi].$$

The first boundary condition gives us

$$c_1 \cos(\mu\pi) - c_2 \sin(\mu\pi) = c_1 \cos(\mu\pi) + c_2 \sin(\mu\pi),$$

which is equivalent to the equation

$$c_2 \sin(\mu\pi) = 0.$$

Hence the first boundary condition is satisfied if we take

$$\mu = \mu_n := n,$$

$n = 1, 2, 3, \cdots$. *It is then easy to check that the second boundary condition is also satisfied. Hence*

$$\lambda_n = n^2$$

$n = 1, 2, 3, \cdots$ *are eigenvalues and corresponding to each of these eigenvalues are two linearly independent eigenfunctions given by* $u_n(\theta) = \cos(n\theta)$, $v_n(\theta) = \sin(n\theta), \quad \theta \in [-\pi, \pi]$.

Final comment

The main reason we are interested in eigenpairs for the Sturm-Liouville operator is we can use them to solve equations of the form

$$Lu = f, \quad \text{where} \quad f \in H_0^1([a, b]).$$

We first solve the SLEP

$$Lu = \lambda u$$

and then write $f = \sum c_i u_i$, which we can do since the eigenvectors span $H_0^1([a, b])$. Then, define

$$u := \sum \frac{c_i}{\lambda_i} u_i \quad \text{for} \quad \lambda_i \neq 0.$$

Then, u is a solution to

$$Lu = f.$$

References

[1] G. Buttazzo, M. Giaquinta, and S. Hildebrandt, *One-dimensional variational problems. An introduction.* Oxford Lecture Series in Mathematics and its Applications, 15. The Clarendon Press, Oxford University Press, New York, 1998.

[2] W. Kelley, A. Peterson, *Topics in Ordinary Differential Equations.* Prentice Hall, To be published.