Three dimensional FC Artin groups are CAT(0)

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Abstract. Building upon earlier work of T. Brady, we construct locally CAT(0) classifying spaces for those Artin groups which are three dimensional and which satisfy the FC (flag complex) condition. The approach is to verify the "link condition" by applying gluing arguments for CAT(1) spaces and by using the curvature testing techniques of M. Elder and J. McCammond.

Keywords: Artin group, Coxeter group, CAT(0) space

AMS Subject Classification: 20F36, 20F65

1. Introduction

The present article addresses the question of whether Artin groups act geometrically on non-positively curved spaces. We give a positive answer for the class of three-dimensional FC Artin groups.

An *Artin group* is any group with a presentation of the form

$$\langle s_1, \dots s_n \mid s_i s_j s_i \dots = s_j s_i s_j \dots \rangle,$$

where each alternating string $s_i s_j s_i \dots$ has $m_{ij} = m_{ji} \geq 1$ letters, with $m_{ij} = 1$ if and only if $i = j$. Some pairs of generators s_i and s_j may share no relation, which is indicated by $m_{ij} = \infty$. The most familiar examples are finitely generated free abelian groups (each $m_{ij} = 2$), finitely generated free groups (each $m_{ij} = \infty$), and braid groups ($m_{i,i+1} = 3$ and $m_{ij} = 2$ for $|i - j| > 1$).

The addition of relations $s_i^2 = 1$ for $i = 1, \dots, n$ to the above presentation defines the *Coxeter group* associated to this Artin group. In the three examples, the associated Coxeter groups are, respectively, a direct sum of n copies of $\mathbb{Z}/2\mathbb{Z}$, a free product of n copies of $\mathbb{Z}/2\mathbb{Z}$, and the symmetric group on n letters.

Let $S = \{s_1, \dots, s_n\}$ be the generating set for an Artin group A, as in the above presentation. Let W be its associated Coxeter group, and again denote its generating set by S. We say that A satisfies the *FC condition*, or say A is FC, if whenever $T \subset S$ and each pair $t_i, t_j \in T$ generates a finite subgroup of W, then T itself generates a finite subgroup of W. The simplest example of a non-FC Artin group has $n = 3$ and each $m_{ij} = 3$.

We say that A has formal dimension $\leq k$ if every subgroup of W generated by $k+1$ elements of S is infinite. The minimal such k is called the *formal dimension* of A. Since there always exists a free abelian subgroup of rank equal to the formal dimension, the formal dimension of A is bounded above by both the cohomological and geometric dimensions of A. If A is FC or has formal dimension equal to 2, then R. Charney and M. Davis [16] have shown that these dimensions are all equal; thus, for such Artin groups, we refer to this common value as the *dimension* of A. For example, finitely generated free groups are one dimensional FC Artin groups, whereas braid groups and finitely generated free abelian groups are n dimensional FC Artin groups.

A metric space (X, d) is a *geodesic metric space* if any two points may be connected by a length minimizing path. Such a path is called a geodesic segment. A triangle, $\Delta \subset X$, is the union of three geodesic segments joining three distinct points. A comparison triangle, $\bar{\Delta} \subset \mathbb{E}^2$, is a Euclidean triangle with the same corresponding edge lengths. If $d(p, q) \leq |\bar{p} - \bar{q}|$ for every $p, q \in \Delta$, where $\bar{p}$ and $\bar{q}$ denote the corresponding points in $\bar{\Delta}$, then we say that Δ satisfies the *CAT(0) inequality*. If every triangle in X satisfies the CAT(0) inequality, then X is called a *CAT(0) space*. Such a space is said to have global non-positive curvature.

A group G acts *geometrically* on a metric space X if the action is proper, cocompact, and via isometries. If G acts geometrically on a CAT(0) space, then we say that G is a CAT(0) group, or simply that G is CAT(0). We can now state the main result.

Main Theorem. *Every three dimensional FC Artin group acts geometrically on a three dimensional CAT(0) space.*

CAT(0) groups are of intrinsic interest. Such groups are finitely presented, they have solvable word and conjugacy problems, and every solvable subgroup is virtually abelian. The book by M. Bridson and A. Haefliger [12] is a standard reference.

Despite their rich theory, it remains a difficult problem to construct interesting examples of CAT(0) groups, especially if it is expected that a group act on a space of dimension ≥ 3. But, quite often, this is the case. If G is virtually torsion-free and acts properly, cocompactly, and cellularly on a CAT(0) cell complex X, then, since CAT(0) spaces are contractible, the dimension of X is bounded below by the virtual cohomological dimension of G. In fact, there are examples of groups (in particular, some three generator Artin groups) where the cohomological dimension is two, the group admits a geometric action on a three dimensional CAT(0) space, but the group does not admit a geometric action on any two dimensional CAT(0) space [11, 6, 26, 28].

The purpose of this article is to provide examples of CAT(0) groups in dimension three and to demonstrate the effectiveness of the "curvature testing" techniques proposed by M. Elder and J. McCammond [23].

In general, it is unknown whether or not every Artin group acts geometrically on a CAT(0) space. The answer is not even known for braid groups on more than four strings. An affirmative answer to the CAT(0) question would give a geometric proof of a number of group-theoretic properties which conjecturally hold for all Artin groups.

Some partial answers are known. R. Charney and M. Davis [16] have studied the action of an Artin group on the universal cover of its Salvetti complex. This is a piecewise Euclidean cube complex, which is CAT(0) if and only if the Artin group is right-angled (each $m_{ij} = 2$ or ∞).

More recently, T. Brady and J. McCammond [9] studied new presentations for Artin groups with formal dimension two. They showed that many of the associated presentation 2-complexes admit locally CAT(0) metrics. So, by the Cartan-Hadamard theorem for CAT(0) spaces, the universal covers of these complexes are (globally) CAT(0). The fundamental group acts geometrically via deck transformations; therefore, such Artin groups are CAT(0).

T. Brady [7] continued this line of investigation for the finite type Artin groups with three generators. These are the three dimensional Artin groups whose associated Coxeter group is an essential finite reflection group on $\mathbb{R}^3$; there are precisely three such Coxeter groups which do not split as a direct product, namely the full symmetry groups of the tetrahedron, the cube, and the dodecahedron. For each such Artin group A, Brady constructed a three dimensional, connected, piecewise Euclidean complex K with $\pi_1(K) \cong A$. He proved that K is locally CAT(0) by applying the the "link condition", i.e. a piecewise Euclidean complex is locally CAT(0) if and only if the geometric link of each of its vertices is a CAT(1) space. (A geodesic metric space is *CAT(1)* if every triangle of perimeter $< 2\pi$ satisfies a comparision inequality with respect to a comparison triangle in the unit sphere.) Again, the universal covering space of K is CAT(0) and the Artin group acts geometrically via deck transformations.

The complexes we consider are amalgamations of the spaces studied by Brady. However, in Brady's case, each link is a spherical suspension of a 1-complex. Since spherical suspensions of CAT(1) spaces are again CAT(1), it sufficed to check that a certain finite metric graphs were CAT(1); this is essentially a combinatorial condition. However, in our complexes, the links are not suspensions. The difficulty, then, is to check that a given piecewise spherical 2-complex is CAT(1). With the exceptions of Gromov's criterion for all-right piecewise spherical complexes [27] and Moussong's Lemma for piecewise spherical complexes with polyhedral cells with edge lengths $\geq \pi/2$ [32], there are no known combinatorial characterizations of CAT(1) 2-complexes. We overcome this difficulty by using gluing arguments for CAT(1) spaces and M. Elder & J. McCammond's curvature testing techniques [23]. When combined with some deep results of B. Bowditch on locally CAT(1) spaces [5], curvature testing is an effective way to study the link.

The construction of these complexes is closely related to the structure of special subgroups of Coxeter groups. Thus, we begin with an overview of Artin groups and their associated Coxeter groups.

2. Artin groups and Coxeter groups

Let S be a finite set of cardinality n. A *Coxeter matrix* for S is an $n \times n$ symmetric matrix with entries $m_{ij} \in \{1, 2, \ldots, \infty\}$ such that $m_{ij} = 1 \iff i = j$. The entries of a Coxeter matrix can be used to define a presentation of an Artin group A, as in the introduction. The pair (A, S) is called an *Artin system.* A relation

$$\overbrace{s_i s_j s_i \ldots}^{m_{ij}} = \overbrace{s_j s_i s_j \ldots}^{m_{ij}}$$

is called an *Artin relation* of length m_{ij}.

Each Artin system determines a Coxeter group W. The pair (W, S) is called a Coxeter system. Conversely, a Coxeter system determines an Artin system. We say that these systems are *associated.* We often suppress any reference to the generating set S and speak of properties of the systems as if they belonged to the underlying groups.

If the Coxeter group associated to an Artin group is finite, we say that the Artin group is *spherical.* If the associated Coxeter group is infinite, then the Artin group is of *infinite type.* (It is common in the literature to find the term “finite type Artin group” instead of “spherical Artin group”.) For example, the braid groups and finitely generated free abelian groups are spherical Artin groups, whereas finitely generated free groups are of infinite type.

For each subset $T \subset S$, we denote by A_T (respectively W_T) the subgroup of A (respectively W) generated by T. These are called the *special subgroups.*

Let M be a Coxeter matrix which defines an Artin system (A, S), and let (W, S) be the associated Coxeter system. For each $T \subset S$, we can define an Artin system $(A(T), T)$ and a Coxeter system $(W(T), T)$ by forming the Coxeter matrix consisting of those entries of M indexed by pairs $(i, j) \in T \times T$. There are natural epimorphisms $A(T) \to A_T$ and $W(T) \to W_T$. In fact, these maps are isomorphisms. Moreover, for every $T, Q \subset S$, $A_T \cap A_Q = A_{T\cap Q}$ and $W_T \cap W_Q = W_{T\cap Q}$. (The proofs of these statements appear in Bourbaki [4] for Coxeter groups and in van der Lek's Ph.D. thesis [30] for Artin groups.) Because of these natural identifications, we say that a special subgroup A_T of A is a *spherical subgroup* when W_T is finite.

The subsets $T \subset S$ which generate finite Coxeter groups will play an important role. We call these the *spherical subsets*, and we write

$$\mathcal{S} = \{\, T \subset S \mid W_T \text{ is finite} \,\}.$$

Suppose that M is a Coxeter matrix for $S = \{s_1, \ldots, s_n\}$. Define Γ as the labeled graph with vertices S and with edges labeled m_{ij} joining s_i and s_j whenever $1 < m_{ij} < \infty$. We call this a *defining graph.* Such a graph contains precisely the same information as a Coxeter matrix. The associated Artin and Coxeter groups are denoted A_Γ and W_Γ, respectively.

Let $\Delta(\Gamma)$ be an abstract simplicial complex with vertices S and declare that a nonempty set of vertices $T \subset S$ spans a simplex and only if $T \in \mathcal{S}$. $\Delta(\Gamma)$ is usually called the *nerve* of the Artin or Coxeter system defined by Γ. The graph Γ (without labels) is precisely the 1-skeleton of $\Delta(\Gamma)$, and the formal dimension of A_Γ is equal to the dimension of Γ plus one. Also, A_Γ is FC if and only if $\Delta(\Gamma)$ is a flag complex; hence the term 'FC'. (Recall that a simplicial complex with vertices S is a flag complex if each subset $T \subset S$ spans a simplex if and only if every distinct pair of vertices $t_i, t_j \in T$ spans an edge.)

Example. Let (A, S) be the Artin system with $S = \{s_1, s_2, s_3\}$ and $m_{ij} = 3$ for $i \neq j$. The associated Coxeter group W can be realized as the subgroup of isometries of the Euclidean plane generated by affine reflections across three lines which meet pairwise, forming an equilateral triangle. The product of two such reflections is a rotation by $2\pi/3$. Thus, each special subgroup indexed by two generators is a dihedral group of order six. But, the group W is not finite— the W-orbit of any equilateral triangle covers the entire plane. So, in terms of the nerve Γ, we have that each distinct pair $\{s_i, s_j\}$ spans a simplex; but $\{s_1, s_2, s_3\}$ does not— the subgroup generated by these, namely all of W, is not finite. Therefore, (A, S) is not FC.

We use the remainder of this section to review some of the theory of Coxeter groups. Proofs can be found in the books by N. Bourbaki [4], K. Brown [14], and J. Humphreys [29].

Theorem 2.1. *(Tits' solution to the word problem) Suppose that* (W, S) *is a Coxeter system and* w *is a word in* S*. Then* w *represents the identity in* W *if and only if it can be transformed into the empty word* *by a finite sequence of moves of the type* $s^2 \to 1$ *or* $sts\cdots \leftrightarrow tst\ldots$ *(the Artin relation between* s *and* t*). Moreover, if* w *is a word in* $T \subset S$*, then the moves only involve letters occurring in* T*.*

Let (W, S) be a Coxeter system. The elements of the set

$$R = \{\, wsw^{-1} \in W \mid w \in W, s \in S \,\}$$

are called *reflections*. The *(reflection) length* of an element $w \in W$, denoted by $\ell(w)$, is the smallest non-negative integer k such that w can be written as a product of k reflections. For each subset $T \subset S$, let

$$R_T = \{\, wtw^{-1} \in W \mid w \in W_T, t \in T \,\}.$$

If $w \in W_T$, we denote its length with respect to R_T by $\ell_T(w)$. The term "reflection" is justified by the following:

Theorem 2.2. *(Geometric Representation) Let* (W, S) *be a Coxeter system, and let* V *be a vector space of dimension* $|S|$*. Then there is a canonical faithful linear representation* $\sigma : W \to GL(V)$*.*

There is a canonical bilinear form preserved by the W-action, given by $B(e_s, e_t) = -\cos(\pi/m_{st})$, where $\{e_s : s \in S\}$ defines a basis for V; the action of W on V is then given by $\sigma(s).v = v - 2B(e_s, v)e_s$. If (W, S) is a spherical Coxeter system, this form is positive definite, and the reflections are precisely the elements of W which act as orthogonal reflections. If (W, S) is of infinite type, this form is not positive definite; nonetheless, the reflections act as "psuedo-reflections": $\sigma(r)$ fixes a codimension 1 hyperplane and has a 1 dimensional (-1)-eigenspace.

By studying the action of W on the dual space V^*, the geometric representation can be reformulated in the language of chamber complexes. For each $T \subset S$, let

$$C_T = \{f \in V^* : f(e_s) > 0, \forall s \notin T; \sigma^*(s)(f) = f, \forall s \in T\},$$

Theorem 2.3. *Suppose* $f \in w.C_T \subset V^*$*. Then the stabilzer of* f *is* wW_Tw^{-1}*.*

If we set $\bar{C} = \cup C_T$, we get a polyhedral cone; $\{C_T\}$ is precisely the set of open faces of $\bar{C}$. The maximal open face $C := C_\emptyset$ is called the *fundamental chamber*. Any W-translate of C is called a *chamber*. A *gallery* is a sequence $w_1C, \ldots w_nC$ of adjacent (i.e. sharing a codimension one face) chambers.

The following is well-known, but we give a proof to demonstrate the utility of chambers and galleries.

Proposition 2.4. *Let* (W, S) *be a Coxeter system and let* $T \subset S$*. Then,* $R_T = R \cap W_T$*.*

Proof. R_T is contained in $R \cap W_T$ by definition. For the other inclusion, suppose $r \in R \cap W_T$. Write $r = s_1 \dots s_k$ as an S-reduced word. By the solution to the word problem, each $s_i \in T$. Let $w_i = s_1 \dots s_i$. The gallery $C, w_1C, \dots, w_kC = rC$ must cross the hyperplane fixed by r; so, there is an open face $w_{i-1}C_{\{s_i\}} = w_iC_{\{s_i\}}$ fixed by r for some i. By Theorem 2.3, $r = w_{i-1}s_iw_{i-1}^{-1} \in R_T$. □

Remark. Every Coxeter group acts geometrically on a very natural piecewise Euclidean complex called its *Davis complex* X (see [20] for a survey). It was shown by Moussong [32] that X is CAT(0) and the elements of $\{r : r \in R_T \text{ for some } T \in \mathcal{S}\}$ act by reflections in the walls of X. Thus, Coxeter groups are CAT(0) groups.

3. Allowable elements and allowable expressions

Let (W, S) be a Coxeter system, and let R be the set of reflections. The reflection length ℓ defines a relation $\leq$ on W as follows:

$$w \leq w' \iff \ell(w) + \ell(w^{-1}w') = \ell(w').$$

Regarding R as a (possibly infinite) generating set for W, we say a word, $w = r_1 \dots r_k$, is *reduced* if $\ell(r_1 \dots r_k) = k$. A *prefix* of a reduced word $r_1 \dots r_k$ is a subword of the form $r_1 \dots r_i$ for some i, $1 \leq i \leq k$. The empty word is also considered a prefix.

Proposition 3.1. *Let (W, S) be a Coxeter system, and suppose $w, w' \in W$. Then $w \leq w'$ if and only if w is prefix of an reduced word representing w'. Thus, the relation, $\leq$, defines a partial order on W.*

Proof. Suppose $w = r_1 \dots r_k$ is reduced and suppose $\ell(w') = m$. If $w \leq w'$, then there is a reduced word $w^{-1}w' = u_1 \dots u_{m-k}$. Thus, $w' = w(w^{-1}w') = r_1 \dots r_k u_1 \dots u_{m-k}$ is reduced and w is a prefix of w'.

Conversely, if w is a prefix of w', then $\ell(w) + \ell(w^{-1}w') = \ell(w')$. So, $w \leq w'$. As the relation "w is a prefix of w'" is a partial order, $\leq$ defines a partial order. □

Since the relation $w \leq w'$ is clearly invariant under conjugation, $w \leq w'$ if and only if w is a *suffix* of a reduced word representing w': $\ell(w^{-1}w') = \ell(w(w^{-1}w')w^{-1}) = \ell(w'w^{-1})$. In particular, if $w \leq w'$, then $w^{-1}w' \leq w'$.

If $u \leq v$, we write $[u, v] = \{w : u \leq w \leq v\}$. Open and half-open intervals have the usual interpretation.

Proposition 3.2. *For each $u \leq v$, there is an order preserving bijection $[u, v] \to [1, u^{-1}v]$ given by $w \mapsto u^{-1}w$.*

Proof. If $u \leq w \leq w' \leq v$, then $u^{-1}w$ is a prefix of an R-reduced word for $u^{-1}w'$, which, in turn, is a prefix of $u^{-1}v$; thus, the map is well-defined and preserves order.

On the other hand, consider the inverse mapping, $w \mapsto uw$. If $1 \leq w \leq u^{-1}v$, then w is a prefix of an R-reduced word for $u^{-1}v$. Since, $u \leq v$, there is a factorization $v = (u)(u^{-1}v) = (u)(w)(w^{-1}u^{-1}v)$, as R-reduced words. Thus, uw is a prefix of v. □

Suppose (W, S) is an Coxeter system. For each $T \in \mathcal{S}$, we have a partial order $\leq_T$ defined by the length function ℓ_T on W_T with respect to the reflections R_T. We will show that these partial orders and length functions coincide under the natural inclusions into $(W, \leq)$ and that these orders and lengths agree on the intersection of spherical subgroups. The proof relies on a theorem of R. Carter; refer to Lemma 2.8 in [15] for a proof. The theorem in the form stated below can be found in [1]. Also, Proposition 2.2 in [10] gives an independent proof.

Theorem 3.3. *(Carter's Lemma) Let (W, S) be a spherical Coxeter system with reflections R and reflection length function ℓ. Suppose $\rho : W \to GL(V)$ is a faithful linear representation of W on a finite dimensional vector space V such that, for every $w \in W$, $codim(ker(\rho(w) - Id)) = 1 \iff w \in R$. Suppose $w \in W$. Then the reflection length of each $w \in W$ is equal to the codimension of its fixed subspace:*

$$\ell(w) = codim(ker(\rho(w) - Id)).$$

The following theorem is due to R. Charney and the author. It is inspired by a similar result for spherical Coxeter groups in [17].

Theorem 3.4. *Let (W, S) be a Coxeter system and let R be the set of reflections. Suppose that $w = r_1 \dots r_k$ is R-reduced. If $w \in W_T$ and $T \in \mathcal{S}$, then $r_i \in R_T$ for all i. In particular, $\ell(w) = \ell_T(w)$ for every $w \in W_T$.*

Proof. Let $n = |S|$ and consider the action of W on $V^* \cong \mathbb{R}^n$. Suppose $w \in W_T$ and $T \in \mathcal{S}$. Write $w = r_1 \dots r_k$ as an R-reduced word; thus, $k \leq \ell_T(w)$. Let $F := \cap_{i=1}^k H_i$, where each H_i is the codimension one hyperplane fixed by r_i. Let $Fix(w) := \{ v \in V^* : w.v = v \}$. Observe that $F \subset Fix(w)$. Carter's Lemma, applied to σ^* restricted to W_T (a spherical Coxeter group), says that $\ell_T(w)$ is equal to the codimension of $Fix(w)$. Because codim$(F) \leq k \leq \ell_T(w)$, the fixed subspaces are equal: $F = Fix(w)$. In particular, each reflection r_i fixes a point $f \in C_T \subset Fix(w)$. By Theorem 2.3, $r_i \in W_T \cap R = R_T$. □

The following is an immediate corollary:

Corollary 3.5. *Let (W, S) be a Coxeter system. For each $T \in \mathcal{S}$, the natural inclusion $W_T \subset W$ induces an isomorphism of posets*

$$(W_T, \leq_T) \cong (W_T, \leq),$$

where the latter poset is ordered by restriction. Further, W_T is a full sub-poset of W: $w \in W_T, w' \leq w \implies w' \in W_T$. If $Q, T \in \mathcal{S}$, then the partial orders $\leq_Q$ and $\leq_T$ and length functions ℓ_Q and ℓ_T agree on $W_Q \cap W_T$. Hence,

$$(W_{Q\cap T}, \leq) \cong (W_Q, \leq) \cap (W_T, \leq).$$

Hereafter, we view $(W_T, \leq_T)$ as a full sub-poset of $(W, \leq)$ whenever $T \in \mathcal{S}$. We will have no further need to distinguish between the partial orders $\leq_T$ and $\leq$.

Suppose (W, S) is a Coxeter system, where $S = \{s_1, \ldots, s_n\}$. An element $x = s_{i_1} \ldots s_{i_n}$, where $\{ i_1, \ldots, i_n \}$ is a permutation of $\{ 1, \ldots, n \}$, is called a *Coxeter element* for (W, S). If W is finite, then all of its Coxeter elements are conjugate.

An *ordered Coxeter system* is a Coxeter system (W, S) together with a total ordering $\prec$ on S. For each $T \in \mathcal{S}$, let $x_T := t_1 \ldots t_k \in W_T$ where $T = \{t_1 \prec \cdots \prec t_k\}$. Thus, $\prec$ determines a Coxeter element x_T in each spherical Coxeter system (W_T, T).

By repeated application of the *(left) shuffle* $x = r_1 r_2 \ldots r_n = r_2(r_2^{-1} r_1 r_2) r_3 \ldots r_n$, it is easy to see that $T \subset [1, x_T]$ and that $x_Q \leq x_T$ whenever $Q \subset T$. Thus, a total ordering of S makes a *consistent* choice of Coxeter elements.

The following definition is due to D. Bessis [1]: Suppose (W, S) is a Coxeter system. A Coxeter element x is *chromatic* with respect to S if

$$x = x_{A,B} := (\prod_{\alpha\in A} s_\alpha)(\prod_{\beta\in B} s_\beta),$$

for some partition $S = A \sqcup B$ such that all the elements in A commute and all the elements in B commute.

A consequence of the classification of spherical Coxeter systems by their Coxeter graphs (forests) is that every spherical Coxeter system has a chromatic Coxeter element.

Proposition 3.6. *Let (W, S) be an ordered Coxeter system. Then, for each $T \in \mathcal{S}$, $R_T \subset (1, x_T]$.*

Proof. The difficult work has already been done by Bessis [1], who proved this assertion in the case that x_T is chromatic. For the general case, choose a chromatic Coxeter element y_T for (W_T, T) and write $x_T = g y_T g^{-1}$ for some $g \in W_T$. Let $r \in R_T$. Since conjugation preserves reflection length, $r \le y_T$ implies that $grg^{-1} \le x_T$. Since conjugation by g defines a permutation of the set R_T, every $r \in R_T$ belongs to $(1, x_T]$. □

Given an ordered Coxeter system, (W, S), and given $T \in \mathcal{S}$, we call the elments of $[1, x_T]$ x_T*–allowable*. We define the *allowable* elements of W thus:

$$Allow(W) := \bigcup_{T \in \mathcal{S}} [1, x_T].$$

Proposition 3.1 and Theorem 3.4 imply that the x_T–allowable elements are precisely the elements of W which can be represented as a prefix (or a suffix) of an R-reduced word for x_T.

Proposition 3.7. *Let (W, S) be an ordered Coxeter system. Suppose $T \in \mathcal{S}$ and $Q \subset S$. Then*

$$[1, x_T] \cap W_Q = [1, x_{T \cap Q}].$$

Proof. Because $W_T \subseteq W$ is a full sub-poset, $[1, x_T] \cap W_Q = [1, x_T] \cap W_{T \cap Q}$. So, we may assume that $Q \subset T$. Since, $x_{T \cap Q} = x_Q \le x_T$, the right hand side contains the left. The other inclusion follows from the observation that $x_Q = x_{T \cap Q}$ is maximal in $[1, x_T] \cap W_Q$:

Suppose x_Q is not maximal. Then there is a reflection $r \in R_Q$ such that $x_Q r$ is reduced and $x_Q r \le x_T$. But this is impossible: $r \le x_Q$ by Theorem 3.6 and so $x_Q r$ cannot be reduced. □

The following corallary is easily deduced from Proposition 3.7.

Corollary 3.8. *Let (W, S) be an ordered Coxeter system and let $T, Q \in \mathcal{S}$. Then*

– *$Allow(W) \cap W_T = [1, x_T]$ and*

– *$[1, x_T] \cap [1, x_Q] = [1, x_{T \cap Q}]$*

Let (W, S) be an ordered Coxeter system, and let $T \in \mathcal{S}$. A sequence of x_T–allowable elements $(w_1, \ldots, w_k)$ defines an x_T*–allowable expression* of length k if $1 < w_1 < w_1 w_2 < \cdots < w_1 \cdots w_k \le x_T$. Denote the set of x_T–allowable expressions of length k by $Expr(x_T; k)$ and all x_T–allowable expressions by $Expr(x_T)$. We define the *allowable expressions* in W to be the set

$$Expr(W) = \bigcup_{T \in \mathcal{S}} Expr(x_T).$$

Corollary 3.9. *Let (W,S) be an ordered Coxeter system and let $T,Q \in \mathcal{S}$. Then*

- $Expr(W;k) \cap (W_T)^k = Expr(x_T;k)$ *and*
- $Expr(x_T;k) \cap Expr(x_Q;k) = Expr(x_{T\cap Q};k)$

Proof. In both equations, it is clear that the left hand side contains the right hand side. The other inclusions follow from Proposition 3.7 and the fact that $x_{T\cap Q} \le x_T, x_Q$. □

Remark. The results of this section build upon the earlier work of several researchers: D. Bessis [1]; D. Bessis, F. Digne, J. Michel [2]; J. Birman, K. Ko, & J. Lee [3]; T. Brady [8]; T. Brady & C. Watts [10]; and M. Picantin [33]. The primary goal of these papers was to develop a dual theory of braid monoids, and thereby obtain new solutions to classical questions about Artin groups. The present treatment differs from theirs in that, here, the results apply to infinite type Artin groups.

4. The Brady–Krammer complex

Suppose that $(\mathcal{P}, \le)$ is a finite poset. Let $(\mathcal{P}', \subseteq)$ denote the poset of of nonempty chains in P ordered by inclusion. This defines an abstract simplicial complex, and we denote a geometric realization by $|\mathcal{P}'|$.

Suppose Γ defines an ordered Coxeter system (W,S). Let P be the poset of allowable elements, $Allow(W)$. An equivalence relation $\sim$ on chains is generated by the identifications of the intervals $[u,v]$ with $[1,u^{-1}v]$ via the poset isomorphisms $w \mapsto u^{-1}w$ (see Proposition 3.2). The cell complex, $K_\Gamma := |\mathcal{P}'|/\sim$, is called the *Brady–Krammer complex.*

The Brady–Krammer complex, $K = K_\Gamma$, has a single vertex since all chains of length one are equivalent to $1 \in W$; we denote this vertex by v_0. The 1-cells of K correspond to nontrival allowable elements, since every chain of length two is equivalent to one of the form $1 < w$. Similarly, the k-cells of K correspond to allowable expressions of length k. We orient each 1-cell and label each by its corresponding nontrivial allowable element. The dimesion of K_Γ is equal to the formal dimension of the Artin group A_Γ.

Remark. The Brady–Krammer complexes for finite Coxeter groups were defined independently by T. Brady and D. Krammer. If W_Γ is a finite Coxeter group with generators $S = \{b \prec a \prec c\}$ such that $m_{ac} = 2$, then K is precisely the complex considered by T. Brady in [7]. If W_Γ is a finite dihedral group, this is exactly the 2-complex considered by T. Brady and J. McCammond [9].

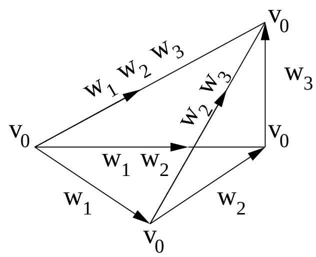

Figure 1. A typical 3-cells in K.

For each $T \subset S$, let $K(T)$ be the Brady-Krammer complex associated to the Coxeter system (W_T, T) together with the total ordering of S restricted to T. Let K_T be the subcomplex of K_Γ whose cells correspond to elements of $Expr(x_T)$.

Proposition 4.1. *Suppose $T, Q \in \mathcal{S}$. The inclusion $W_T \subset W$ induces a cellular isomorphism $K(T) \cong K_T \subset K_\Gamma$; further, $K_{T\cap Q} = K_T \cap K_Q$.*

Proof. This statement is simply a reformulation of Corollaries 3.8 and 3.9 in terms of the Brady-Krammer complex. □

From the 2-skeleton, we obtain the following presentation for the fundamental group of K:

$$\langle \{[w] : 1 \neq w \in Allow(W)\} \mid \{[u][u^{-1}v] = [v] : 1 < u < v\}\rangle.$$

The generator $[w]$ is called a *lift* of the allowable element w. Brackets are used to distinguish an element of the fundamental group from an element of the Coxeter group.

We will prove that the map $s \mapsto [s]$ defines an isomorphism $A_\Gamma \cong \pi_1(K_\Gamma)$. The computations for spherical Artin systems with $|S| = 2, 3$ were first obtained by T. Brady and J. McCammond [9, 7]. These computations were extended to type A_n by T. Brady [8] and to type B_n and D_n spherical Artin systems by T. Brady and C. Watts [10]. The computation for all spherical Artin systems was done independently of this work by D. Bessis [1]; however, some of the proof are still case by case.

Following Bessis, a *dual Coxeter system* is a triple (W, R, x) such that (W, S) is a spherical Coxeter system, R is the set of reflections, and x is a Coxeter element (not necessarily chromatic with respect to S). A pair of reflections r, q are *non-crossing* if $rq \leq x$ or $qr \leq x$. The *dual braid group* $G(R, x)$ is defined by the following presentation:

$$\langle\, \{[r] : r \in R\} \mid [r][q] = [rqr^{-1}][r]\,\rangle.$$

The relations range over all pairs r, q of non-crossing reflections.

Theorem 4.2. *(Bessis [1]) Every dual Coxeter system* (W, R, x) *satisfies a dual Matsumoto property: two* R*-reduced words* $r_1 \dots r_k$ *and* $q_1 \dots q_k$ *represent the same* $w \leq x$ *if and only if there is a sequence of applications of dual braid relations transforming* $[r_1] \dots [r_k]$ *into* $[q_1] \dots [q_k]$*.*

Lemma 4.3. *Suppose* (W, S) *is a spherical ordered Coxeter system with* $x = x_S$*. Then, the identity map on* $\{[r] : r \in R\}$ *extends to an isomorphism* $G(R, x) \to \pi_1(K)$*.*

Proof. If r and q are non-crossing reflections, say $rq \leq x$, then the dual braid relation is a consequences of the relations $[r][q] = [rq]$ and $[rqr^{-1}][r] = [rq]$, which hold in $\pi_1(K)$. Thus, the map is well-defined. The map is surjective since, by the application of Tietze transformations, it is easy to see that $\{[r] : r \in R\}$ is a generating set for $\pi_1(K)$. Likewise, if $[u][u^{-1}v] = [v]$ is a relation appearing in the presentation for $\pi_1(K)$, we can write this in terms of the generating reflections; the dual Matsumoto property (Theorem 4.2) then implies that this relation is a consequence of the dual braid relations. Hence, the map is injective. $\square$

Theorem 4.4. *(Bessis [1]) Let* (A, S) *be a spherical Artin system,* (W, S) *the associated Coxeter system,* R *the set of reflections, and* y *a chromatic Coxeter element with respect to* S*. Then the inclusion* $S \subset R$ *induces a group isomorphism* $A \cong G(R, y)$*.*

Theorem 4.5. *Let* Γ *define an ordered Coxeter system and let* A_Γ *be the associated Artin group. Then map* $s \mapsto [s]$ *defines and isomophism* $A_\Gamma \to \pi_1(K_\Gamma)$*.*

Proof. If $T = \{s, t\} \in \mathcal{S}$, then the Artin relation of length $m_{s,t}$ holds between $[s]$ and $[t]$. This follows from Lemma 2.2.1 of [1] or from the analysis of 2-generator Artin groups in [9]. Thus, the map is a well-defined homomorphism. That the map is bijective will follow from from the case when A_Γ is spherical; for, each generator and relation of $\pi_1(K_\Gamma)$ comes from a subcomplex K_T for some $T \in \mathcal{S}$.

Suppose (A, S) is spherical and let $x = x_S$. According to Lemma 4.3, it suffices to show that the map $s \mapsto [s]$ defines an isomorphism $\phi : A \to G(R, x)$. The map ϕ is a well-defined homomorphim for the same reasons as for the map $A \to \pi_1(K_\Gamma)$. If x is chromatic with respect to S, then Bessis's Theorem 4.4 says that ϕ is an isomorphism.

If x is not chromatic with respect to S, then choose a chromatic Coxeter element y and element $g \in W$ such that $gxg^{-1} = y$. Let $\beta : G(R, x) \to G(R, y)$ be the isomorphism taking $[r]$ to $[grg^{-1}]$. This map is well-defined since conjugation sends x-allowable elements to y-allowable elements, and it is obviously invertible. Now, let α :

$G(R, y) \to A$ be the inverse of the isomorphism $s \mapsto [s]$ from $A \to G(R, y)$ of Theorem 4.4.

We now show that ϕ is surjective. Suppose $[r_1]$ is a generator of $G(R, x)$. Then there is an R-reduced word $r_1 \cdots r_n = x$. By the dual Matsumoto property (Theorem 4.2), there is a sequence of braid relations trasforming the $[s_1] \cdots [s_n]$ into $[r_1] \cdots [r_n]$, where $x = s_1 \cdots s_n$. Since each application of a dual braid relation, $[r][q] = [rqr^{-1}][r]$, allows for the solution of $[rqr^{-1}]$ in terms of $[r]$ and $[q]$ (and their inverses), each reflection $[r_i]$, admits a solution in terms of the $[s_i]$'s (and their inverses).

Finally, consider the composite $\alpha\beta\phi : A \to A$. This is a surjective homomorphism. But spherical Artin groups are Hopfian: spherical Artin groups are finitely generated and linear [19, 22] and finitely generated linear groups are Hopfian [31] (i.e. every surjective homomorphism is an isomorphism). Since α and β are isomorphisms, so is ϕ. $\square$

5. A metric on the Brady–Krammer complex

For the rest of this paper, we will restrict our attention to Brady-Krammer complexes of dimension ≤ 3. When specifying an Artin group A_Γ, we will tacitly assume that Γ defines an ordered Coxeter system with generating set S.

We define a piecewise Euclidean structure on K_Γ by assigning a length of $\sqrt{k}$ to each edge labeled by an allowable element of length k. The metric on each cell is then determined. We will study the geometry of K_Γ within the formal framework of M_κ- polyhedral and simplicial complexes.

Let M_κ^n denote the complete simply connected Riemannian manifold of constant curvature κ and dimension n. Thus, M_0^n is Euclidean n-space, M_1^n is the unit n-sphere, and M_{-1}^n is the hyperbolic n-space. A M_κ-complex is a cell complex constructed by gluing convex polyhedral cells (in M_κ^N) along isometric faces. If X is an M_κ complex, we will denote a convex n-dimensional cell by C_λ^n and its attaching map by $q_\lambda : C_\lambda^n \to X$. M. Bridson proved that every M_κ complex having only finitely many isometry types of cells (referred to as *finite shapes*) is a complete geodesic metric space with respect to the intrinsic length metric. The reader is referred to [12] for a proof of this theorem. The Brady-Krammer complex K_Γ, with its cells metrized as above, is an M_0-polyhedral complex. By Bridson's theorem, this complex is a complete geodesic metric space.

An important class of these complexes is the M_κ simplicial complexes. In this case, cells are required to be simplices and the attaching maps are required to be injective. Note that, while K_Γ is not simplical, it is clear that its universal cover is simplicial. Therefore, the link of the unique 0-cell in K_Γ is an M_1-simplicial complex.

6. The link of v_0 in K_Γ

Let $L_\Gamma := Lk(v_0, K_\Gamma)$ denote the link of v_0 in K_Γ. For each $T \in \mathcal{S}$, let $L(T) = Lk(v_0, K(T))$, where $K(T)$ is the Brady-Krammer complex defined by restricting the ordered Coxeter system to (W_T, T). Let L_T be the full subcomplex of L spanned by vertices arising from edges in K_T. A consequence of Proposition 4.1 is the following:

Proposition 6.1. *Suppose $T, Q \in \mathcal{S}$. The embedding $K(T) \to K_\Gamma$ induces a simplicial isomorphism $L(T) \cong L_T \subset L_\Gamma$. Additionally, $L_T \cap L_Q = L_{T\cap Q}$.*

The *geometric link* of v_0 in $K = K_\Gamma$ is, by definition, the cell complex defined by the unit tangent vectors based at vertices in each of the convex polyhedral cells which are attached to the vertex v_0 in K_Γ. The identification of this geometric link with the (combinatorial) link of v_0 equips L_Γ with the structure of an M_1-simplicial complex.

Each 1-cell of K_Γ contributes exactly two vertices to L. Suppose that a 1-cell C^1_λ is oriented from a vertex v_1 to a vertex v_2 and that its edge is labeled by the allowable element w. The attaching map q_λ maps both vertices to v_0 in K_Γ. Thus, $Lk(v_0, q_\lambda(C^1_\lambda))$ consists of two vertices: $(w, 1)$ for the initial tangent vector of the geodesic path from v_1 to v_2 and $(w, -1)$ for the initial tangent vector of the reverse path. (Refer to Figure 2.) As every vertex of L arises in this way, the vertices of L_Γ are in bijective correspondence with the set $(Allow(W) - \{1\}) \times \{\pm 1\}$. Given a vertex (w, ϵ) of L, we say it has *length* $\ell(w)$ and *sign* ϵ.

Remark. We regard $Allow(W) \times \{\pm 1\}$ as a poset via reverse lexicographic ordering: $(w_1, \epsilon_1) \le (w_2, \epsilon_2) \iff \epsilon_1 < \epsilon_2$ or $\epsilon_1 = \epsilon_2$ and $w_1 \le w_2$. We can use this description to uniquely label the cells of L_Γ in terms of their vertices.

Suppose C^2_λ is a 2-cell in K_Γ. C^2_λ is a Euclidean triangle indexed by an allowable expression $\lambda := (w_1, w_2)$ of length 2. Suppose the vertices of C^2_λ are v_1, v_2, and v_3, and suppose the directed edge from v_i to v_{i+1} is labeled by w_i for $i = 1, 2$. The directed edge from v_1 to v_3 is labeled by $w_1 w_2 \in W$. The attaching map q_λ maps all of the vertices to v_0 and maps each directed edge to the 1-cell with the same label and orientation. Thus, the link of of a 2-cell of K_Γ consists three disjoint

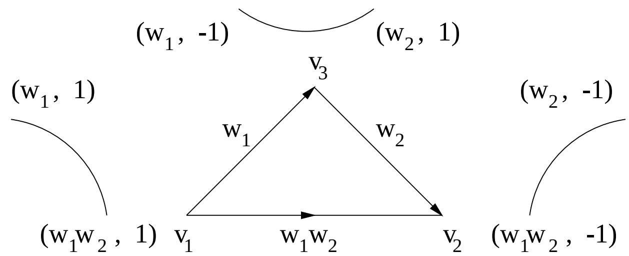

Figure 2. Each vertex of C^2_λ contributes a 1-cell in L.

arcs (refer to Figure 2):

$$Lk(v_0, q_\lambda(C^2_\lambda)) = \bigsqcup_{i=1,2,3} Lk(v_i, C^2_\lambda).$$

The vertices of a 1-cell in L_Γ are related by the reverse lexicographic ordering on $Allow(W) \times \{\pm\epsilon\}$. Making the convention that vertices are listed in ascending order, we can list the 1-cells according to their vertex set as follows:

$$[(w_1, 1), (w_1 w_2, 1)],\ [(w_1, -1), (w_2, 1)],\ \text{and } [(w_2, -1), (w_1 w_2, -1)].$$

This is a complete list if we range over all ordered pairs $(w_1, w_2) \in Expr(W; 2)$.

Proposition 6.2. *Let $w_1, w_2 \in Allow(W)$.*

1. *The vertices $\{(w_1, 1), (w_2, 1)\}$ or $\{(w_1, -1), (w_2, -1)\}$, span a 1-cell in $L \iff w_1, w_2 \in (1, x_T]$, for some $T \in \mathcal{S}$ and either $w_1 < w_2$ or $w_2 < w_1$.*
2. *The vertices $\{(w_1, -1), (w_2, 1)\}$ span a 1-cell in $L \iff (w_1, w_2) \in Expr(W; 2)$.*

Notation. We adopt the convention that reflections are labeled by the letters p, q, r, s or t. The *rotations* (elements in W of length two) are indicated by the letters y or z. And, the letters x or x_T are reserved for elements of length three in W.

In Figure 3, we list the three different oriented, metric 2-cells of K_Γ. They correspond to expressions of the form $(r, s), (y, t)$, and (q, y) in $Expr(x; 2)$. Recall that the lengths of the edges are 1 for a reflection, $\sqrt{2}$ for a rotation, and $\sqrt{3}$ for and element of length three. The first triangle is an isoceles right triangle. The angles in the second two triangles are indicated, where $\alpha = \arctan(\sqrt{2})$ and $\beta = \arctan(1/\sqrt{2})$. So, $0 < \beta < \pi/4 < \alpha < \pi/2$.

The left 2-cell contributes the following 1-cells to the link:

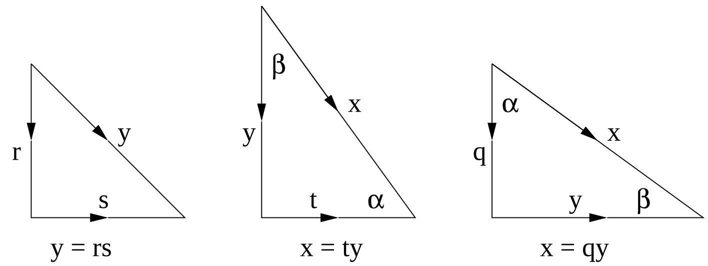

Figure 3. The metric 2-cells of K.

- $[(r, 1), (y, 1)]$ of length $\pi/4$; (algebraically: $r < y$)
- $[(r, -1), (s, 1)]$ of length $\pi/2$; ($rs = y$ is a reduced expression)
- $[(s, -1), (y, -1)]$ of length $\pi/4$; ($s < y$)

The middle 2-cell contributes:

- $[(y, 1), (x, 1)]$ of length β; ($y < x$)
- $[(y, -1), (t, 1)]$ of length $\pi/2$; ($yt = x$ is reduced)
- $[(t, -1), (x, -1)]$ of length α; ($t < x$)

And the right 2-cell contributes:

- $[(q, 1), (x, 1)]$ of length α; ($q < x$)
- $[(q, -1), (y, 1)]$ of length $\pi/2$; ($qy = x$ is reduced)
- $[(y, -1), (x, -1)]$ of length β; ($y < x$)

Thus, if we consider all unordered pairs of vertices $\{ (w, \epsilon), (w', \epsilon') \}$ up to their length and signs $\{ (\ell(w), \epsilon), (\ell(w'), \epsilon') \}$, we get exactly nine different 1-cells in L. This list is complete because every 1-cell in L necessarily arises from a link of one of the three different oriented, metric 2-cells in Figure 3 .

We repeat this analysis for the 2-cells of L_Γ. There is a unique isometry type of 3-cell in K_Γ, and each 3-cell of K_Γ correspondends to an allowable expressions of length three. For each allowable expression $\lambda := (r, s, t)$, we get four 2-cells in L by considering $Lk(v_0, q_\lambda(C^3_\lambda)) = \bigsqcup_{i=1,\ldots,4} Lk(v_i, C^3_\lambda)$. (Refer to Figure 4.)

We enumerate the 2-cells of L_Γ. (Refer to Figure 5). Clockwise from the upper left corner, they are, respectively, the links of v_1, v_3, v_4, and v_2. These are illustrated in the order shown to suggest how the 2-cells fit together. Two 2-cells are glued along a face if and only if they have the same vertices (labeled by the same allowable element and sign). In

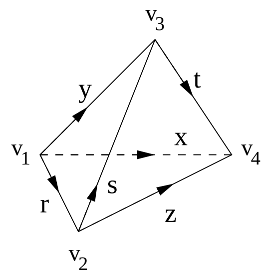

Figure 4. Each vertex of a 3-cell contributes a different 2-cell to the link.

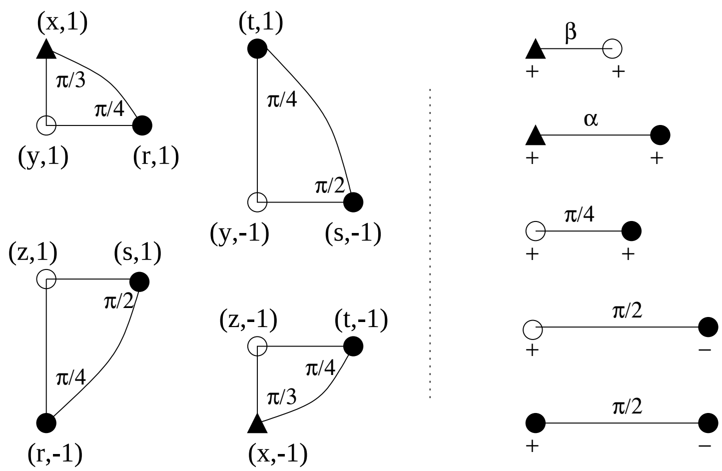

Figure 5. The metric 2-cells of L_Γ.

particular, such vertices must have the same length. We have illustrated the length of a vertex as follows: a reflection is symbolized by a solid circle, a rotation by an open circle, and an element of length three by a solid triangle.

When convenient, we indicate the sign of the vertex by adding a $+$ or $-$ symbol to the diagram, as in the list on the right of Figure 5. This list shows the edges of L_Γ and their lengths. We can recover our complete listing of 1-cells in L (up to length and sign of the vertices) if we change all the $+$ signs to $-$ signs. Note that the bottom 1-cell does not give rise to a new 1-cell if we change the signs— it is characterized as a pair of vertices of length one with opposite signs.

The 2-cells in L_Γ are spherical triangles. From the spherical law of cosines or by considering the dihedral angles between the faces of the model polyhedral 3-cell of K_Γ, one can compute their angles. The measures of the angles in each spherical triangle is indicated beside each vertex. The unlabeled angles are understood to be $\pi/2$.

Again, we can list the vertices of each 2-cells in ascending order with respect to the ordering on $Allow(W) \times \{\pm 1\}$:

- $Lk(v_1, C^3) = [(r, 1), (y, 1), (x, 1)]$; (algebraically: $r < y < x$)
- $Lk(v_2, C^3) = [(r, -1), (s, 1), (z, 1)]$; ($rz = x$ and $s < z$)
- $Lk(v_3, C^3) = [(s, -1), (y, -1), (t, 1)]$; ($s < y$ and $yt = x$)
- $Lk(v_4, C^3) = [(t, -1), (z, -1), (x, -1)]$; ($t < z < x$)

Thus, we have the following:

Proposition 6.3. *Given vertices $\{(w_1, \epsilon_1), \ldots, (w_3, \epsilon_3)\}$. These vertices span a 2-cell in $L \iff$*

1. *all the vertices have the same sign and the vertices are totally ordered: $w_i \leq w_j \leq w_k$ for some permutation (i, j, k) of $(1, 2, 3)$,*
2. *or exactly two vertices, $w_i \leq w_j$, are positive and $w_k w_j = x_T$ for some $x_T \in Allow(W; 3)$,*
3. *or exactly two vertices, $w_i \leq w_j$, are negative and $w_j w_k = x_T$ for some $x_T \in Allow(W; 3)$.*

In each of the last two, the negative vertex right multiplied by the positive vertex gives an allowable element.

7. CAT(0) spaces and the link condition

Let κ be a real number. Let $D_\kappa := \pi/\sqrt{\kappa}$ if $\kappa > 0$ and let $D_\kappa = \infty$ if $\kappa \leq 0$. A metric space, (X, d), is *D_κ-geodesic* if every two points $x, y \in X$ of distance less than D_κ may be joined by a geodesic segment. (Though these geodesics, in general, are not unique, we will conveniently denote such a segment by $[x, y]$.) Each model space, M^n_κ, is uniquely D_κ-geodesic.

Suppose that (X, d) is a D_κ-geodesic metric space. A triangle $\Delta = [x, y] \cup [y, z] \cup [x, z]$ satisfies the *CAT(κ) inequality* if for each point p in the arc $[y, z]$, $d(x, p) \leqq |\bar{x} - \bar{p}|$, where $\bar{x}$ and $\bar{p}$ are the comparison points on a comparison triangle $\bar{\Delta} \subset M^2_\kappa$. If every triangle in X of perimeter $< 2D_\kappa$ satisfies the CAT(κ) inequality, we say that X is a *CAT(κ) space*.

A geodesic metric space (X, d) is *locally CAT(κ)* if each point has a open neighborhood in which all triangles satisfy the CAT(κ) inequality. Locally CAT(κ) spaces are said to have *curvature* $\leq \kappa$.

Theorem 7.1. *(Local to Global) An M_κ-polyhedral complex X, with finite shapes, is (globally) CAT(κ) if and only if X is locally CAT(κ) and contains no isometrically embedded circles of length less than $2D_\kappa$. In particular, an M_0-polyhedral complex is CAT(0) if and only if it is locally CAT(0) and simply connected.*

Theorem 7.2. *(Link Condition) A M_κ-polyhedral complex X, with finite shapes, is a locally CAT(κ) space if and only if for every vertex v of X, the geometric link, $Lk(v, X)$, is CAT(1) space.*

Theorem 7.1, in the case of $\kappa \leq 0$, follows from the Cartan-Hadamard Theorem for complete locally CAT(0) spaces. Theorem 7.2 is a consequence of Berestovski's Theorem which states that the κ-cone on the link of a vertex is CAT(κ) $\iff$ the link is CAT(1). In particular, a sufficiently small neighborhood of the vertex is CAT(κ) $\iff$ the link is CAT(1). A thorough discussion, as well as proofs, can be found in [12].

Together, these two theorems reduce the question of whether K_Γ is locally CAT(0) to the question of whether the link L_Γ is locally CAT(1) and contains no isometrically embedded circles of length $< 2\pi$. Such circles are parameterized by closed (local) geodesics. Recall, that a path $\gamma : [a, b] \to X$ is a *local geodesic* if it is locally an isometric embedding. This path defines a *closed local geodesic* if $\gamma(a) = \gamma(b)$ and the induced map from $[a, b]/(a \sim b) \to X$ defines a local isometric embedding with respect to the quotient metric. We will refer to the image of a closed local geodesic of length $< 2\pi$ as a *short loop.*

In the case of an M_κ-complex X with finite shapes, a path defines a local geodesic if and only if for each $a \leq t \leq b$, the distance in $Lk(\gamma(t), X)$ between the incoming and outgoing unit vectors is $\geq \pi$. This is a practical way to decide if a given path is locally geodesic because such a path must necessarily 'look' like a geodesic in M^n_κ when restricted to an open cell.

We will blur the distinction between the the path $\gamma : [a, b] \to X$ and its trace, $\gamma([a.b]) \subset X$. So, given a subspace $Y \subseteq X$, we might say that γ "intersects" or "meets" Y. Likewise, we may say that $\gamma \cap Y = \emptyset$ if γ does not meet Y.

8. Basic gluing of CAT(1) spaces

A subspace Y of a geodesic metric space (X, d) is *r-convex* if every pair of points $x, y \in Y \subset X$ such that $d(x, y) < r$ may be joined by a geodesic segment, and, moreover, every such segment lies in Y.

Lemma 8.1. *(Basic Gluing of CAT(1) Spaces) Let X_1 and X_2 be CAT(1) spaces and let Y be a complete metric space. Suppose we are given π-convex subspaces $Y_i \subset X_i$ and isometries $\phi_i : Y \to Y_i \subset X_i$ for $i = 1, 2$. Then the space obtained by gluing X_1 and X_2 along Y, denoted by $X := X_1 \sqcup_Y X_2$, is CAT(1). Moreover, X_1 and X_2 are π-convex subspaces of X.*

The proof may be found in [12]. The idea is to use Aleksandrov's Lemma. The lemma says that if a triangle of perimeter $< 2\pi$ can be divided into two triangles, each satisfying the CAT(1) inequality, then the original triangle satisfies the CAT(1) inequality. The hypotheses of Lemma 8.1 guarantee that every triangle in X of perimeter $< 2\pi$ may be decomposed into two triangles which each lie in either X_1 or X_2.

By applying Basic Gluing, we will prove that L_Γ is CAT(1) whenever the Coxeter graph Γ is sufficiently nice. For such a Coxeter graph, we can identify nice subcomplexes Y_i and prove they are π-convex by using the following lemma:

Lemma 8.2. *Let Y and X_1 be connected CAT(1) spaces. Suppose we are given a continuous bijection $\phi_1 : Y \to Y_1 \subset X_1$ which takes local geodesics to local geodesics. If Y has diameter $\leq \pi$ then ϕ_1 is an isometry and Y_1 is a π-convex subspace of X_1.*

Proof. Let $x, y \in Y$. Let λ parameterize a geodesic segment from x to y. Then $\phi_1 \circ \lambda$ parameterizes a locally geodesic segment in X of length $\leq \pi$. As X is CAT(1), this segment is, in fact, a geodesic. (In a CAT(1) space, local geodesics of length $\leq \pi$ are (global) geodesics.) Hence, ϕ is an isometry. As geodesics of length $< \pi$ in a CAT(1) space are unique, Y_1 is a π-convex subspace of X_1. □

9. Tree-like complexes

The following summarizes the results of T. Brady and J. McCammond on the curvature of L_Γ in the case of a spherical Artin groups:

Theorem 9.1. *(Brady, Brady-McCammond) If Γ defines a spherical Artin group of dimension ≤ 3, then L_Γ is CAT(1). In fact, L_Γ is a spherical suspension of a CAT(1) space; thus, L_Γ has diameter π.*

Basic Gluing is valid along the subcomplexes L_T of L_Γ, if A_Γ is a spherical Artin group of dimension ≤ 3:

Lemma 9.2. *Suppose that Γ defines a spherical Artin group of dimension ≤ 3. For each $T \subset S$, L_T is a π-convex subspace of L_Γ.*

Proof. If $T = S$, then, trivially, $L_T = L_\Gamma$ is a π-convex subspace of itself. Suppose that Γ is two or three dimensional and $|T| = 1$. Then it is straightforward to check that the two vertices which comprise L_T are the endpoints of a locally geodesic edge path in L_Γ of length π (see Lemma 9.3 below). As L_Γ is CAT(1), this local geodesic is, in fact, a geodesic. Therefore, L_T is π-convex.

Similarly, if $|T| = 2$, by using Lemma 9.3 (below) we can construct between any two points in L_T a path in L_T which is a local geodesic in L_Γ. Because L_T has diameter π and because L_Γ is CAT(1), these local geodesics are, in fact, geodesics. Thus, by uniqueness of geodesics of length $< \pi$ in CAT(1) spaces, L_T is a π-convex subspace. $\square$

We use the following in-line notation for edges in L_Γ. The five types of edges as shown in Figure 5 are abbreviated as follows:

$$\blacktriangle - \circ, \quad \blacktriangle - \bullet, \quad \circ - \bullet, \quad \circ - - \bullet, \quad \text{and} \quad \bullet - - \bullet.$$

Thus, one dash denotes an edge of length $\beta, \pi/4$, or α; two dashes denotes an edge of length $\pi/2$. The first three edges have vertices of the same sign; the other two edges have vertices of opposite sign. As usual, the symbols $\bullet, \circ$, and $\blacktriangle$ denote vertices of length $1, 2$, and 3, respectively.

Lemma 9.3. *Suppose that* Γ *defines a spherical Artin group of dimension* 2 *or* 3. *Then the following edge paths in* L_Γ *are local geodesics:*

$$\circ - \bullet - - \bullet \quad \textit{and} \quad \bullet - \circ - \bullet\,.$$

Proof. If A_Γ is 2-dimensional, then L_Γ is a graph; so there is nothing to prove. If A_Γ is 3-dimensional, it suffices to show that the two edges in each of the above paths subtend an angle of $\geq \pi$ at their common vertex, (w, ϵ). Equivalently, we prove that the points in $Lk((w, \epsilon), L_\Gamma)$ corresponding to two edges in each path are distance $\geq \pi$ apart. There are two cases, depeding on the sign of the vertex. We prove the positive cases, leaving the other cases to the reader.

First, we address the right edge path. Suppose the common vertex $\bullet$ is labelled by $(r, 1)$, where r is a reflection. The vertices adjacent to $(r, 1)$ in L_Γ are of the form $(w, 1)$ with w allowable and $w > r$ or of the form $(w, -1)$ with wr allowable. There is a unique vertex of length 3 of the first type, namely $w = x_S$. There is a unique vertex of length 2 of the second type, namely $w = x_S r^{-1}$. The rest of the vertices adjacent to $(r, 1)$ arise in pairs: $(y, 1)$ is adjacent $(y > r)$ if and only if $(yr^{-1}, -1)$ is adjacent. The pair of vertices is joined by an edge $\circ - - \bullet$. Each $(y, 1)$ is joined by $\circ - \blacktriangle$ to $(x_S, 1)$; likewise, each $(yr^{-1}, -1)$ is joined by $\bullet - \circ$ to $(x_S r^{-1}, -1)$. This gives a combinatorial description of $Lk(\bullet, L_\Gamma)$. The metric, however, is different— the link of the link is metrized according the angles subtended by edges which share the vertex $(r, 1)$. Thus, the edges $\blacktriangle - \circ, \circ - - \bullet$, and $\bullet - \circ$ inherit lengths $\pi/4, \pi/2$ and $\pi/4$, respectively. (Refer to the left column of Figure 5.) This gives a complete description of $Lk(\bullet, L_\Gamma)$. One readily computes that the angle subtended by $\circ - \bullet - - \bullet$ is equal to π. (The angle is equal to the distance between $(\circ, 1)$ and $(\bullet, -1)$ in the link of the link.)

For the left path, suppose that the common vertex $\circ$ is labeled by $(y, 1)$. The vertices adjacent to $(y, 1)$ in L_Γ are of the form $(w, 1)$ with w allowable and $w > y$, of the form $(w, 1)$ with w allowable and $y > w$, or of the form $(w, -1)$ with wy allowable. The first and last vertices are unique: $(x_S, 1)$ and $(x_S y^{-1}, -1)$, respectively. The other vertices have the form $(r, 1)$, where $r < y$. These vertices are joined as follows: $(x_S, 1)$ by $\blacktriangle - \bullet$ to $(r, 1)$ and $(r, 1)$ by $\bullet - - \bullet$ to $(x_S y^{-1}, -1)$. The link of the link induces a length of $\pi/2$ on these edges. Thus, the link of the link is a spherical suspension of the set $\{(r, 1) : r < y\}$, with poles $(x_S, 1)$ and $(x_S y^{-1})$. Now it is easy to see that the angle subtended by $\bullet - \circ - \bullet$ is equal to π. $\square$

Let $\mathcal{T}$ denote the smallest class of finite simplicial flag complexes such that

1. $\Delta^n \in \mathcal{T}$ for all n and
2. if $K = K_1 \cup K_2$, $K_1 \cap K_2 = \Delta^k$ for some k, and $K_1, K_2 \in \mathcal{T}$, then $K \in \mathcal{T}$.

We say that the complexes in $\mathcal{T}$ are *tree-like.*

Recall that the nerve of a defining graph Γ is denoted by $\Delta(\Gamma)$. We say that A_Γ is *tree-like* if $\Delta(\Gamma) \in \mathcal{T}$.

Theorem 9.4. *If A_Γ is a tree-like Artin group of dimension ≤ 3, then L_Γ is CAT(1). Moreover, for each $T \in \mathcal{S}$, L_T is a π-convex subspace of L_Γ.*

Proof. We apply induction on the number of vertices in $\Delta(\Gamma)$. Theorem 9.1 and Lemma 9.2 imply that the theorem is true if the number of vertices is ≤ 3.

Suppose that $\Delta(\Gamma)$ has more than three vertices. Choose a cover of Γ by subgraphs Γ_1 and Γ_2 so that $\Delta(\Gamma) = \Delta(\Gamma_1) \cup \Delta(\Gamma_2)$ and $\Delta(\Gamma_1) \cap \Delta(\Gamma_2) = \Delta^k$ for some k. This is possible because $\mathcal{T}$ was defined as the smallest class of flag complexes satisfying conditions 1 and 2 in the definition above. Therefore, every $\Delta(\Gamma) \in \mathcal{T}$ has such a splitting.

By induction on the number of vertices in $\Delta(\Gamma)$ combined with basic gluing (which is applicable because of the second statement in the theorem), we obtain that L_Γ is CAT(1). It remains to show that if $T \in \mathcal{S}$, then L_T is a π-convex subcomplex of L_Γ. Necessarily, $L_T \subset L_{\Gamma_i}$ for $i = 1$ or 2. By the inductive step, L_T is π convex in L_{Γ_i}. By Basic Gluing, L_{Γ_i} is π-convex in L_Γ. Therefore, L_T is π-convex in L_Γ. $\square$

We say that $T \in \mathcal{S}$ is a maximal spherical subset if $T \subset Q \in \mathcal{S}$ implies that $T = Q$. We say that an Artin group A_Γ has *small diameter* if it has at most three maximal spherical subsets. If the complex Γ, defines a three dimensional FC Artin group with small diameter, then it is not hard to see that A_Γ is tree-like.

Corollary 9.5. *If Γ defines a three dimensional FC Artin group with small diameter, then L_Γ is CAT(1).*

When applying Theorem 9.4 and Corollary 9.5, we will be implicitly using the following observation: if γ is a locally geodesic loop which is contained in a subcomplex $L_{\Gamma'} \subset L_\Gamma$ and if Γ' defines a tree-like Artin group, then γ has length $\geq 2\pi$. We are simply using the fact that γ, viewed as a curve in $L_{\Gamma'}$ (with its intrinsic metric), is also a local geodesic.

10. Local curvature of L_Γ

Theorem 10.1. *Suppose Γ defines an FC Artin group of dimension ≤ 3. Then the link L_Γ is locally CAT(1).*

Proof. We prove that L_Γ is locally CAT(1) by verifying the link condition at each of its vertices. By the classification of the simplices in L_Γ (Propositions 6.2 and 6.3), $Lk((w,\epsilon), L_\Gamma) = Lk((w,\epsilon), L_{star(T(w))})$, where $T(w)$ is the minimal $T \in \mathcal{S}$ such that $w \in W_T$ and where $star(T)$ denotes the defining graph specified by the 1-skeleton of $star(\Delta(T))$. (Recall that the *star* of a simplex σ is the subcomplex spanned by all simplices containing σ.)

If w has length three, then $w = x_T$ for some $T \in \mathcal{S}$ of cardinality three. Thus, $star(T(w)) = T$. Now apply Brady and McCammond's result (Theorem 9.1).

If w has length two, then, because K_Γ has dimension ≤ 3 and $|T| \geq 2$, $star(\Delta(T))$ is tree-like. Therefore, Theorem 9.4 applies.

Finally, if w has length one, then we prove that link of the link is CAT(1) directly. It suffices to prove that there cannot exist any circuits of length $< 2\pi$. There are two cases depending upon the sign of (w, ϵ); we treat the case of $\epsilon = 1$, the other case being similar. Write $w = r$, where r is a reflection. As explained in the proof of Lemma 9.3, the vertices of L_Γ which are adjacent to $(r,1)$ are of the following types: "uniquely determined" vertices $(x_T,1)$ and $(x_T r^{-1},-1)$, where $x_T > r$, or pairs of adjacent vertices $(y,1)$ and $(yr^{-1},-1)$, where $y > r$. The situation is complicated by the fact that there may be several distinct Coxeter elements x_T, x_Q, etc. greater than r. However, we completely understand when these vertices share an edge. The edges $\blacktriangle - \circ, \circ - - \bullet$, and $\bullet - \circ$ are the building blocks circuits in $Lk((r,1),L_\Gamma)$; the lengths $\pi/4, \pi/2$, and $\pi/4$ are induced on each edge, respectively.

Every circuit can be decomposed into pieces of the following three types: $\circ - \blacktriangle - \circ$, $\circ - - \bullet$, or $\bullet - \circ - \bullet$. Each piece has length $\pi/2$. No piece, nor a union of two pieces can form a circuit. If a circuit consists of three pieces, then it is either $\circ - \blacktriangle - \circ - \blacktriangle - \circ - \blacktriangle - \circ$, where $x_i = x_{T(i)}$ is the Coxeter element assigned to each $\blacktriangle$, or it is $\bullet - \circ - \bullet - \circ - \bullet - \circ - \bullet$, where $x_i r^{-1}$ is the allowable element assigned to each $\circ$.

The vertices of $T(1), \ldots, T(3)$ are pairwise adjacent. By the FC condition, they span a 3-simplex, contradicting the fact that Γ is two dimensional complex. Thus, every circuit consists of at least four segments. Hence, every circuit has length $\geq 2\pi$. (Notice that the FC hypothesis is essential– without this hypothesis , L_Γ is not even locally CAT(1).) $\square$

11. Locally geodesic edge loops in L_Γ

An *edge path* is a path which lies entirely in the 1-skeleton of a complex. We prove that certain edge paths in L_Γ are not locally geodesic.

Lemma 11.1. *Suppose A_Γ is a 3-dimensional Artin group. A locally geodesic edge path in L_Γ cannot contain a subpath of the form* $\blacktriangle - \circ - \blacktriangle - \circ - \blacktriangle$.

Proof. Suppose that the allowable elements which label the vertices are (from left to right) $x_{T(1)}, y_1, x_{T(2)}, y_2$, and $x_{T(3)}$. We prove the case where all vertices have positive sign; the other case is analagous.

A local geodesic cannot "double back" along an edge just traversed. Therefore, $x_{T(1)} \neq x_{T(2)}, x_{T(2)} \neq x_{T(3)}$ and y_1 and y_2. According to Proposition 4.1, $y_1 = x_{T(1)\cap T(2)}$ and $y_2 = x_{T(2)\cap T(3)}$.

Suppose that $T(2) = \{a \prec b \prec c\}$ and $x_{T(2)} = abc$. Then each y_i must be one of ab, bc or ac. These vertices fit together in the 2-cell of $L_{T(2)}$ shown in Figure 6.

The path $y_1 - x_{T(2)} - y_2$ makes an angle of $2\pi/3$ at $x_{T(2)}$. So, this path is not a local geodesic. $\square$

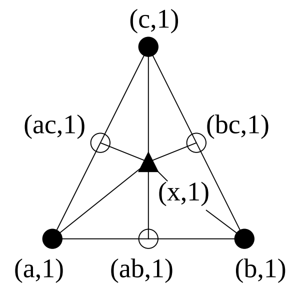

Figure 6. The 2-cells of L form an all-right spherical triangle.

Lemma 11.2. *Suppose A_Γ is a 3-dimensional Artin group; and suppose y is an allowable rotation. Then,*

$$Lk((y,1), L_\Gamma) \cong \{(x_T, 1), (x_T y^{-1}, -1) : x_T > y\} * \{r : y > r\}.$$

Here, $X*Y$ denotes the spherical suspension. We interpret $\emptyset * Y = Y$. If we consider the link of $(y,-1)$, the statement is nearly identical. Just replace $x_T y^{-1}$ with $y^{-1} x_T$ and change the signs of all the vertices. The proof is the same as the analysis of $Lk((y,1), L_\Gamma)$ done in the proof of Lemma 9.3. The only difference is that there may be several spherical subsets $T \in \mathcal{S}$ for which $x_T > y$.

Lemma 11.2 implies that if a geodesic edgepath in L_Γ terminates at a vertex (y, ϵ), then any locally geodesic continuation of this path must initially be an edgepath. This is because $Lk((y,\epsilon), L_\Gamma)$ is either discrete or diameter π.

On the other hand, there are a continuum of different ways to extend a path geodesically through a vertex (r, ϵ) of length one or a vertex (x, ϵ) of length three; the links of these vertices in L_Γ do not have diameter π. For these reasons, we say that vertices of length 1 or 2 are *singular* points of L_Γ. All other points are *non-singular*. A path α in L_Γ is *non-singular* if $\alpha(t)$ is non-singular for all t.

The locally geodesic edge paths appearing in Figure 7 are called *basic pieces*. We have displayed the sign of the vertices. These polarities may be reversed, changing all positive signs to negative signs. Observe that every basic piece has length at least $\pi/2$.

Proposition 11.3. *Every locally geodesic edge loop in L_Γ can be decomposed into basic pieces each of which is contained in some maximal subcomplex L_T (meaning $T \in \mathcal{S}$ and T maximal). These basic pieces intersect only at vertices.*

Proof. The following two edge paths are not locally geodesic:

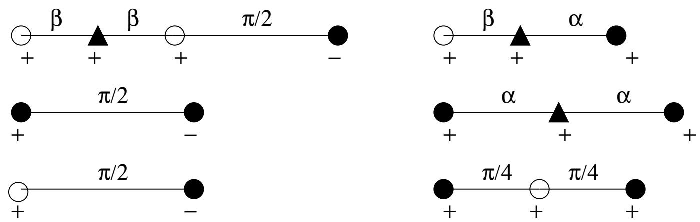

Figure 7. The basic pieces in $L^{(1)}$.

1. $\bullet - \circ - \blacktriangle$; all the vertices in this configuration have the same sign. The two edges make a right angle at the center vertex. (The path belongs to the boundary of the 2-cell $[(r,1),(y,1),(x,1)]$.)

2. $\bullet - \circ - -\bullet$; the two ends have opposite signs and the double dash denotes an edge of length $\pi/2$. The two edges make a right angle at the center vertex. (The edges belong to the boundary of the 2-cell $[(q,-1),(r,1),(y,1)]$.)

Any other combination of two edge paths appears on the list of basic pieces. With the exception of the upper left piece, it is clear that the basic pieces in a locally geodesic loop only intersect at vertices. The pieces in the right column are each contained in a subcomplex indexed by a maximal $T \in \mathcal{S}$ such that $w \in W_T$ and w labels the vertex of longest length. The lower two pieces in the left column are each contained in a maximal L_T, with $w \in W_T$ and w equals the product of the negative vertex and the positive vertex.

The piece $\circ - \blacktriangle - \circ - -\bullet$ needs a special explanation. Suppose a locally geodesic edge loop contains $\circ - \blacktriangle - \circ$. By Lemma 11.1, the geodesic must extend to $\circ - -\bullet$ at one of its ends. Moreover, if both ends extend to $\circ - -\bullet$, then at least one of these extensions is contained in L_T, where x_T is the label of $\blacktriangle$. For otherwise, the path would make an angle of $2\pi/3$ at $\blacktriangle$, as in Figure 6. By the same considerations, a path $\circ - \blacktriangle - \circ - \blacktriangle - \circ$ must extend to $\circ - -\bullet$ at each end and these extensions must be contained in the maximal subcomplex indexed by the neighboring $\blacktriangle$. Note that the path could not have already closed to form a loop because it would be contained in the link of an Artin group with small diameter. □

Theorem 11.4. *Suppose Γ defines an FC Artin group of dimension ≤ 3. Then L_Γ does not contain any short loops in its 1-skeleton.*

Proof. According to Proposition 11.3, every geodesic loop can be decomposed into basic pieces each of which is contained in a maximal subcomplex of L_Γ. If there are fewer than four basic pieces, then the geodesic loop is contained in a subcomplex $L_{\Gamma'}$ defined by an Artin group with small diameter. Thus, by Corollary 9.5, the loop has length $\geq 2\pi$. On the other hand, any loop of four or more basic pieces has length $\geq 2\pi$. $\square$

12. Developing galleries onto the sphere

The following ideas are due to M. Elder & J. McCammond (see [23] and [24]). Suppose $\gamma : [a, b] \to X$ defines a local geodesic in an M_1-simplicial complex X. Let $(\sigma_1, \ldots, \sigma_k)$ be the sequence of closed simplices $\sigma \subset X$ such that $\mathring{\sigma} \cap \gamma \neq \emptyset$ (if σ is a vertex, then we define $\mathring{\sigma} = \sigma$). These simplices are ordered according to the order in which γ meets each one. Let $\mathcal{G}$ denote the M_1-simplicial complex defined by gluing σ_i to σ_j if σ_i is a proper face of σ_j and $j = i - 1$ or $i + 1$, where $1 \leq i, j \leq k$. This complex is called the (linear) *gallery* determined by γ. For each gallery $\mathcal{G}$, there is a unique locally geodesic path defined by gluing the maps $\gamma|_{\gamma^{-1}(\sigma_i)}$. The resulting path, $\hat{\gamma} : [a, b] \to \mathcal{G}$, is called the *lift* of γ.

Suppose $\gamma : [0, h] \to X$ defines a local geodesic and $\gamma(0)$ belongs to an edge or vertex of X. Let $\mathcal{G}$ be its gallery and $\hat{\gamma}$ its lift. Fix a point p in the unit sphere and fix an oriented great arc from p to the antipodal point $-p$. Define a map $\phi : \mathcal{G} \to M_1^2$ by first mapping $\hat{\gamma}(0)$ onto the midpoint of the oriented great arc. We insist that $\phi(\hat{\gamma})$ trace a local geodesic and that it make an (oriented) angle of 90 degrees with the oriented great arc; the map ϕ is then determined. We say that ϕ *develops* $\mathcal{G}$ onto the unit sphere. By design, $\phi(\hat{\gamma})$ traces a great arc (or circle) on the unit sphere. In particular, if $\phi(\hat{\gamma})$ meets any other great arc in two points, then γ must have length $\geq \pi$.

Suppose that $\gamma : (-\delta, \delta) \to L_\Gamma$ is a local geodesic, $\gamma(0)$ is a non-singular vertex, and $\gamma(t)$ does not belong to the 1-skeleton of L_Γ for $t \neq 0$. Thus, the gallery determined by γ is $\mathcal{G} = (\sigma_1, \ldots, \sigma_3)$, where σ_2 is a vertex of length two and σ_1 and σ_3 are 2-simplices. We construct a complex $\mathcal{TG}$, called a *thickening*, such that $\mathcal{G}$ is a subcomplex and the additional simplices of $\mathcal{TG}$ are uniquely determined by $\mathcal{G}$.

Up to the sign of the vertices of L_Γ, the simplices in $\mathcal{G}$ fall into three cases. Suppose that σ_2 is the vertex $(y, 1) \in L_\Gamma$. The 2-simplices σ_1 and σ_3 may be of two types– either they contain a vertex of length three or not. If both contain a vertex of length three, say $\sigma_1 = [(r, 1), (y, 1), (x_T, 1)]$ and $\sigma_3 = [(q, 1), (y, 1), (x_Q, 1)]$, then the 2-simplices $[(q, 1), (y, 1), (x_T, 1)]$ and $[(r, 1), (y, 1), (x_Q, 1)]$ exist and glue to form a square as in Figure 8 (left square). This follows from the fact that $r, q < y$.

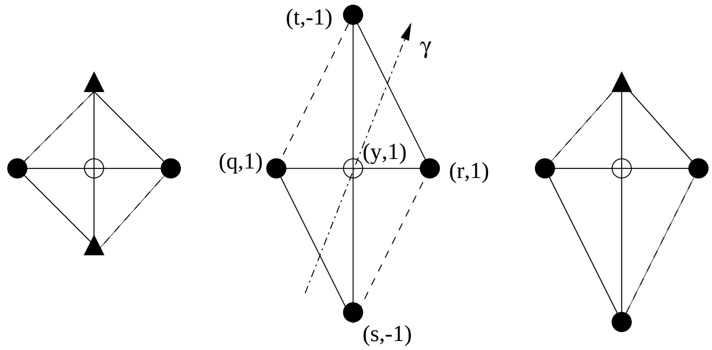

Figure 8. Galleries which contain non-singular vertices may be thickened.

If both do not contain a vertex of length three, say $\sigma_1 = [(s,-1), (q,1), (y,1)]$ and $\sigma_3 = [(t,-1), (r,1), (y,1)]$, then the 2-simplices $[(s,-1), (r,1), (y,1)]$ and $[(t,-1),(q,1),(y,1)]$ exist and glue to form a square as in the center square in Figure 8. Again, this follows from the fact that $r, q < y$. The third case is when the simplices are of mixed type (the right square in Figure 8). In all cases, the complex obtained from $\mathcal{G}$ by gluing in two additional 2-simplices near each non-singular vertex is called a thickening.

Most (thickened) galleries in L_Γ develop in a very special way onto the 2-sphere. Let Σ denote the unit 2-sphere together with the following simplicial structrure: First, divide the sphere into eight spherical triangles by intersecting with the usual coordinate planes. Each of these triangles is an all-right spherical triangle (all edges and angles measure 2π). Second, pass to the barycentric subdivision. The resulting M_1-simplicial complex has 48 spherical triangles, each of which is isometric to the 2-cell of L_Γ with edge lengths $\beta, \pi/4$, and α (the 2-cells labeled by allowable elements satisfying $r < y < x$). The other 2-cells of L_Γ are isometric to subcomplexes of Σ. The 2-cells with edge lengths $\pi/4, \pi/2, \pi/2$ are isometric to one half of an all-right triangle.

Every closed geodesic which does not lie entirely in the 1-skeleton of L_Γ admits a parameterization so that it begins in one of the following *general positions*:

1. There exists a $\delta > 0$ such that $\gamma(0)$ is a vertex of length three and $\gamma(t) \notin L^{(1)}$ for all $0 < t < \delta$,

2. or there exists a $\delta > 0$ such that $\gamma(0)$ belongs to a segment of type $\bullet - \circ - \bullet$ or $\bullet - - \bullet$ and $\gamma(t) \notin L^{(1)}$ for all $0 < t < \delta$.

Proposition 12.1. *Let A_Γ be a 3-dimensional Artin group. Suppose $\alpha : [0,h] \to L_\Gamma$ is local geodesic which does not contain any edges. If α*

is non-singular, except possibly at its endpoints, then α determines a (thickened) gallery which develops onto a subcomplex of Σ.

Proof. Parametrize α so that it begins in general position. Let $\mathcal{G}$ be the gallery determined by α, and let $\mathcal{TG}$ be the gallery obtained by thickening $\mathcal{G}$ at any non-singular vertices which α meets. Let $\phi : \mathcal{G} \to \Sigma$ be the developing map. The simplices which are glued to $\mathcal{G}$ to form $\mathcal{TG}$ fit together to form "squares" which are isometric to convex spherical cells. Because $\phi(\hat{\alpha})$ defines a local geodesic on the sphere, the developing map extends continuously to $\phi : \mathcal{TG} \to \Sigma$. We can adjust ϕ so that the image is a subcomplex. Choose ϕ so that the first 2-simplex in $\mathcal{G}$ is a 2-simplex in Σ. The developing map is then determined, and each simplex of $\mathcal{TG}$ fits onto a subcomplex of Σ. $\square$

Proposition 12.1 is certainly not true if α contains singular vertices in its interior. The distance between incoming and outgoing tangent vectors at a singular vertex may be strictly greater than π in the link of L_Γ. Thus, the gallery determined by such a geodesic might not develop onto a subcomplex of Σ.

13. Extra-short loops

A closed geodesic is an *extra-short loop* if it has length $\leqq \pi$. We first prove that L_Γ does not contain any extra-short loops.

Proposition 13.1. *Let A_Γ be a 3-dimensional Artin group. Suppose $\gamma : [0, h] \to L_\Gamma$ is a closed geodesic, $\gamma(0)$ is singular, and $\gamma(t)$ does not belong to the 1-skeleton for $0 < t < \delta$. Then γ contains a subarc α of length $\geq \pi$ such that $\alpha(t)$ is non-singular for $0 < t < \pi$.*

Proof. By hypothesis, there exists a $\delta > 0$ so that $\gamma(t)$ is non-singular for $0 < t < \delta$. Let α a maximal non-singular subarc containing this initial portion. Using Proposition 12.1, we develop the thickened gallery determined by α onto a subcomplex of Σ. The subcomplexes, depending on whether $\gamma(0) = \bullet$ or $\blacktriangle$, are described by Figure 9. Observe that α cannot meet another singular vertex unless it has length $\geq \pi$. $\square$

In Figure 10, we have sketched the initial few cells of typical galleries (developed onto Σ) determined by local geodesics of L_Γ beginning in general position. Either we develop the local geodesic beginning at a vertex of length three or we develop beginning at a point in a great circle in the 1-skeleton of Σ.

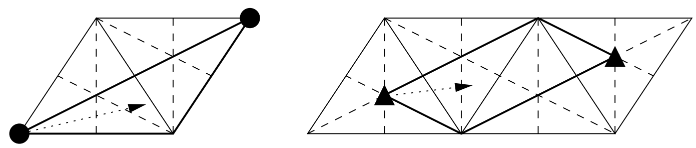

Figure 9. The (thickened) galleries are develop onto subcomplexes of Σ.

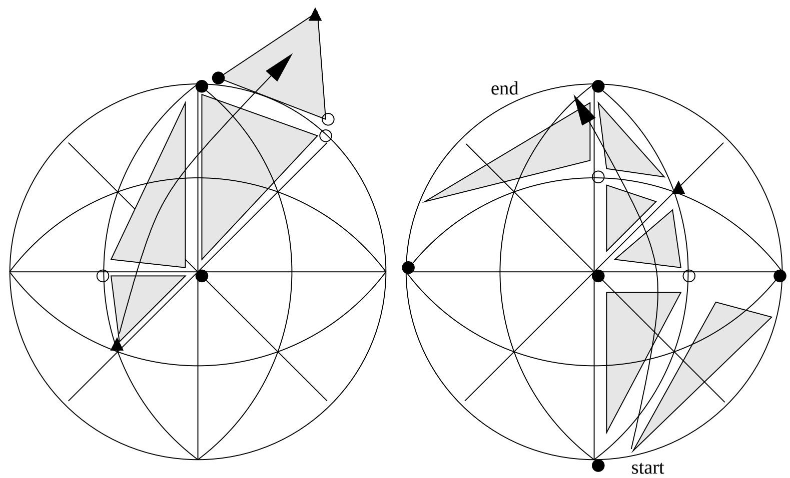

Figure 10. Typical galleries of local geodesics in general position.

Theorem 13.2. *If Γ defines a three dimensional FC Artin group, then L_Γ does not contain any extra-short loops. Moreover, every closed geodesic in L_Γ which is not contained in the 1-skeleton has a subarc $\alpha : [0, \pi] \to L_\Gamma$ of length π so that either*

1. *$\alpha(t)$ is non-singular for all t,*
2. *or $\alpha(t)$ is singular if and only if $t = 0$ or π.*

Proof. Suppose γ is a closed geodesic in L_Γ. We may assume that γ is not an edge path, and so we parameterize γ so that it begins in general position.

Suppose that γ does not contain any singular vertices. Choose a maximal subarc α which does not contain any edges of L_Γ. Thicken the gallery determined by α and develop it onto Σ using Proposition 12.1. The thickened gallery develops onto subcomplexes of the all-right triangles as depicted in Figure 11. Fix one such all-right triangle Δ. The simplices in L_Γ which develop onto Δ all belong to the same maximal subcomplex L_T for some $T \in \mathcal{S}$. The only edges which are common to two distinct maximal subcomplexes belong to a piece $\bullet - \circ - \bullet$ or $\bullet - - \bullet$.

The spherical subset T is determined by either a vertex of length three or by the product of a length one and a length two vertex with opposite sign. So, if γ is extra short, then it is contained in a subcomplex $L_{\Gamma'}$ defined by an Artin group with small diameter. But, by Corollary 9.5, such a γ has length $\geq 2\pi$. Hence, we can find a subarc α of type 1 above.

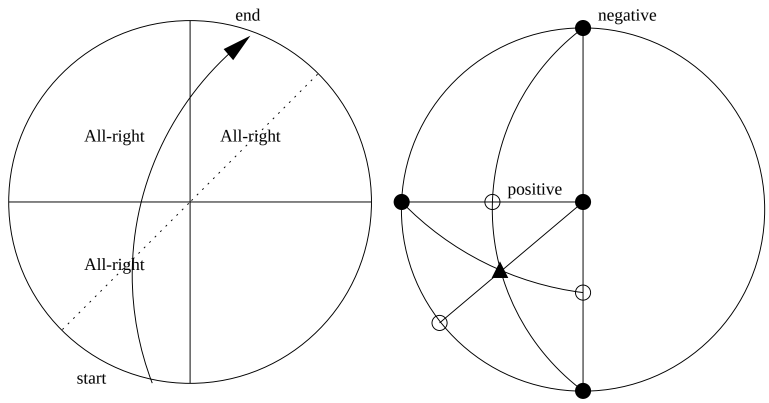

Figure 11. The all-right triangles encode the maximal subcomplexes L_T which contain the local geodesic γ. The lift of a closed geodesic of length $\leq \pi$ meets at most 3 all-right triangles. Each vertex of length 3 and each edge $\circ - - \bullet$ determines a maximal subcomplex.

If γ contains a singular vertex, we choose the parametrization so that $\gamma(0)$ is singular. By Proposition 13.1, γ has length $\geq \pi$. If $\gamma(\pi)$ is singular, then $\alpha = \gamma|_[0, \pi]$ is a subarc of type 2. If $\gamma(\pi)$ is non-singular, then, by tracing the curve a little bit farther and deleting the initial segment, we obtain a subarc of type 1. In either case, we can develop the thickened gallery determined by α onto Σ, the image lying in at most three all-right triangles. If γ was extra-short, then $\gamma = \alpha$ would be a geodesic loop in a subcomplex defined by an Artin group with small diameter, contradicting Corollary 9.5. $\square$

14. Shrinking and rotating local geodesics

It remains to show that L does not contain any isometrically embedded circles of length $< 2\pi$ which do not lie entirely in the 1-skeleton. The arguments are inspired by an alternate characterization of CAT(1) spaces due to B. Bowditch [5]. The actual implementation of Bowditch's ideas are in the spirit of the curvature testing techniques in [23] and, especially, the more recent paper by M. Elder, J. McCammond, and J.Meier [25]. The following theorems of B. Bowditch [5] apply to $X = L_\Gamma$:

Theorem 14.1. *(Bowditch) Let X be a compact locally CAT(1) space. If X is not CAT(1), then there exists a minimal length closed geodesic of length $< 2\pi$.*

A homotopy $H : [a, b]\times I \to X$ is a *monotone homotopy* if length(H_s) $\leq$ length(H_t) for $s < t$. We say that H_0 is *monotonically homotopic* to H_1. (This is not a symmetric relation.) A loop which is monotonically homotopic (through loops) to a constant loop is said to be *shrinkable.*

Theorem 14.2. *(Bowditch) Let X be a compact locally CAT(1) space. If γ is a closed geodesic in X of length $< 2\pi$, then γ is not shrinkable.*

Theorem 14.3. *(Bowditch) Let X be a compact locally CAT(1) space. If γ is a loop in X of length $< 2\pi$, then either γ is shrinkable or γ is monotonically homotopic to a closed geodesic γ'.*

The above theorems use a reformulation of the locally CAT(1) condition in terms of the length of a minimal closed geodesic. Bowditch defines a space to be ϵ-CAT(1) if every triangle of perimeter $< 2\epsilon$ satisfies the CAT(1) inequality. To prove Theorem 14.1, he shows that X is ϵ-CAT(1) and contains an isometrically embedded circle of length 2ϵ. For Theorems 14.2 and 14.3 he uses the Birkhoff curve shortening process. This process takes a closed loop and iterates the process by which we subdivide the loop into segments, join the midpoints of adjacent segments by geodesics, and consider this new loop as the next input.

Suppose that A_Γ is a 3-dimensional FC Artin group. If L_Γ is CAT(1), we are done; otherwise, by Theorem 14.1, we may assume there exists a closed geodesic γ in L_Γ of length $< 2\pi$. By Theorem 11.4, γ is not an edge loop. By Theorem 13.2, γ has length $> \pi$ and contains a subarc α of length equal to π of type 1 or 2.

A *rotation* of α is a length preserving homotopy of α which leaves its endpoints fixed. The loop $rot(\gamma)$ obtained by removing the arc α and replacing it by the rotated arc is said to be *obtained by rotating the arc α*. In particular, γ and $rot(\gamma)$ have the same length; however, in general, $rot(\gamma)$ is not a geodesic.

If $\alpha(t)$ is non-singular for all t (type 1), then we may rotate the arc α by a small amount. First, develop the thickened gallery determined by α onto Σ. Then rotate the lift of α by a small amount within the developed gallery. The length is preserved and the endpoints are fixed because the lift of α is, by construction, a great arc of length π in Σ. Pasting these homotopies, cell by cell, we obtain a rotation of α in L_Γ.

The loop $rot(\gamma)$ obtained by rotation fails to be geodesic at the non-singular endpoint of α. (This need not be the case if the endpoints were singular!) Choose small balls about each endpoint so that $rot(\gamma)$ meets each ball in two points. Then join each pair of points by geodesics. The resulting loop γ' has length strictly less than the length of γ. Moreover, we may realize this reduction by a sequence of monotone homotopies leaving the endpoints fixed. Thus, γ is monotonically homotopic to a loop γ' of strictly smaller length. According to Theorems 14.3, γ' is either shrinkable or monotonically homotopic to a closed geodesic. It cannot be shrinkable; for, otherwise, γ is shrinkable, contradicting Theorem 14.2. But it cannot be monotonically homotopic to a closed geodesic either— γ was minimal. This is a contradiction.

If α is of type 2, then $\alpha(t)$ is non-singular for $0 < t < \pi$ and singular for $t = 0$ and π. As above, we consider the thickened gallery determined by α and develop it onto a subcomplex of Σ. If $\alpha(0)$ is a vertex of length one, then we may rotate α into the 1-skeleton of L_Γ. Observe, in Figure 9 (right), that we may rotate $\phi(\hat{\alpha})$ to the boundary of the thickened gallery. Pasting these homotopies cell by cell, we obtain a rotation of α.

Similarly, if $\alpha(0)$ is a vertex of length three, then we may rotate $\phi(\hat{\alpha})$ into the top arc of the boundary of the developed gallery in Figure 9 (left). This homotopy induces a rotation of α into the 1-skeleton of L_Γ.

Now consider the rotated loop $rot(\gamma)$. The subarc of the loop which was fixed by the rotation $(\gamma - \alpha)$ still joins two singular vertices. By the same reasoning as in the proof of Theorem 13.2, this subarc also has length $\geq \pi$. Therefore, γ had length $\geq 2\pi$. This is a contradiction. Therefore, L_Γ contains no short loops. Thus, we have completed the proof of the following:

Theorem 14.4. *If Γ defines a 3-dimensional FC Artin system, then the link L_Γ is CAT(1).*

15. Results and concluding remarks

Theorem 14.4, combined with the local to global theorem, completes the proof of the Main Theorem:

Theorem 15.1. *If Γ defines a 3 dimensional FC Artin system, then $A_\Gamma \cong \pi_1(K_\Gamma, v_0)$ is CAT(0): it acts geometrically on the universal cover of K_Γ by deck transformations.*

The proof given also shows that FC Artin systems of dimension ≤ 3 are CAT(0). Two dimensional FC Artin groups were shown to be

CAT(0) by T. Brady and J. McCammond in [9]. Brady and McCammond used the same cell complex K, but with a different metric: every edge was assigned length one, so that boundary of every 2-cell was an equilateral triangle. Checking the link condition was equivalent to deciding if L contained any edge loops of fewer than six edges.

Recently, W. Choi [18] has proven that for most spherical Artin groups of higher dimensions, the Brady–Krammer complex is not CAT(0), at least with respect to any "obvious" metric. Still, the Brady–Krammer complexes are good candidate for $K(\pi, 1)$ spaces. Meanwhile, the question of whether Artin groups are CAT(0) remains open.

Acknowledgements

I would especially like to thank my advisor, Ruth Charney, for her advice, criticism, and encouragement. She was a daily source of inspiration while researching this problem at The Ohio State University. I would also like to thank Mike Davis and Jon McCammond for many helpful conversations and for their encouragement on this project. And finally I would like to thank the reviewer for his careful reading of this manuscript and for his many helpful suggestions and improvements; in particular, his contributions to sections 3 and 11 clarified and simplified many of the original arguments.

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