INJECTIONS OF ARTIN GROUPS

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January 5, 2005

Abstract. We study those Artin groups which, modulo their centers, are finite index subgroups of the mapping class group of a punctured sphere. In particular, we show that any injective homomorphism between these groups is parameterized by a homeomorphism of a punctured disk together with a homomorphism to the integers. The technique, following Ivanov, is to prove that every superinjective map of the complex of curves of a sphere with at least 5 punctures is induced by a homeomorphism.

1. Introduction

We investigate injective homomorphisms between those Artin groups which, modulo their center, embed with finite index in the mapping class group of a punctured sphere. S will always denote a sphere with $m \geq 5$ punctures. The *extended mapping class group* of a surface F is the group of isotopy classes of homeomorphisms of F:

$$\mathrm{Mod}^{\pm}(F) = \pi_0(\mathrm{Homeo}(F)).$$

The *mapping class group* $\mathrm{Mod}(F)$ is the subgroup of orientation preserving *mapping classes* (elements of $\mathrm{Mod}^{\pm}(F)$).

Main Theorem. *Suppose G is a finite index subgroup of* $\mathrm{Mod}(S)$. *Then every injective homomorphism $\rho : G \to \mathrm{Mod}(S)$ is of the form $\rho(g) = fgf^{-1}$ for some $f \in \mathrm{Mod}^{\pm}(S)$.*

2000 *Mathematics Subject Classification.* Primary: 20F36; Secondary: 57M07.

Key words and phrases. Artin groups, mapping class groups, complex of curves, co-Hopfian.

This first author is partially supported by a VIGRE postdoctoral position under NSF grant number 0091675 to the University of Utah. The second author is supported by an NSF postdoctoral fellowship.

In particular, the Main Theorem applies to four infinite families of Artin groups modulo their centers: $A(A_n)/Z$, $A(B_n)/Z$, $A(\widetilde{C}_{n-1})$, and $A(\widetilde{A}_{n-1})$ (defined below) where $n = m-2$. Throughout, Z denotes the center of the ambient group; the groups $A(\widetilde{C}_{n-1})$ and $A(\widetilde{A}_{n-1})$ have trivial center.

The Main Theorem and Corollary 2 (below) were proved for the group $G = A(A_n)/Z$ in an earlier paper [2]. These results can also be viewed as a generalization of the work of Charney and Crisp, who computed the automorphism groups of the aforementioned Artin groups using similar techniques [7].

To prove the Main Theorem, we translate the problem into one about the complex of curves $\mathrm{C}(S)$, which is an abstract simplicial complex with vertices corresponding to isotopy classes of curves in S and edges corresponding to disjoint pairs of curves. To this end, we focus on particular elements of G—powers of Dehn twists; each such element is associated to a unique isotopy class of curves in S. We show that the injection ρ must take a power of a Dehn twist to a power of a Dehn twist, thus giving an action $\rho_\star$ on the vertices of $\mathrm{C}(S)$. Since $\rho_\star$ is easily shown to be superinjective in the sense of Irmak (i.e. $\rho_\star$ preserves disjointness and nondisjointness), we will be able to derive the Main Theorem from the following theorem.

Theorem 1. *Suppose that $m \geq 5$. Then every superinjective map of $\mathrm{C}(S)$ is induced by an element of $\mathrm{Mod}^{\pm}(S)$.*

The proofs of both theorems are modeled on previous work of Ivanov, who showed that the abstract commensurator of $\mathrm{Mod}(F)$, for F a closed surface of genus at least 3, is isomorphic to $\mathrm{Mod}^{\pm}(F)$ [21]. To do this, he applied his theorem that $\mathrm{Aut}(\mathrm{C}(F)) \cong \mathrm{Mod}^{\pm}(F)$ in these cases. His method has been used to prove similar theorems by Korkmaz [27], Ivanov and McCarthy [23], Schmutz Schaller [31], Irmak [18] [17] [16], Margalit [28], Irmak, Ivanov, and McCarthy [19], Farb and Ivanov [12], McCarthy and Vautaw [29], Brendle and Margalit [5], and Irmak and Korkmaz [20]. In particular, Korkmaz proved that every element of $\mathrm{Aut}\,\mathrm{C}(S)$ is induced by an element of $\mathrm{Mod}^{\pm}(S)$ [27], and Irmak showed that every superinjective map of $\mathrm{C}(F)$, for higher genus F, is induced by an element of $\mathrm{Mod}^{\pm}(F)$, thus obtaining the analog of the Main Theorem for higher genus surfaces [18, 17].

Artin groups. Before we explain the applications of the Main Theorem to Artin groups, we recall the basic definitions. An *Artin group* is any group with a finite set of generators $\{s_1, \ldots, s_n\}$ and, for each $i \neq j$, a defining relation of the form

$$s_i s_j \cdots = s_j s_i \cdots ,$$

where $s_i s_j \cdots$ denotes an alternating string of $m_{ij} = m_{ji}$ letters. The value of m_{ij} must lie in the set $\{2, 3, \ldots, \infty\}$ with $m_{ij} = \infty$ signifying that there is no defining relation between s_i and s_j.

It is convenient to define an Artin group by a *Coxeter graph*, which has a vertex for each generator s_i, and an edge labelled m_{ij} connecting the vertices corresponding to s_i and s_j if $m_{ij} > 2$. The label 3 is suppressed. The Coxeter graphs A_n, B_n, $\widetilde{C}_{n-1}$, and $\widetilde{A}_{n-1}$ for the Artin groups $A(A_n)$, $A(B_n)$, $A(\widetilde{C}_{n-1})$, and $A(\widetilde{A}_{n-1})$ are displayed in Figure 1.

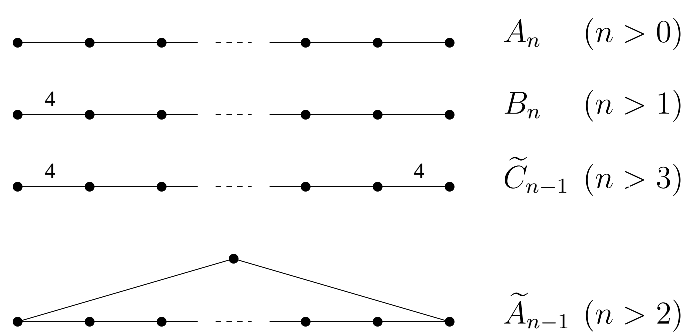

FIGURE 1. Coxeter graphs with n vertices.

Artin groups and mapping class groups. The Artin group $A(A_n)$ is better known as the braid group on $n+1$ strands. Artin's classical work on braids identifies $A(A_n)/Z$ with the the subgroup of $\mathrm{Mod}(S)$ consisting of elements which fix one particular puncture (see [3]).

The pure braid group on $n+1$ strands $P(A_n)$ is defined as the kernel of the map from $A(A_n)$ to the symmetric group on $\{1, \ldots, n+1\}$ which sends each s_i to the transposition switching i and $i+1$. Thus, the group $P(A_n)/Z$ can be identified with the *pure mapping class group* $\mathrm{PMod}(S)$, which is the subgroup of $\mathrm{Mod}(S)$ consisting of mapping classes which fix every puncture.

The work of Allcock, Charney, Crisp, Kent, and Peifer shows that $A(\widetilde{A}_{n-1})$, $A(B_n)/Z$, and $A(\widetilde{C}_{n-1})$ are also isomorphic to finite index subgroups of $\mathrm{Mod}(S)$ [1] [7] [8] [9] [24]. A complete description of these isomorphisms appears in the recent paper of Charney and Crisp [7].

$$\begin{array}{ccccccccccccccc} & & & & & & & & & & A(\widetilde{A}_{n-1}) & & & & \\ & & & & & & & & & & \downarrow & & & & \\ \mathrm{PMod}(S) & \longrightarrow & G_{m-2} & \longrightarrow & \cdots & \longrightarrow & G_4 & \longrightarrow & A(\widetilde{C}_{n-1}) & \longrightarrow & A(B_n)/Z & \longrightarrow & A(A_n)/Z & \longrightarrow & \mathrm{Mod}(S) \end{array}$$

FIGURE 2. The diagram for Corollary 3.

Applications. We now give some consequences of the Main Theorem for injective maps between Artin groups. A group is *co-Hopfian* if each of its injective endomorphisms is an isomorphism.

Corollary 2. *For $n \geq 3$, all finite index subgroups of $\mathrm{Mod}(S)$ are co-Hopfian; in particular, the groups $A(A_n)/Z$, $A(B_n)/Z$, $A(\widetilde{C}_{n-1})$, $A(\widetilde{A}_{n-1})$, and $P(A_n)/Z$ are co-Hopfian.*

For each $0 \leq k \leq m$, let G_k be the subgroup of $\mathrm{Mod}(S)$ consisting of elements which fix k given punctures. Note that $G_0 = \mathrm{Mod}(S)$, $G_1 = A(A_n)/Z$, and $G_{m-1} = G_m = \mathrm{PMod}(S)$. Also, $G_2 = A(B_n)/Z$ and $G_3 = A(\widetilde{C}_{n-1})$ (see [7]).

Corollary 3. *Suppose $n \geq 3$ and let G and H be any of the groups in Figure 2. Then there exists an injection $\rho : G \to H$ if and only if there is directed path from G to H in Figure 2. Further, any such injection is of the form $\rho(g) = fgf^{-1}$ for some fixed $f \in \mathrm{Mod}^{\pm}(S)$.*

We are also able to characterize injections between the groups $A(A_n)$, $A(B_n)$, and $P(A_n)$ (with their centers). There are inclusions: $P(A_n) \to A(B_n) \to A(A_n)$ (see Section 5); all other injections between these groups are described by the following result:

Theorem 4. *Suppose $n \geq 3$. Let G be a finite index subgroup of $A(A_n)$. If $\rho : G \to A(A_n)$ is an injection, then there exists an $f \in \mathrm{Mod}^{\pm}(S)$ and a homomorphism $t : G \to \mathbb{Z}$ such that*

$$\rho(g) = fgf^{-1}\ z^{t(g)}$$

where z generates the center of $A(A_n)$.

Theorem 4 was proven for $G = A(A_n)$ in the authors' earlier work [2]. In this case, t is an integral multiple of the *length homomorphism* $L : A(A_n) \to \mathbb{Z}$, defined by $s_i \mapsto 1$ for each i.

Finally, as a corollary of Theorem 4, we will prove the following:

Corollary 5. *Suppose $n \geq 3$ and let $N = \binom{n+1}{2}$. Then*

$$\mathrm{Aut}(P(A_n)) \cong \mathrm{Mod}^{\pm}(S) \ltimes \mathbb{Z}^{N-1}.$$

It follows that $\mathrm{Out}(P(A_n)) \cong (\Sigma_{n+2} \times \mathbb{Z}_2) \ltimes \mathbb{Z}^{N-1}$, where Σ_{n+2} is the symmetric group on $n+2$ letters.

Outline. Section 2 contains preliminary definitions and ideas used in the paper. Section 3 gives a proof of the Main Theorem, assuming Theorem 1. We prove Theorem 4 in Section 4, and use Theorem 4 to give a complete classification of injections of $P(A_n)$, $A(B_n)$, and $A(A_n)$ into $A(A_n)$ in Section 5. Finally, Section 6 contains a proof of Theorem 1.

Acknowledgements. The authors would like to thank Benson Farb for bringing the recent work of Charney and Crisp to our attention and for his continued encouragement on this project. Both Joan Birman and Luis Paris were generous with their time and energy; for this we are thankful. We are also grateful to Mladen Bestvina, Tara Brendle, Ruth Charney, and Michah Sageev for helpful conversations.

2. Background

Curves. By a *curve* in a surface F, we mean the isotopy class of a simple closed curve in F which does not bound a disk or a once punctured disk. We will often not make the distinction between a representative curve and its isotopy class. A curve is *peripheral* if it is represented by a component of ∂F.

We denote by $i(a, b)$ the *(geometric) intersection number* between two curves a and b. We say that a and b are *disjoint* if $i(a, b) = 0$. A maximal collection of pairwise disjoint nonperipheral curves in S is called a *pants decomposition.* Any pants decomposition of S or a disk with $m-1$ punctures has $m-3$ curves.

Complex of curves. The *complex of curves* $\mathrm{C}(F)$ for a surface F (defined by Harvey [14]) is an abstract simplicial flag complex with a vertex for each nonperipheral curve in F and an edge for each pair of disjoint curves.

Superinjective maps. A simplicial map ϕ of $\mathrm{C}(F)$ to itself is called *superinjective* if it preserves nondisjointness: that is, if v and w are vertices not connected by an edge, then $\phi(v)$ and $\phi(w)$ are not connected (the corresponding curves intersect). It is straightforward to show that superinjective maps are injective.

Dehn twists. A *Dehn twist* about a curve a is the mapping class T_a which has support on an annular neighborhood N of a, and is described on N by Figure 3.

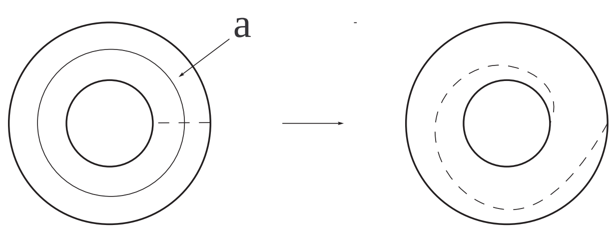

FIGURE 3. Dehn twist about a curve a.

For each $f \in \mathrm{Mod}^{\pm}(S)$, let $\epsilon(f) = 1$ if f preserves orientation and $\epsilon(f) = -1$ if not. We will use the following connection between the topology and algebra of Dehn twists in $\mathrm{Mod}(S)$:

Fact 6. *Suppose* $f \in \mathrm{Mod}^{\pm}(S)$. *Then* $fT_af^{-1} = T_{f(a)}^{\epsilon(f)}$. *In particular,* $[f, T_a] = 1$ *implies* $f(a) = a$, *and powers of Dehn twists commute if and only if the curves are disjoint.*

For a group Γ, we define its *rank*, $\mathrm{rk}(\Gamma)$, to be the maximal rank of a free abelian subgroup of Γ. It follows from work of Birman, Lubotzky, and McCarthy that for any surface F, $\mathrm{rk}\,\mathrm{Mod}(F)$ is realized by any subgroup generated by powers Dehn twists about curves forming a pants decomposition for F [4]; thus, $\mathrm{rk}\,\mathrm{Mod}(S) = m-3$. The following theorem of Ivanov gives another connection between the algebra and topology of $\mathrm{Mod}(S)$ [21].

Theorem 7. *Let* P *be a finite index subgroup of* $\mathrm{PMod}(S)$. *An element* g *of* P *is power of Dehn twist if and only if* $Z(C_P(g)) \cong \mathbb{Z}$ *and* $\mathrm{rk}\, C_P(g) = m-3$.

Remark. Ivanov's theorem holds more generally for any surface F of negative Euler characteristic, with $\mathrm{PMod}(S)$ replaced by any finite index subgroup of $\mathrm{Mod}(F)$ consisting of "pure" mapping classes [23]. For a surface of arbitrary genus, a mapping class f is called pure if whenever $f^n(c) = c$, where c is a curve in F and $n \neq 0$, we have $f(c) = c$. In the genus zero setting, though, it is a theorem of Irmak, Ivanov, and McCarthy that the pure mapping classes of $\mathrm{Mod}(S)$ are exactly the elements of $\mathrm{PMod}(S)$ [19].

We now state a group theoretical lemma, due to Ivanov and McCarthy [23], which will be used in Proposition 9.

Lemma 8. *Let $\rho : \Gamma \to \Gamma'$ be an injection, where $\mathrm{rk}\,\Gamma' = \mathrm{rk}\,\Gamma < \infty$. Let $G < \Gamma$ be a free abelian subgroup of maximal rank, and let $g \in G$. Then*

$$\mathrm{rk}\, Z(C_{\Gamma'}(\rho(g))) \leq \mathrm{rk}\, Z(C_\Gamma(g)).$$

Note that Lemma 8 applies whenever g is a power of a Dehn twist and Γ and Γ' are finite index subgroups of $\mathrm{PMod}(S)$.

3. Proof of Main Theorem

Let $\rho : G \to \mathrm{Mod}(S)$ be an injective homomorphism, where G is a finite index subgroup of $\mathrm{Mod}(S)$.

Proposition 9. *For each curve a in S, there are nonzero integers k and k' and a curve a' such that $\rho(T_a^k) = T_{a'}^{k'}$.*

Proof. Let $Q = \mathrm{PMod}(S)$, and let $P = \mathrm{PMod}(S) \cap \rho^{-1}(Q)$. Since P is a finite index subgroup of $\mathrm{Mod}(S)$, we can choose a k so that $g = T_a^k$ belongs to P. By Theorem 7, $Z(C_P(g)) \cong \mathbb{Z}$. Lemma 8 and the fact that ρ is injective imply that $Z(C_Q(\rho(g)) \cong \mathbb{Z}$. Since $\mathrm{rk}\,\rho(C_P(g)) = \mathrm{rk}\,\mathrm{Mod}(S)$, Theorem 7 says that $\rho(g)$ must be a power of a Dehn twist.

□

By Proposition 9, ρ induces a well-defined action $\rho_\star$ on curves given by

$$\rho(T_a^k) = T_{\rho_\star(a)}^{k'}.$$

Proposition 10. *The map $\rho_\star$ is a superinjective map of $\mathrm{C}(S)$.*

Proof. We make repeated use of Fact 6. First, the map $\rho_\star$ is simplicial since if a and b are disjoint curves, then $\rho(T_a^k)$ and $\rho(T_b^k)$ commute, and so the curves $\rho_\star(a)$ and $\rho_\star(b)$ are disjoint. The map $\rho_\star$ is superinjective since if $\mathrm{i}(a, b) \neq 0$ then T_a^k and T_b^k do not commute; since ρ is an injection, $[T_a^k, T_b^k] \neq 1$, and so $\rho_\star(a)$ and $\rho_\star(b)$ are not disjoint.

□

We are now ready to complete the proof of the Main Theorem, assuming Theorem 1.

Proof. By Proposition 9 and 10, the injection ρ gives rise to a map $\rho_\star$ of $\mathrm{C}(S)$, which by Theorem 1 is induced by some $f \in \mathrm{Mod}^{\pm}(S)$; that is to say, $\rho_\star(c) = f(c)$ for every curve c. To see that $\rho(g) = fgf^{-1}$ for every $g \in G$, we check that $fg(c) = \rho(g)f(c)$ for any curve c:

$$T_{fg(c)}^{k'} = \rho(T_{g(c)}^{k}) = \rho(gT_c^k g^{-1}) =$$

$$\rho(g)\rho(T_c^k)\rho(g)^{-1} = \rho(g)T_{f(c)}^{k''}\rho(g)^{-1} = T_{\rho(g)f(c)}^{k''}$$

Thus, $T_{fg(c)}^{k'} = T_{\rho(g)f(c)}^{k''}$, which immediately implies that $fg(c) = \rho(g)f(c)$.

□

4. Proof of Theorem 4

We now turn our attention to injections between the groups $P(A_n)$, $A(B_n)$, and $A(A_n)$ for $n \geq 3$. These groups also have topological descriptions. In particular, if D_{n+1} is the disk with $n+1$ punctures, and $\mathrm{Homeo}^+(D_{n+1}, \partial D_{n+1})$ is the space of homeomorphisms of D_{n+1} which are the identity on the boundary, then

$$A(A_n) = \pi_0(\mathrm{Homeo}^+(D_{n+1}, \partial D_{n+1})).$$

The group $A(B_n)$ is isomorphic to the subgroup of $A(A_n)$ fixing one given puncture, and $P(A_n)$ is the subgroup fixing all punctures (see [1] or [7]). Both $A(B_n)$ and $P(A_n)$ are finite index subgroups of $A(A_n)$. The center of $A(A_n)$, denoted Z, is generated by z, the Dehn twist about a curve isotopic to ∂D_{n+1}. Both $A(B_n)$ and $P(A_n)$ inherit the same center.

The *interior* of a curve a in D_{n+1} is the component of $D_{n+1} - a$ which does not contain ∂D_{n+1}. A curve in D_{n+1} is a *k-curve* if it has exactly k punctures in its interior.

Lemma 11. *If f is a finite order element of $\mathrm{Mod}(D_{n+1})$, where $n \geq 3$, and f fixes each curve in a pants decomposition $\mathcal{P}$ of D_{n+1}, then f is the identity.*

Proof. We can consider two cases, as any pants decomposition of D_{n+1} either contains two 2-curves or a 2-curve nested inside a 3-curve.

In the first case, since f fixes the curves of $\mathcal{P}$, we can collapse the 2-curves and ∂D_{n+1} to points, and f induces a finite order homeomorphism $\bar{f}$ of this new surface $\bar{S}$. By the Nielsen realization theorem, we can choose a metric for $\bar{S}$ so that $\bar{f}$ has a representative, which we also call $\bar{f}$, that is an isometry [25]. It is then a classical theorem of Kerékjártó, Brouwer, and Eilenberg that $\bar{f}$ is conjugate (via homeomorphism) to a Euclidean isometry [6] [11] [26]. Since $\bar{f}$ fixes three points (the collapsed curves), it follows that $\bar{f}$ is the identity. Since f is finite order, it must also be the identity.

In the latter case, if we collapse the 2-curve and ∂D_{n+1} to points, then, as above, f induces a finite order homeomorphism $\bar{f}$ fixing three points (one of which is the puncture in the interior of the 3-curve but not the 2-curve). By the same argument, f is the identity.

□

As with the Main Theorem, the proof of Theorem 4 requires the existence of a superinjective map $\rho_\star$ of $\mathrm{C}(D_{n+1})$ which is induced by ρ in the sense that for any curve a we have

$$\rho(T_a^k) = T_{\rho_\star(a)}^{k'} z^{k''}$$

for some integers k, k', and k'' (k and k' nonzero). The argument is exactly the same as in Proposition 9, with Theorem 7 replaced by the following theorem, which has the same proof as Theorem 7.

Theorem 12. *Let P be a finite index subgroup of $P(A_n)$. An element g of P is the product of a central element and a nontrivial power of a noncentral Dehn twist if and only if $Z(C_P(g)) \cong \mathbb{Z}^2$ and $\operatorname{rk} C_P(g) = n$.*

We now prove Theorem 4.

Proof. As in the statement of the theorem, let G be a finite index subgroup of $A(A_n)$. We know that G has nontrivial center $Z(G)$ since it is finite index in $A(A_n)$. Further we have $Z(G) \cong \mathbb{Z}$. Indeed, if y is an element of $Z(G)$, then y must fix every curve in D_{n+1} by Fact 6 and the fact that G is finite index; hence y is a power of z.

Let ζ denote a generator of $Z(G)$. We now show that $\rho(Z(G)) < Z$ by showing $\rho(\zeta) \in Z$. Since $\operatorname{rk} G = \operatorname{rk} A(A_n)$, we have that $\rho(\zeta^k) \in Z$ for some nonzero k. Thus, $\rho(\zeta)$ is a root of a central element of $A(A_n)$.

Choose a pants decomposition $\mathcal{P}$ of D_{n+1}. As in Section 3, we know that $\rho_\star(\mathcal{P})$ is also a pants decomposition. Further, since ζ is central and ρ is injective, it follows that $\rho(\zeta)$ fixes each element of $\rho_\star(\mathcal{P})$. Let π be the quotient map $A(A_n) \to A(A_n)/Z$. Since $\pi(\rho(\zeta))$ is finite order in $A(A_n)/Z$, it follows from Lemma 11 that $\pi(\rho(\zeta)) = 1$, and so $\rho(\zeta) \in Z$.

Moreover, we have that $\rho^{-1}(Z) < Z(G)$, by the injectivity of ρ. Thus, ρ induces a well-defined injection $\bar{\rho} : G/Z(G) \to A(A_n)/Z$. Since $G/Z(G)$ is finite index in $A(A_n)/Z$, we may apply the Main Theorem. Interpreted appropriately, this means that there is an $f \in \mathrm{Mod}^\pm(D_{n+1})$ so that

$$\bar{\rho}(g) = fgf^{-1}$$

for all $g \in G/Z(G)$. It follows that for any $g \in G$, there is an integer $t(g)$ so that $\rho(g) = fgf^{-1}z^{t(g)}$. That t is a well-defined homomorphism follows from the fact that ρ is a homomorphism.

□

5. Catalogue of injections

We now use Theorem 4 to list all injections of the groups $P(A_n)$, $A(B_n)$, and $A(A_n)$ into $A(A_n)$. As in the previous section, we denote the generator of Z by z.

$A(A_n)$ and $A(B_n)$ are defined via the presentations given by Figure 1; here we denote the generators for $A(A_n)$ by $\sigma_1, \ldots, \sigma_n$, and the generators for $A(B_n)$, from left to right, by $s_1, \ldots, s_n$. The usual inclusion $A(B_n) \to A(A_n)$ is given by $s_1 \mapsto \sigma_1^2$ and $s_i \mapsto \sigma_i$ for $i > 1$. The standard generators of $P(A_n)$ are

$$a_{i,j} = (\sigma_{j-1} \cdots \sigma_{i+1})\sigma_i^2(\sigma_{j-1} \cdots \sigma_{i+1})^{-1}$$

where $1 \le i < j \le n+1$. The defining relations of $P(A_n)$ are of the form $xa_{i,j}x^{-1} = ya_{i,j}y^{-1}$ (see [3]).

In what follows, t is the integral homomorphism defined by the statement of Theorem 4.

Injections of $A(A_n)$. For $A(A_n)$ the solution was given in the previous paper of the authors [2]. In that case, since all standard generators are conjugate, we have that $t(\sigma_i)$ is the same for all i. Thus, any choice of $t = t(\sigma_1)$ determines a homomorphism. This map can alternately be described as

$$g \mapsto fgf^{-1}z^{L(g)t}$$

for some $f \in \mathrm{Mod}^{\pm}(D_{n+1})$, where L is the length homomorphism. Further, any such choice of t does lead to an injection. Indeed, if $fgf^{-1}z^{L(g)t} = 1$, it follows that $g \in Z$, and that $L(g)t + 1 = 0$. Since $z = (\sigma_1 \cdots \sigma_n)^{n+1}$, we have $L(z) = n(n+1)$, so there is no choice of t which satisfies the last equation. Thus, we have an injection for any t; moreover, the map is not surjective when $t \neq 0$: the preimage of Z is Z, yet $z \mapsto z^{1+n(n+1)}$, so nothing maps to z.

It follows that $\mathrm{Aut}(A(A_n)) \cong \mathrm{Mod}^{\pm}(D_{n+1})$. This was first proven by Dyer and Grossman [10]. Ivanov was the first to compute $\mathrm{Aut}(A(A_n))$ from the perspective of mapping class groups [22].

Injections of $A(B_n)$. For $A(B_n)$, since the s_i are conjugate for $i > 1$, we have that $t(s_i)$ is the same for these i. If $u = t(s_1)$ and $v = t(s_2)$, then we have a well-defined homomorphism from $A(B_n)$ to $A(A_n)$ given on generators by

$$s_1 \mapsto fs_1f^{-1}\ z^u \qquad s_i \mapsto fs_if^{-1}\ z^v \quad \text{for } i > 1$$

for any $f \in \mathrm{Mod}^{\pm}(D_{n+1})$. If $g \mapsto 1$, we again have that $g \in Z$. Since $z = (s_1 \cdots s_n)^n$, we have $z \mapsto z^{1+nu+n(n-1)v}$. But there are no u and v which make the latter trivial (as n and $n(n-1)$ are not relatively prime), so every choice of u and v leads to an injection.

It follows immediately from Theorem 4 that all injections $A(B_n) \to A(B_n)$ are of the same form, where f is required to fix the puncture fixed by all of $A(B_n)$. We now answer the question of which u and v give automorphisms of $A(B_n)$. Again we note that the preimage of z must be central, say z^q. But we have $z^q \mapsto z^{q(1+nu+n(n-1)v)}$. So for z to be in the image, we need to choose u and v so that $nu + n(n-1)v = 0$. For such u and v, the map is an isomorphism, as the elements $(f^{-1}s_1f)z^{-u}$ and $(f^{-1}s_if)z^{-v}$ (for $i > 1$) map to the standard generators.

The above analysis appears in the work of Charney and Crisp; they prove that $\mathrm{Aut}(A(B_n)) \cong (G_1 \times \mathbb{Z}) \rtimes (\mathbb{Z}_2 \times \mathbb{Z}_2)$ [7].

Injections of $P(A_n)$. Since all of the defining relations of $P(A_n)$ are of the form $xa_{i,j}x^{-1} = ya_{i,j}y^{-1}$, it follows that even if the $t(a_{i,j})$ are all different, we have a well-defined homomorphism from $P(A_n)$ to $A(A_n)$ given by

$$a_{i,j} \mapsto fa_{i,j}f^{-1}z^{t(a_{i,j})}$$

for any $f \in \mathrm{Mod}^{\pm}(D_{n+1})$. Again, the kernel must be contained in Z. In the generators of $P(A_n)$, z can be written as

$$(a_{1,2}\, a_{1,3}\, \cdots\, a_{1,n+1}) \cdots (a_{n-1,n}\, a_{n-1,n+1})\, (a_{n,n+1}).$$

Thus, we see that

$$z \mapsto z^{1+\sum t(a_{i,j})}.$$

Hence, there is an affine hyperplane in $\mathbb{Z}^N$, where $N = \binom{n+1}{2}$, corresponding to noninjective homomorphisms of $P(A_n)$ into $A(A_n)$.

Again, we have that injections of $P(A_n)$ into itself are also of the above form. We now decide when these injections are automorphisms. As before, for z to lie in the image, we need $z \mapsto z$, that is, $\sum t(a_{i,j}) = 0$. Now we need to check if, under this condition, the other generators of $P(A_n)$ are in the image. As before, this is no problem: $(f^{-1}a_{i,j}f)z^{-t(a_{i,j})} \mapsto a_{i,j}$.

Proof of Corollary 5. Charney and Crisp define the *transvection subgroup* $T(P(A_n))$ of $\mathrm{Aut}(P(A_n))$ as the collection of automorphisms of the form $x \mapsto xz^{t(x)}$. Above, we have shown that $T(P(A_n)) \cong \mathbb{Z}^{N-1}$.

There is an exact sequence

$$1 \to T(P(A_n)) \to \mathrm{Aut}(P(A_n)) \to \mathrm{Aut}(P(A_n)/Z) \to 1.$$

Using Korkmaz's theorem that $\mathrm{Aut}(P(A_n)/Z)$ is isomorphic to $\mathrm{Mod}^{\pm}(S)$ [27], we define a splitting by sending $f \in \mathrm{Mod}^{\pm}(S)$ to the automorphism of $P(A_n)$ mapping x to fxf^{-1}. It immediately follows that $\mathrm{Aut}(P(A_n)) \cong \mathrm{Mod}^{\pm}(S) \ltimes \mathbb{Z}^{N-1}$. We note that the action of $\mathrm{Mod}^{\pm}(S)$ on $T(P(A_n))$ is given by

$$f \cdot (x \mapsto xz^{t(x)}) = (x \mapsto xz^{t(f^{-1}xf)}).$$

Remark. The *abstract commensurator* $\mathrm{Comm}(G)$ of a group G is the collection of isomorphisms of finite index subgroups of G, where two such isomorphisms are equivalent if they agree on some common finite index subgroup. We note for a given group $A(A_n)$, $A(B_n)$, or $P(A_n)$, different choices of homomorphism t give rise to a distinct elements of $\mathrm{Comm}(A(A_n))$. In particular, the elements of $T(P(A_n))$ give interesting examples of elements of $\mathrm{Comm}(A(A_n))$.

6. Proof of Theorem 1

Let S be a sphere with $m \geq 5$ punctures, and let ϕ be a superinjective map of $\mathrm{C}(S)$. We will prove Theorem 1, i.e. that ϕ is induced by an element of $\mathrm{Mod}^{\pm}(S)$. The proof is broken up into a series of lemmas, all of which have direct analogues in the work of Ivanov, Korkmaz, Irmak, and Brendle and Margalit.

A *side* of a curve a in S is a connected component of $S - a$. In this setting, a is called a *k-curve* if the minimum of the numbers of punctures on each side is k. Two curves a and b in S are said to be *adjacent* if $i(a, b) = 2$.

Lemma 13 (Sides)**.** *If a and b are curves which lie on the same side of a curve w, then $\phi(a)$ and $\phi(b)$ lie on the same side of $\phi(w)$.*

Proof. Choose a curve d which intersects a and b, but not w. Since ϕ is superinjective, $\phi(d)$ intersects $\phi(a)$ and $\phi(b)$, but not $\phi(w)$, and so the lemma follows.

□

Lemma 14 (2-curves)**.** *If a is a 2-curve, then $\phi(a)$ is a 2-curve.*

Proof. Choose a pants decomposition $\{a = a_1, a_2, \ldots, a_{m-3}\}$. Applying Lemma 13, we see that $\phi(a_2), \ldots, \phi(a_{m-3})$ must all lie on the same side of $\phi(a_1)$. It follows that $\phi(a_1)$ is a 2-curve.

□

Lemma 15 (k-curves)**.** *If w is a k-curve, then $\phi(w)$ is a k-curve. Further, if a is a curve on a side of w with k punctures, then $\phi(a)$ is a curve on a side of $\phi(w)$ with k punctures.*

Proof. First, we assume that w has an even number of punctures on at least one of its sides. In this case, we can choose a pants decomposition $\mathcal{P} = \{a_1, \ldots, a_{k-2}, w, b_1, \ldots, b_{m-2-k}\}$ so that the a_i are on one side of w and the b_i are on the other side, and $\mathcal{P}$ contains the maximal number of disjoint 2-curves on S (namely, $\lfloor \frac{m}{2} \rfloor$ 2-curves). By the superinjectivity of ϕ and Lemma 14, $\phi(\mathcal{P})$ is also a pants decomposition of S containing a maximal number of 2-curves. By Lemma 13, the $\phi(a_i)$ lie on a common side of $\phi(w)$ and the $\phi(b_i)$ lie on a common side of $\phi(w)$. Thus, there are two possibilities: either $\phi(w)$ is a 2-curve, or $\phi(w)$ is a k-curve. By Lemma 14 and the fact that a pants decomposition contains at most $\lfloor \frac{m}{2} \rfloor$ disjoint 2-curves, it follows that $\phi(w)$ can only be a 2-curve if w is a 2-curve.

In the case that w has an odd number of punctures on both of its sides, we again choose a pants decomposition $\mathcal{P}$ which contains w and the greatest number of 2-curves disjoint from w. In this case, $\mathcal{P}$ has $\frac{1}{2}(m-2)$ 2-curves $c_1, \ldots, c_{\frac{m-2}{2}}$, which is not a maximal collection of 2-curves on S. As above, $\phi(w)$ must be either a k-curve or a 2-curve. However, since $\mathcal{P}$ does not contain a maximal collection of disjoint 2-curves, we need a new argument that $\phi(w)$ is not a 2-curve.

To this end, we choose a 2-curve x in S which is adjacent to one of the 2-curves of the c_i, say c_1, but is disjoint from the other c_i. Set $c_0 = w$. If $\phi(w)$ is a 2-curve, then $\{\phi(c_0), \ldots, \phi(c_{\frac{m}{2}})\}$ is a maximal collection of disjoint 2-curves on S. Applying superinjectivity of ϕ and Lemma 14, we see that since $\phi(x)$ can only intersect $\phi(c_i)$ for $i = 1$, $\phi(x)$ and $\phi(c_1)$ must have the same punctures on each of their twice-punctured sides. It follows that $\phi(x) \cup \phi(c_1)$ separates the set $\{\phi(c_0), \phi(c_2), \ldots, \phi(c_{\frac{m}{2}})\}$ into at least two nonempty disjoint subsets, by which we mean that there exists a j and a k (both not 1) so that every curve which intersects $\phi(c_j)$ and $\phi(c_k)$ must intersect at least one of $\phi(x)$ and $\phi(c_1)$ (otherwise, $\phi(x)$ and $\phi(c_1)$ would be isotopic). However, since there is always a curve which intersects c_j and c_k but not x or c_1, this contradicts superinjectivity.

Both statements of the lemma follow.

□

Lemma 16 (Adjacency)**.** *If a and b are adjacent 2-curves, then $\phi(a)$ and $\phi(b)$ are adjacent 2-curves.*

Proof. First assume $m > 5$. We claim that 2-curves a and b are adjacent if and only if there exists a 3-curve w and curves x and y so that: a and b lie on a thrice-punctured side of w, x intersects a and w but not b and not y, and y intersects b and w but not a and not x. By Lemma 15 and the definition of superinjectivity, all of these properties are preserved by ϕ, and thus the lemma will follow.

One direction is easy: if a and b are adjacent, then we can find curves w, x, and y which satisfy the given properties. Now suppose that there exist curves w, x, and y which satisfy the given properties. We restrict our attention to the side of w containing a and b. On this subsurface S', x and y are collections of disjoint arcs. Note that on a thrice-punctured disk, there can be at most three families of disjoint parallel arcs. However, since a is a curve disjoint from y, arcs of y can only appear in one of these families. The same is true for x, and we see that the arcs of x are not parallel to the arcs of y. Thus, a must lie in the component of $S' - y$ which is a twice punctured disk. There is only one such curve. Likewise, there is only one choice for b, and we see that a and b are adjacent.

For $m = 5$, the proof is exactly the same, except w is chosen to be a 2-curve.

□

There is a natural bijection between 2-curves in S and (isotopy classes of) arcs in S joining two punctures. A collection of three pairwise adjacent arcs joining three punctures is an *ideal triangle* if each arc lies in a regular neighborhood of the other two. An *ideal triangulation* of S is a maximal collection of disjoint arcs in S. Two ideal triangulations are related by a *basic move* if they differ by one arc.

The following theorem was first proven by Harer [13]. Later, Hatcher and Mosher gave more elementary proofs [15] [30].

Theorem 17. *Any two ideal triangulations of S are related by a finite sequence of basic moves.*

Lemma 18 (Triangles). *If a, b, and c form an ideal triangle, then $\phi(a)$, $\phi(b)$, and $\phi(c)$ form an ideal triangle.*

Proof. The condition that a, b, and c form an ideal triangle is equivalent to the conditions that a, b, and c are pairwise adjacent and that they lie on a thrice-punctured side of a 3-curve (2-curve in the case $m = 5$). By Lemmas 15 and 16, these properties are preserved by ϕ, and the lemma follows.

□

We are now ready to prove the Theorem 1. In the proof, a *chain* of curves in S is a collection $\{a_1, \ldots, a_k\}$ such that each a_i is adjacent to a_{i+1} and $\mathrm{i}(a_i, a_j) = 0$ otherwise.

Proof. Let $\mathcal{T}$ be an ideal triangulation of S. By Lemma 18, $\phi(\mathcal{T})$ is also an ideal triangulation of S. Since $\mathcal{T}$ and $\phi(\mathcal{T})$ are abstractly isomorphic as simplicial complexes, it follows that there is an $f \in \mathrm{Mod}^{\pm}(S)$ so that $f(\mathcal{T}) = \phi(\mathcal{T})$. If $\mathcal{T}'$ differs from $\mathcal{T}$ by a basic move, then it follows from Lemma 18 that $f(\mathcal{T}') = \phi(\mathcal{T}')$. Since any 2-curve belongs to some ideal triangulation, Theorem 17 implies that f agrees with ϕ on all 2-curves.

Now let x be a curve in S which is not a 2-curve. We wish to show that $f(x) = \phi(x)$. Choose maximal chains $\mathcal{C}_1$ and $\mathcal{C}_2$ of 2-curves on the two sides of x. By Lemma 16 and the superinjectivity of ϕ, $\phi(\mathcal{C}_1)$ and $\phi(\mathcal{C}_2)$ are disjoint chains in S. By maximality of the chains, we see that there is a unique curve x' which is disjoint from both $\phi(\mathcal{C}_1)$ and $\phi(\mathcal{C}_2)$. Thus, $\phi(x) = x'$. Also, we see that $f(x) = x'$, since f also preserves disjointness.

□

References

[1] Daniel Allcock. Braid pictures for Artin groups. *Trans. Amer. Math. Soc.*, 354(9):3455–3474 (electronic), 2002.

[2] Robert W. Bell and Dan Margalit. Braid groups and the co-Hopfian property. ArXiv:math.GT/0403145.

[3] Joan S. Birman and Tara E. Brendle. Braids: A Survey. ArXiv:math.GT/0409205.

[4] Joan S. Birman, Alex Lubotzky, and John McCarthy. Abelian and solvable subgroups of the mapping class groups. *Duke Math. J.*, 50(4):1107–1120, 1983.

[5] Tara E. Brendle and Dan Margalit. Commensurations of the Johnson Kernel. *Geom. Top.*, 8:1361–1384, 2004.

[6] L.E.J. Brouwer. Über die periodischen Transformationen der Kugel. *Math. Ann.*, 80:39–41, 1919.

[7] Ruth Charney and John Crisp. Automorphism groups of some affine and finite type Artin groups. ArXiv:math.GR/0408412.

[8] Ruth Charney and David Peifer. The $K(\pi, 1)$-conjecture for the affine braid groups. *Comment. Math. Helv.*, 78(3):584–600, 2003.

[9] John Crisp. Injective maps between Artin groups. In *Geometric group theory down under (Canberra, 1996)*, pages 119–137. de Gruyter, Berlin, 1999.

[10] Joan L. Dyer and Edna K. Grossman. The automorphism groups of the braid groups. *Amer. J. Math.*, 103(6):1151–1169, 1981.

[11] S. Eilenberg. Sur les transformations périodiques de la surface de sphère. *Fund. Math.*, 22:28–41, 1934.

[12] Benson Farb and Nikolai V. Ivanov. The Torelli geometry and its applications.

[13] John L. Harer. Stability of the homology of the mapping class groups of orientable surfaces. *Ann. of Math. (2)*, 121(2):215–249, 1985.

[14] W. J. Harvey. Boundary structure of the modular group. In *Riemann surfaces and related topics: Proceedings of the 1978 Stony Brook Conference (State*

Univ. New York, Stony Brook, N.Y., 1978), volume 97 of *Ann. of Math. Stud.*, pages 245–251, Princeton, N.J., 1981. Princeton Univ. Press.

[15] Allen Hatcher. On triangulations of surfaces. *Topology Appl.*, 40(2):189–194, 1991.

[16] Elmas Irmak. Complexes of Nonseparating Curves and Mapping Class Groups.

[17] Elmas Irmak. Superinjective Simplicial Maps of Complexes of Curves and Injective Homomorphisms of Subgroups of Mapping Class Groups II.

[18] Elmas Irmak. Superinjective simplicial maps of complexes of curves and injective homomorphisms of subgroups of mapping class groups. *Topology*, 43(3):513–541, 2004.

[19] Elmas Irmak, Nikolai V. Ivanov, and John D. McCarthy. Automorphisms of surface braid groups. Preprint.

[20] Elmas Irmak and Mustafa Korkmaz. Automorphisms of the Hatcher-Thurston complex.

[21] N. V. Ivanov. Automorphisms of Teichmüller modular groups. In *Topology and geometry—Rohlin Seminar*, volume 1346 of *Lecture Notes in Math.*, pages 199–270. Springer, Berlin, 1988.

[22] Nikolai V. Ivanov. *Subgroups of Teichmüller modular groups*, volume 115 of *Translations of Mathematical Monographs*. American Mathematical Society, Providence, RI, 1992. Translated from the Russian by E. J. F. Primrose and revised by the author.

[23] Nikolai V. Ivanov and John D. McCarthy. On injective homomorphisms between Teichmüller modular groups. I. *Invent. Math.*, 135(2):425–486, 1999.

[24] Richard P. Kent, IV and David Peifer. A geometric and algebraic description of annular braid groups. *Internat. J. Algebra Comput.*, 12(1-2):85–97, 2002. International Conference on Geometric and Combinatorial Methods in Group Theory and Semigroup Theory (Lincoln, NE, 2000).

[25] Steven P. Kerckhoff. The Nielsen realization problem. *Ann. of Math. (2)*, 117(2):235–265, 1983.

[26] B. de Kerékjártó. Über die periodischen Tranformationen der Kreisscheibe und der Kugelfläche. *Math. Annalen*, 80:3–7, 1919.

[27] Mustafa Korkmaz. Automorphisms of complexes of curves on punctured spheres and on punctured tori. *Topology Appl.*, 95(2):85–111, 1999.

[28] Dan Margalit. Automorphisms of the pants complex. *Duke Math. J.*, 121(3):457–479, 2004.

[29] John D. McCarthy and William R. Vautaw. Automorphisms of Torelli groups.

[30] Lee Mosher. Tiling the projective foliation space of a punctured surface. *Trans. Amer. Math. Soc.*, 306(1):1–70, 1988.

[31] Paul Schmutz Schaller. Mapping class groups of hyperbolic surfaces and automorphism groups of graphs. *Compositio Math.*, 122(3):243–260, 2000.

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