Stable degenerations of symmetric squares of curves

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1 Introduction

Suppose $C \to \Delta'$ is a family of smooth curves of genus greater than two over a punctured disk. Denote by $C^{(2)}_{\Delta'}$ the *fibered symmetric square* of this family, that is, the quotient of the fibered product $C \times_{\Delta'} C$ by the $\mathbf{Z}_2$ action swapping the factors. The theory of moduli of stable surfaces ensures that (after possibly taking a base change by a cover of Δ totally ramified over 0) this family of surfaces may be completed to a family of surfaces $S \to \Delta$ satisfying

1. S_0 has *semi-log canonical* (slc) singularities;
2. some reflexive power $\omega^{[N]}_{S/\Delta}$ of the relative dualizing sheaf of the family is locally free;
3. $\omega^{[N]}_{S_0}$ is an ample line bundle.

By analogy with the theory of moduli of stable curves, the process of finding $S \to \Delta$ is called *stable reduction*.

The main result of this article is a geometric description of these stable reductions. The first step is completing $C \to \Delta'$ to a family of stable curves over Δ (I will ignore base changes in this non-technical discussion). The family $C^{(2)}_{\Delta} \to \Delta$ is not stable in general. A partial resolution of this family is given by the Hilbert scheme $\mathrm{Hilb}_2(C/\Delta)$ of length two subschemes in the fibers. Geometrically, this replaces the points on $C^{(2)}_{\Delta}$ coming from the quotient of a product of a node with itself by rational curves. The fibers of $\mathrm{Hilb}_2(C/\Delta)$ have slc singularities, and the relative dualizing sheaf is locally free, but the special fiber does not yet in general have an ample dualizing sheaf. The last step of stabilization is an exercise in geometry of products and symmetric squares of smooth curves.

This explicit description of the stable degenerations yields a description of the irreducible component of the moduli space of stable surfaces which is the closure of the moduli space of symmetric squares of curves of a given genus.

This article continues the program of studying moduli spaces of stable surfaces using compact moduli spaces of simpler objects to complete families of surfaces. Like [vOa], the moduli space of stable curves is used here to ensure that a degeneration of smooth curves may be replaced by a degeneration with special fiber at worst nodes. Moduli of stable curves with group action was used to study surfaces isogenous to a product in [vOb]. One could conceivably use the methods of Abramovich and Vistoli [AV00] to study stable degenerations of Kodaira fibrations (see [BHPVdV04], V.14). This method has been employed by La Nave in the case of elliptic fibrations [Nav]. Studying symmetric squares of smooth curves by stable degeneration of curves has yielded results on the cone of effective divisors of such surfaces; the article [CK99] contains excellent pictures which may help the reader in visualizing the constructions used here.

2 Stable surfaces and their moduli

This section contains the relevant references and definitions from the theory of moduli of stable surfaces which are necessary in what follows. Since ampleness of the canonical class is more important for us than nonsingularity, we will consider families whose general member is a *canonically polarized surface*, that is, a projective surface with rational double points and ample dualizing sheaf.

Definition 2.1. A *stable surface* is a projective, reduced surface S with semi-log canonical (slc) singularities such that $\omega^{[N]}_S$ (the reflexive hull of the N-th power of the dualizing sheaf of S) is an ample invertible sheaf for some N. The smallest such N is called the index of S.

Remark 2.2. For the definition of slc singularities, see [KSB88], Definition 4.17. In the same article, all slc surface singularities are classified, and we will use the classification more than the definition. For our purposes, the following information suffices:

1. Normal crossings singularities are slc.

2. Products of slc singularities are slc (in particular, the product of nodal curves has slc singularities – these are special cases of *degenerate cusps*).

3. Slc singularities are those which appear on the fibers of relative canonical models of semistable reductions (see below).

It turns out that the naive definition of a family of stable surfaces – a flat, proper morphism whose fibers are stable surfaces – does not lead to a separated moduli functor. One needs a further condition on the family.

Definition 2.3. A *family of stable surfaces* is a flat, proper morphism $f : X \to B$ whose fibers are stable surfaces and whose relative dualizing sheaf $\omega_{X/B}$ is **Q**-Cartier, i.e., some reflexive power is a line bundle. One also says that the morphism f is **Q**-Gorenstein.

In [Kol90], a stronger condition is required on a family of stable surfaces. For the purposes of this article, the weaker condition given here suffices, since the weaker condition given here implies Kollár's stronger condition if the family has a smooth curve as base and canonically polarized general fiber.

An essential fact for moduli theory of stable surfaces, which is proved using Mori theory is the following:

Theorem 2.4 (Stable reduction). *Let $X \to B$ be a one-parameter family with a smooth base such that X_b is a canonically polarized surface for all $b \neq 0$ for some point $0 \in B$. Then there exists a finite map $B' \to B$ totally ramified over 0 and a family of stable surfaces $X' \to B'$ extending the pullback of the family $X|_{B\setminus\{0\}}$ to B'. The special fiber of X' is uniquely determined by the original family.*

Sketch of proof. After a base change, the original family admits a semistable resolution, so we may assume $X \to B$ is a family with smooth total space and normal crossings divisors for fibers. In this case, a unique relative canonical model for the morphism $X \to B$ exists (see Chapter 7 of [KM98] for a proof). This relative canonical model is a family of stable surfaces. The uniqueness of the special fiber follows from the facts that every pair of semistable resolutions is dominated by a third, and the uniqueness of relative canonical models. □

Once one has established the existence of a coarse moduli space for stable surfaces (using the notion of family given above), this theorem implies that the moduli space is proper (and in particular, separated). We will only need the uniqueness part of the theorem, since our work will prove existence for the special class of families which we consider.

A one-parameter family $X \to B$ with smooth base and canonically polarized general fiber as in the theorem will be called a *degeneration*. If a degeneration $X \to B$ is a family of stable curves or stable surfaces, it will be called a *stable degeneration*.

3 Punctual Hilbert schemes and Chow varieties of nodal curves

Let $C \to B$ be a stable degeneration of curves. Note that the fibered symmetric square $C_B^{(2)}$ is isomorphic to the relative Chow variety $\mathrm{Chow}_{0,2}(C/B)$ of dimension zero, degree two cycles in the fibers of $C \to B$. Abbreviate $C_B^{(2)}$ by X. The points of X fall into four classes:

1. Cycles consisting of two smooth points (possibly equal): such points are smooth on X and on its fibers.

2. Cycles consisting of a smooth point and a node: such points are smooth on X and are normal crossings on its fibers.

3. Cycles consisting of two different nodes: such points are isolated singular points on X (cone over a quadric surface) and degenerate cusps on its fibers.

4. Cycles consisting of the same node taken twice: such points are isolated singular points on X ($\mathbf{Z}_2$ quotients of the cone over a quadric surface). Such singularities are analytically isomorphic to the vertex of the cone over the cubic scroll. They are thus singular points of the fibers as well.

The cone over the cubic scroll is not **Q**-Gorenstein, so the family $X \to B$ cannot be **Q**-Gorenstein either, if the original family $C \to B$ contained nodal curves. The points of X of the fourth type considered above will be called the *bad points* of X.

Let $\bar{X} \to X$ be the blowup of all bad points of X. It is easy to check by computing in local coordinates that the fibers of $f : \bar{X} \to B$ have only slc singularities (indeed, the only singular points in the fibers are the normal crossings and degenerate cusps described as points of type 3 above). The stable reduction of $X \to B$ will be the relative canonical model of this morphism, that is,

$$\operatorname{Proj} \bigoplus_{n=0}^{\infty} f_* \omega_{\bar{X}}^n.$$

A first step towards a geometric description of this model is the relative canonical model of $\bar{X} \to X$. Call this $\tilde{X}$. Example 2.7 of [KM98] describes how to obtain $\tilde{X}$ directly from X. This variety $\tilde{X}$ is a resolution of the singularities of X with as small as possible exceptional locus, in this case, consisting of rational curves rather than divisors. There is a preferred choice for such a resolution, which is distinguished by having an ample relative canonical class.

Let us recall the local description of $\tilde{X}$ from *loc. cit.*. A bad point has a local analytic description as $xy - uv = 0$ modulo the action of $\mathbf{Z}_2$ taking x to $-x$ and u to $-u$. In these coordinates, the parameter on the base is given by $xv - yu$, up to a constant factor. In these local coordinates, we obtain $\tilde{X}$ by blowing up the ideal (x,u) in $\mathbf{C}[x,y,u,v]/(xy-uv)$, and then taking the quotient by the $\mathbf{Z}_2$-action described above (which clearly extends to the blowup). The resulting variety has coordinate ring

$$\mathbf{C}[a_1,\ldots,a_7]/(a_5a_6 - a_4a_7, a_3a_6 - a_1a_7, a_2a_6 - a_3a_7, a_3a_4 - a_1a_5, a_2a_4 - a_3a_5, a_1a_2 - a_3^2),$$

where the variables a_6 and a_7 are homogeneous. The central fiber of this new family is obtained by setting $a_3 = a_4a_5$. One easily checks in coordinates that the only singularities of the central fiber are normal crossings. So the stable reduction improves the singularities of the central fiber. The total space of the new family is smooth, and the relative canonical sheaf of the map to the original family is ample ([KM98]).

3.1 Coordinate-free description of $\tilde{X}$

There is another way to obtain this model, which is easier to describe. However, for computations, the description given above it indispensible. Let C be a smooth curve. Then it is well-known that the Hilbert scheme $\operatorname{Hilb}_2(C)$ parameterizing length two subschemes of C is isomorphic to the Chow variety $\operatorname{Chow}_{0,2}(C)$ of dimension zero, degree two cycles. If C has nodes, or more generally, plane curve singularities, each singularity supports a $\mathbf{P}^1$ (the projectivization of the tangent space to this singularity) of length two structures, so the associated Hilbert-to-Chow morphism can be described exactly as the morphism $\tilde{X} \to X$ is described above. In fact, following our notation above,

Theorem 3.1.

$$\tilde{X} \cong \operatorname{Hilb}_2(C/B)$$

Proof. By construction, the varieties $\tilde{X}$ and $\operatorname{Hilb}_2(C/B)$ are birational over B. Since $\tilde{X}$ is a relative canonical model, there is actually a morphism $f : \operatorname{Hilb}_2(C/B) \to \tilde{X}$, which if not an isomorphism, must blow down the exceptional $\mathbf{P}^1$s in the morphism $\operatorname{Hilb}_2(C/B) \to X$.

The morphism f is projective and dominant, hence surjective. Consequently the $\mathbf{P}^1$s occuring on $\operatorname{Hilb}_2(C/S)$ are not blown down by this morphism. □

3.2 A third description of $\tilde{X}$

We will need a description of the special fiber $\tilde{X}_0$, or alternately, $\operatorname{Hilb}_2(C_0)$. In the fiber X_0, a sufficiently small analytic germ at a bad point has three irreducible components (see Figure 1). One of these components is distinguished in that it meets both of the others along curves; the other pair of components meet only at the bad point. The local computation given above shows that the rational curve on $\tilde{X}_0$ which replaces the bad point lies on the inverse image of this component, and meets the other components in a single point. Figure 2 sums this up much more clearly.

The product of C with itself has components $C_i \times C_j$ for each pair C_i, C_j of irreducible components of C. The $\mathbf{Z}_2$-quotient turns each product $C_i \times C_i$ into a component of $C^{(2)}$ isomorphic to the symmetric square of C_i. The other components are not always, however, products of curves. The component $C_i \times C_j$ ($i \neq j$) of C^2 is identified with the

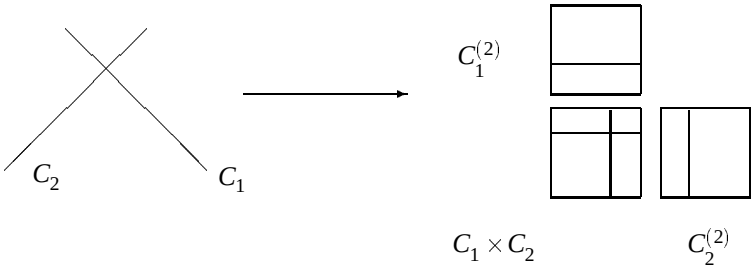


Figure 1: The components of the symmetric square of a nodal curve and their incidences.

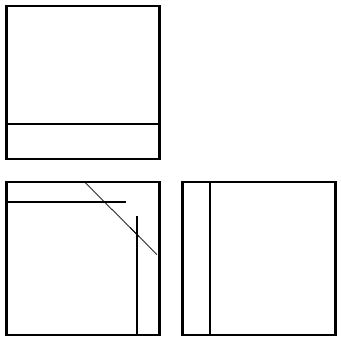


Figure 2: Blowing up and regluing.

component $C_j \times C_i$ by the $\mathbf{Z}_2$-action, but because of incidences with other components, some points of the resulting component are pinched together. The components of the normalization, however, are not pinched, and are indeed isomorphic to products of curves blown up at the points corresponding to bad points of X_0.

The simplest example of this pinching is the symmetric square of a curve $C = C_1 \cup C_2$ with two components glued to each other at two points. The symmetric square $C^{(2)}$ is Cohen-Macaulay, but the component of $C^{(2)}$ which is not a symmetric square is not. However, the normalization of $C^{(2)}$ is the disjoint union of $C_1^{(2)}$, $C_2^{(2)}$ and $C_1 \times C_2$.

In the case of two irreducible components joined at a single node corresponding to a point p on C_1 and q on C_2, one may describe the symmetric square of C as $C_1 \times C_2$ blown up in the point (p,q) with $C_1^{(2)}$ glued along the strict transform of $C_1 \times \{q\}$ and $C_2^{(2)}$ glued along the strict transform of $\{p\} \times C_2$.

4 Stability

4.1 Products and symmetric products of pointed curves

In this section, the symbols C or C_i will denote smooth curves of genus g or g_i, and δ or δ_i will denote reduced divisors on these curves. We denote the degree of δ by $|\delta|$, since we have no occasion to refer to the complete linear system usually notated thus. It is well known that the pair (C,δ) is of log-general type (that is, $K_C + \delta$ is big, which is the same as ample for curves) if $2g-2+|\delta| > 0$. We define log surfaces related to these pointed curves.

Definition 4.1. The *log product* of (C_1,δ_1) with (C_2,δ_2) is the log surface $(C_1 \times C_2, \delta_1 \times C_2 \cup C_1 \times \delta_2)$. Since we will never associate more than a single divisor to a single curve, the notation $C_1 \times_{\log} C_2$ for the log product will not be confusing.

The *log symmetric square* of (C,δ), denoted $C^{(2)}_{\log}$, is defined as follows. Let $\pi : C^2 \to C^{(2)}$ be the quotient map for the $\mathbf{Z}_2$-action swapping the factors. Then $C^{(2)}_{\log}$ is the surface $(C^2, \pi_*(\delta \times C \cup C \times \delta))$.

Recall that the log-canonical class of a log variety (X,D) is defined to be $K_X + D$, where K_X is the canonical class of X. D is often called a *boundary divisor*. It is clear that the log-canonical class of $C_1 \times_{\log} C_2$ is ample if and only if the log-canonical classes of (C_1,δ_1) and (C_2,δ_2) are ample. The case of symmetric squares requires more work.

Proposition 4.2. *The log symmetric square of (C,δ) has ample log-canonical class exactly in the following situations:*

1. *C is rational and $|\delta| > 3$;*
2. *C is elliptic and $|\delta| > 2$;*
3. *C is genus 2 and $|\delta| > 1$;*
4. *C is genus 3 hyperelliptic and $|\delta| > 0$;*
5. *C is non-hyperelliptic of genus 3;*
6. *C is genus 4 or higher.*

The log-canonical class is nef if

1. *C is rational and $|\delta| > 2$;*
2. *C is elliptic and $|\delta| > 1$;*
3. *C is genus 2 or higher.*

Proof. First, some notation is in order. Let g be the genus of C. Denote by B the boundary divisor of $C^{(2)}_{\log}$. Denote its irreducible components by B_i, where i runs from 1 to $2|\delta|$. Let $\pi : C^2 \to C^{(2)}$ be the quotient map, and Δ the diagonal of C^2. Finally, let D_i be the inverse image of B_i under π for every i, and D the union of the D_i. The D_i are fibers of the projections from C^2 to its factors.

Since π is a finite map, and $\pi^*(K_{C^{(2)}}+B) = K_{C^2}+D-\Delta$, it suffices to check that $K_{C^2}+D-\Delta$ is ample on C^2. For brevity, we denote K_{C^2} simply as K. We will use the Nakai-Moishezon criterion: a divisor is ample if and only if its self-intersection is positive and it is positive on every irreducible curve. The following are easy to check:

$$\begin{aligned} K^2 &= 2(2g-2)^2 \\ D^2 &= 2|\delta|^2 \\ \Delta^2 &= 2-2g \\ K.D &= 2|\delta|(2g-2) \\ K.\Delta &= 4g-4 \\ D.\Delta &= 2|\delta| \end{aligned}$$

It follows that

$$(K+D-\Delta)^2 = 2(2g-2)^2 + (4|\delta|-5)(2g-2) + 2|\delta|(|\delta|-2).$$

This yields the following necessary conditions for ampleness:

1. If $g=0$, then $|\delta|>3$.
2. If $g=1$, then $|\delta|>2$.
3. If $g=2$, then $|\delta|>0$.

First, we show that the conditions for $g=0$ or 1 are also sufficient. If $g=0$, then $K+D-\Delta$ is a divisor of type $(|\delta|-3,|\delta|-3)$ on $\mathbf{P}^1\times\mathbf{P}^1$, hence ample as soon as $|\delta|>3$. If $g=1$, one has that a divisor on an abelian surface with positive self-intersection and positive intersection with some other curve (in our case, take Δ, for example) is ample (Corollary 2.2 of [Kan94]).

Suppose now that C is hyperelliptic, and let Γ be the graph of the hyperelliptic involution in C^2. We have $\Gamma^2 = 2-2g$, and by the Hurwitz formula

$$(K+D-\Delta).\Gamma = -6+2g+2|\delta|$$

which proves the necessity of all of the conditions given in the hypotheses.

If C is a genus 2 curve, then $C^{(2)}$ is the blowup of an abelian surface at a single point, and Γ covers the exceptional curve for this blowup. It follows that $K-\Delta$ is numerically equivalent to Γ. Since Γ is effective, $\Gamma+D$ is positive on any curve except possibly its own components as soon as $|\delta|>0$, since then D contains a fiber in each direction. The computations above show that $\Gamma+D$ is positive on Γ once $|\delta|>1$, and for any component D_i of D,

$$(K+D-\Delta).D_i = 2g-4+|\delta|$$

by adjunction and geometric considerations. This finishes the genus 2 case.

Finally, it is well-known that the symmetric square of a genus 3 nonhyperelliptic curve, or of a curve of genus 4 or higher already has an ample canonical class, so no condition is necessary. Further, it is known that the only -2-curve on $C^{(2)}$ for a genus 3 hyperelliptic curve is the curve covered by Γ, so Γ is the only curve to check for positivity on $K+D-\Delta$, and this has been checked above.

The assertions about when the log-canonical class is nef follow from the above computations. □

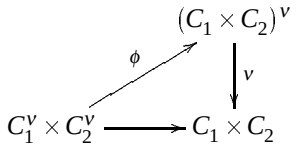
4.2 Main theorems on stability

Let C be a nodal curve with components C_i. To each C_i, associate a number δ_i which equals the number of nodes of C_i, with nodes resulting from self-intersection counted twice. By genus of an irreducible nodal curve, we mean the genus of the normalization, not the arithmetic genus. The following lemma is necessary to reduce much of the proof of the main theorem to the results of the previous section.

Lemma 4.3. *The normalization of a product of irreducible nodal curves is the product of the normalizations of the factors. The normalization of the symmetric square of an irreducible nodal curve is the symmetric square of the its normalization.*

Proof. All normalization morphisms will be denoted ν. The normalization of a variety X will be denoted X^ν. Let C, C_1, and C_2 be irreducible nodal curves.

By the universal property of normalization, there exists a unique morphism ϕ completing the diagram



commutatively. By Zariski's main theorem, the fibers of ϕ are connected. By commutativity of the diagram, any positive dimensional fiber of ϕ lies in a fiber of $C_1^\nu \times C_2^\nu \to C_1 \times C_2$, which is impossible. We conclude that ϕ is a homeomorphism, and hence an isomorphism, since $(C_1 \times C_2)^\nu$ is normal.

We now recycle the notation ϕ to denote the morphism $C^\nu \times C^\nu \to (C \times C)^\nu$. Via ϕ, we have an action of $\mathbf{Z}_2$ on $(C \times C)^\nu$, and the normalization morphism is equivariant. Denote by π the induced morphism from $(C \times C)^\nu/\mathbf{Z}_2$ to $(C \times C)/\mathbf{Z}_2$. By an argument exactly like the one just given, $(C \times C)^\nu/\mathbf{Z}_2$ is the normalization of $(C \times C)/\mathbf{Z}_2$. It follows that $C^{(2)}$ is normalized by $(C^\nu)^{(2)}$. □

Theorem 4.4. *Let C be a nodal curve. Then* $\mathrm{Hilb}_2(C)$ *is a stable surface if and only if*

1. *for every genus 0 component C_i, $\delta_i > 3$,*
2. *for every genus 1 component C_i, $\delta_i > 2$,*
3. *for every genus 2 component C_i, $\delta_i > 1$,*
4. *for every component C_i with genus 3 hyperelliptic normalization, $\delta_i > 0$.*

Proof. We have already seen that $\mathrm{Hilb}_2(C)$ has slc singularities, and that it is Gorenstein. It remains to check that its dualizing sheaf ω is ample.

ω is ample if and only if its restriction to every irreducible component of the normalization of $\mathrm{Hilb}_2(C)$ is ample. These components are products and symmetric squares of curves or possibly blowups of these. The restriction of ω to a component which is a product or symmetric product is exactly the log canonical divisor on these surfaces considered in 4.2. Therefore it remains to consider the blown-up components.

On the other hand, K_X is positive on the exceptional curves of the blowups by the description of $\mathrm{Hilb}_2(C)$ as the relative canonical model of the Hilbert-Chow morphism. □

Remark 4.5. From 4.2, we can see that the morphism $\mathrm{Hilb}^2_S(C) \to S$ is relatively minimal unless there is a genus 1 component C_i with $\delta_i = 1$. In this case, the symmetric square $C_i^{(2)}$ is a ruled surface over C_i, and there is a unique component C_j of C meeting C_i. Running the relative minimal model program collapses $C_i^{(2)}$ to its curve of intersection with the blown up product $C_i \times C_j$. In general, this will make the (-1)-curve on $C_i \times C_j$ zero on K_X, so it will be blown down upon taking the relative canonical model.

From the preceeding theorem and remark, we have an explicit geometric description of the relative minimal model for $\mathrm{Hilb}^2_S(C)$ over S. An explicit description of the canonical model is more complicated; we are content to see a few examples.

Example 4.6. The simplest case to examine is when C is a genus 3 hyperelliptic curve. In this case, the graph of the hyperelliptic involution covers a rational curve of self-intersection -2 on $C^{(2)}$. This is blown down to obtain the canonical model. If the original family has all hyperelliptic members, then these (-2)-curves sweep out a divisor which is collapsed by taking the relative canonical model.

Example 4.7. If C has a genus 2 component C_i with $|\delta_i| = 1$, then again, there is a rational curve covered by the hyperelliptic involution (the unique rational curve on $C_i^{(2)}$), this time with self-intersection -1. This curve, however, is zero on K_X, so taking a minimal model does not contract it. It is contracted to a smooth point by taking the canonical model.

For a concrete example, suppose C is a genus 3 stable curve with two components, a smooth genus 2 curve C_1 and an elliptic tail C_2. The relative MMP collapses $C_2^{(2)}$ to its intersection with the blowup of $C_1 \times C_2$. Now taking

the relative canonical model blows down the (-1)-curve on $C_1 \times C_2$ and the (-1)-curve on $C_1^{(2)}$. Thus the stable limit of some smoothing of $C^{(2)}$ is the product $C_1 \times C_2$ glued to an abelian surface (the Jacobian of C_1) along a curve isomorphic to C_2. The canonical class is ample: its restriction to $C_1 \times C_2$ is the tensor product of pullbacks of ample divisors from each factor, and its restriction to $C_1^{(2)}$ is the class of a genus 2 curve, which has positive self-intersection by the adjunciton formula, hence is ample or anti-ample. It is easy to see that the class of this curve is ample.

Example 4.8. In general, taking the canonical model introduces more singularities. If C has a rational component with only three special points, $\mathrm{Hilb}_2(C)$ has a component isomorphic to $\mathbf{P}^2$, and such that the canonical class restricted to this component is the canonical class of $\mathbf{P}^2$ twisted by the coordinate axes (up to linear equivalence), hence trivial. Therefore the morphism to the relative canonical model collapses this component to a point, therefore also collapsing the curves of intersection of this component with another component to points. The reader may find it amusing to find the stable limit of a degeneration of smooth genus 3 curves to a curve obtained by gluing a rational curve to an elliptic curve in three distinct points.

5 Global moduli results

The study of this seemingly special degeneration gives all possible degenerations of symmetric squares. Indeed, let $X \to B$ be a degeneration whose general fiber is the symmetric square of a curve C. By the results of [Fan94], away from the special point $0 \in B$, X comes from a family $C \to B\backslash\{0\}$ of smooth curves. One then (possibly after base change) completes this family to a family of stable curves $C \to B$. Then, the relative Hilbert scheme may be formed, and the stable reduction process applied as above. For a curve C, denote the Hilbert scheme $\mathrm{Hilb}_2(C)$ by $C^{[2]}$ and the stable model of this surface by $C^{\{2\}}$.

Let $g > 2$. Denote by $M_{g^{(2)}}$ the irreducible component of the moduli space of stable surfaces containing the moduli point of the symmetric square of a smooth genus g curve. Let M_g be the moduli space of genus g stable curves.

Theorem 5.1. *There is a surjective, birational morphism $M_g \to M_{g^{(2)}}$ which is an isomorphism over the locus of M_g parameterizing smooth curves.*

Proof. Let $H \to M_g$ be a surjective finite morphism from a smooth scheme H induced by a family of stable curves. The existence of H is guaranteed by [HM98], Lemma 3.89. Let $C \to H$ be the family inducing $H \to M_g$. Let $X \to H$ be relative Hilbert scheme of length two subschemes of the fibers of $C \to H$.

By [Kar00], Lemma 3.1, the relative canonical model $\tilde{X} \to H$ of $X \to H$ exists. By separatedness of the moduli functor, the fibers of this relative canonical model coincide with those obtained by taking relative canonical models of one-parameter subfamilies.

$\tilde{X}$ induces a morphism $H \to M_{g^{(2)}}$ which clearly descends to the desired morphism $\phi : M_g \to M_{g^{(2)}}$. The injectivity of ϕ on the locus of smooth curves in M_g is the Torelli theorem when $g = 3$ and a theorem of Martens for higher genus curves [CS93]. The aforementioned result of [Fan94] shows that the image of the locus of smooth curves in M_g is a dense open set in $M_{g^{(2)}}$. □

Is ϕ also an isomorphism on the boundary? For a curve C, denote by $C^{[2]}$ the special fiber of a relatively minimal model of the relative Hilbert scheme for some smoothing as considered above. Denote the special fiber of the relative canonical model by $C^{\{2\}}$. Since minimal models are isomorphic in codimension two, it follows that if $C^{\{2\}} \cong D^{\{2\}}$ for some curve C and D, then $C^{[2]} \cong D^{[2]}$. Therefore we can avoid describing these relative canonical models in our investigation of ϕ.

Lemma 5.2. *Suppose C_i and C_i' are smooth curves, $i = 1, 2$. Then if $C_1 \times C_2 \cong C_1' \times C_2'$, either*

1. *after possibly renumbering, $C_1 \cong C_1'$ and $C_2 \cong C_2'$, or*
2. *the C_i and C_i' are all elliptic curves.*

Proof. First, it is clear that up to reordering, the genera of C_i and C_i' are equal. By inclusion of a fiber, the isomorphism $C_1 \times C_2 \to C_1' \times C_2'$, and projection onto factors, we get finite covers between the curves, for example $C_1 \to C_1'$. Let g be the common genus of C_1 and C_1'. Then by Hurwitz,

$$2g - 2 = n(2g - 2) + r$$

for positive integers n and r. This is only possible if $2g-2=0$ or if $n=1$ and $r=0$, from which the lemma follows. □

Lemma 5.3. *Suppose C and D are smooth curves, and $C^{(2)} \cong D^{(2)}$. Then either $C \cong D$ or C and D are genus 2.*

Proof. See, e.g. [CS93]. □

Lemma 5.4. *Suppose C_1, C_2, and D are smooth curves. Then $C_1 \times C_2 \ncong D^{(2)}$.*

Proof. The invariants $\chi(\mathcal{O})$ and K^2 of a product of curves are related by $K^2 = 8\chi$. This holds for symmetric squares only in the case $g=1$ where $K^2 = \chi = 0$. The only product of curves with such invariants is a product of elliptic curves, and the Kodaira dimension of a product of elliptic curves differs from that of a symmetric square of an elliptic curve. □

Remark 5.5. Note, however, that it is possible that the blowup of a product of elliptic curves be isomorphic to the symmetric square of a genus 2 curve.

Theorem 5.6. *The morphism ϕ defined above is an isomorphism.*

Proof. Let C be a stable curve. We reconstruct C from $C^{[2]}$. For simplicity assume that the components of C are all smooth; the argument for self-intersecting components is similar but longer. Write S for $C^{[2]}$ in what follows. The components of S arising from symmetric squares of the components of C are uniquely determined unless C has an elliptic tail, since the only component which can "look like" a symmetric square of a smooth curve is a product of elliptic curves blown up at one point. Such a component exists only if an elliptic component of C is attached to the rest of C in a single point.

On the other hand, if C has an elliptic tail E, the component of $C^{(2)}$ corresponding to the symmetric square of E is blown down in taking the relative minimal model, which leaves the canonical class of S negative on the rational curve on some product of E with another component of C resulting from blowing up. Therefore, there are no products of elliptic curves with a single point blown up occuring among the components of S.

C can be reconstructed by studying the incidences between the symmetric squares of its components thus recovered, if C has no elliptic tails. If C has elliptic tails, the tails can be recovered from the components of S corresponding to (blown up) products of the tails with non-elliptic components. If the curve consists solely of elliptic components, although the product components do not necessarily determine the elliptic components of the curve, the loci along which they are glued to each other are isomorphic to the components of the curve, and the curve can be reconstructed from the surface. □

Therefore the moduli space of stable degenerations of a symmetric square of a genus g curve is simply $\overline{M_g}$ as long as $g \geq 3$.

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