REU Project
Asymptotic Properties of GARCH

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1 Introduction and the existence of GJR–GARCH

It has been observed that the variance of log returns of stock values is not constant and it describes the volatility of the market. Engle (1982) and Bollerslev (1986) introduced the GARCH$(1,1)$ model to model the volatility in time series data. They assume that the log returns satisfy the equations

$$y_k = \sigma_k \epsilon_k, \quad -\infty < k < \infty \tag{1.1}$$

and

$$\sigma_k^2 = \omega + \alpha y_{k-1}^2 + \beta \sigma_{k-1}^2, \quad -\infty < k < \infty, \tag{1.2}$$

where (ω, α, β) is the parameter of the process. It is assumed that the errors (innovations) $\epsilon_k, -\infty < k < \infty$ are independent identically distributed random variables. Nelson (1990) found the necessary and sufficient condition for the existence of $(y_k, \sigma_k^2), -\infty < k < \infty$. Lumsdaine (1996) used the quasi-maximum likelihood method to estimate the parameters.

Glosten, Jagannathan and Runke (1993) modified the GARCH$(1,1)$, giving larger weight to negative returns in the volatility. In the GJR-GARCH$(1,1)$, (1.2) is replaced by

$$\sigma_k^2 = \omega + \alpha_1 y_{k-1}^2 I\{y_{k-1} < 0\} + \alpha_2 y_{k-1}^2 I\{y_{k-1} \geq 0\} + \beta \sigma_{k-1}^2, \tag{1.3}$$

$-\infty < k < \infty$,where $\boldsymbol{\theta} = (\omega, \alpha_1, \alpha_2, \beta)$ is the parameter of the process.

We assme that

$$\omega > 0, \beta \geq 0, \alpha_1 \geq 0 \text{ and } \alpha_2 \geq 0 \tag{1.4}$$

and

$$\epsilon_k \quad -\infty < k < \infty \text{ are independent identically distributed random variables.} \tag{1.5}$$

Using the recursion in (1.3) we get

$$\begin{aligned} \sigma_k^2 &= \omega + \alpha_1 y_{k-1}^2 I\{y_{k-1} < 0\} + \alpha_2 y_{k-1}^2 I\{y_{k-1} \geq 0\} + \beta \sigma_{k-1}^2 \\ &= \omega + \alpha_1 \epsilon_{k-1}^2 \sigma_{k-1}^2 I\{\epsilon_{k-1} < 0\} + \alpha_2 \epsilon_{k-1}^2 \sigma_{k-1}^2 I\{\epsilon_{k-1} \geq 0\} + \beta \sigma_{k-1}^2 \\ &= \omega + \beta \sigma_{k-1}^2 + \eta_{k-1} \sigma_{k-1}^2, \end{aligned} \tag{1.6}$$

where

$$\eta_{k-1} = \alpha_1 \epsilon_{k-1}^2 I\{\epsilon_{k-1} < 0\} + \alpha_2 \epsilon_{k-1}^2 I\{\epsilon_{k-1} \geq 0\}.$$

Using the recursion in (1.3) backwards we get

$$
\begin{aligned}
\sigma_k^2 &= \omega + \eta_{k-1}\sigma_{k-1}^2 \\
&= \omega + \eta_{k-1}(\omega + \eta_{k-2}\sigma_{k-2}^2) \\
&= \omega + \omega\eta_{k-1} + \eta_{k-1}\eta_{k-2}\sigma_{k-2}^2 \\
&= \omega + \omega\eta_{k-1} + \eta_{k-1}\eta_{k-2}(\omega + \eta_{k-3})\sigma_{k-3}^2 \\
&= \omega + \omega\eta_{k-1} + \omega\eta_{k-1}\eta_{k-2} + \eta_{k-1}\eta_{k-2}\eta_{k-3}\sigma_{k-3}^2 \\
&\vdots \\
&= \omega + \omega\eta_{k-1} + \omega\eta_{k-1}\eta_{k-2} + \cdots + \eta_{k-1}\cdots\eta_{k-N}\sigma_{k-N}^2.
\end{aligned} \tag{1.7}
$$

If there is a solution it must be in the form of

$$\sigma_k^2 = \omega\left[1 + \sum_{j=1}^{\infty}\prod_{i=1}^{j}\eta_{k-i}\right].$$

Our first result gives a condition for the existence of σ_k^2.

Theorem 1.1 *If $E\log\eta_0 < 0$, then*

$$P\left\{1 + \sum_{j=1}^{\infty}\prod_{i=1}^{j}\eta_{-i} < \infty\right\} = 1.$$

PROOF: Let $\gamma = E\log\eta_0$. By the Law of Large Numbers (cf. Durrett (1996)) we have

$$\frac{1}{j}\sum_{i=1}^{j}\log\eta_{-i} \longrightarrow \gamma \text{ almost surely.}$$

The Strong Law of Large Numbers means that there exists a random variable j_0 such that

$$\sum_{i=1}^{j}\log\eta_{-i} < \frac{\gamma}{2}j, \text{ if } j \geq j_0.$$

This yields

$$
\begin{aligned}
\sum_{j=1}^{\infty}\prod_{i=1}^{j}\eta_{-i} &= \sum_{j=1}^{\infty}\exp\left(\sum_{i=1}^{j}\log\eta_{-i}\right) \\
&= \sum_{j=1}^{j_0}\exp\left(\sum_{i=1}^{j}\log\eta_{-i}\right) + \sum_{j=j_0}^{\infty}\exp\left(\sum_{i=1}^{j}\log\eta_{-i}\right) \\
&\leq \sum_{j=1}^{j_0}\exp\left(\sum_{i=1}^{j}\log\eta_{-i}\right) + \sum_{j=j_0}^{\infty}\exp\left(\frac{\gamma}{2}j\right).
\end{aligned}
$$

Since $0 < e^{\gamma/2} < 1$ by the convergence of the geometric series we conclude

$$\sum_{j=1}^{\infty}\exp\left(\frac{\gamma}{2}j\right) < \infty,$$

completing the proof of Theorem 1.1.

Using Theorem 1.1 we have a necessary condition for the existence of a uniqe solution of (1.3).

Theorem 1.2 *If* $\gamma = E \log \eta_0 < 0$, *then*

$$\sigma_k^2 = \omega \left[1 + \sum_{j=1}^{\infty} \prod_{i=1}^{j} \eta_{k-i} \right]$$

is the unique stationary solution of GJR–GARCH.

PROOF: We showed that σ_k^2 exists. It is clear that it is a stationary sequence, since it is composed of independent identically distributed random variables. The argument before the proof of the existence of σ_0^2 gives that it is a solution. Alternately, just plug into the equation.

Next we prove the uniqueness. Assume that there exists another solution, τ_k^2 satisfying

$$y_k = \tau_k \epsilon_k$$

and

$$\tau_k^2 = \omega + \eta_{k-1} \tau_{k-1}^2.$$

Using the recursions again we conclude

$$\sigma_0^2 - \tau_0^2 = \eta_{-1} \eta_{-2} \cdots \eta_{-N} (\sigma_{-N}^2 - \tau_{-N}^2).$$

Using again the Law of Large Numbers and the condition $\gamma < 0$ we obtain

$$\eta_{-1} \eta_{-2} \cdots \eta_{-N} \longrightarrow 0 \text{ a.s.}$$

as $N \to \infty$. Since σ_k^2 and τ_k^2 are stationary sequences, σ_{-N}^2 and τ_{-N}^2 are bounded sequences in probability, we get

$$\sigma_0^2 - \tau_0^2 \longrightarrow 0$$

in probability, as $N \to \infty$. This implies $P\{\sigma_0^2 = \tau_0^2\} = 1$.

Conjecture If GJR–GARCH has a unique stationary solution, then $\gamma = E \log \eta_0 < 0$. It follows from our proof of Theorem 1.1 that there is no solution if $\gamma > 0$. We have to consider the case of $\gamma = E \log \eta_0 = 0$ only.

2 The moments of GJR–GARCH

By Theorem 1.2 it is enough to study the moments of

$$X = \sum_{j=1}^{\infty} \prod_{i=1}^{j} \eta_{-i}.$$

We will use the following well–known inequality (cf. Hardy, Lettlewood and Pólya (1959)):

Minkowski's inequality *Let* $X_1, X_2, \dots$ *be non–negative random variables. If* $EX_i^\nu < \infty$, $1 \leq \nu$, *then*

$$\sum_{i=1}^{n} EX_i^\nu \leq E\left(\sum_{i=1}^{n} X_i\right)^\nu \leq \left(\sum_{i=1}^{n} (EX_i^\nu)^{1/\nu}\right)^\nu$$

and

$$\sum_{i=1}^{\infty} EX_i^\nu \leq E\left(\sum_{i=1}^{\infty} X_i\right)^\nu \leq \left(\sum_{i=1}^{\infty} (EX_i^\nu)^{1/\nu}\right)^\nu .$$

Using Minkowski's inequality, we find the necessary and sufficient condition for the existence of EX^ν.

Theorem 2.1 *We assume that (1.4) and (1.5) hold and* $E\log\eta_0 < 0$.
(i) If $E\eta_0^\nu < 1$, *then* $EX^\nu < \infty$.
(ii) If $E\eta_0^\nu \geq 1$, *then* $EX^\nu = \infty$.

Proof: By Minkowski's inequality we have

$$E\left(\sum_{j=1}^{\infty}\prod_{i=1}^{j}\eta_{-i}\right)^\nu \leq \left(\sum_{j=1}^{\infty}\left(E\left(\prod_{i=1}^{j}\eta_{-i}\right)^\nu\right)^{1/\nu}\right)^\nu .$$

By condition (1.5) we get

$$E\left(\prod_{i=1}^{j}\eta_{-i}\right)^\nu = E\prod_{i=1}^{j}\eta_{-i}^\nu = \prod_{i=1}^{j}E\eta_{-i}^\nu = (E\eta_0^\nu)^j .$$

Using again the properties of the geometric series we conclude that

$$\sum_{j=1}^{\infty}\left(E\left(\prod_{i=1}^{j}\eta_{-i}\right)^\nu\right)^{1/\nu} = \sum_{j=1}^{\infty}\left((E\eta_0^\nu)^{1/\nu}\right)^j < \infty,$$

completing the proof of (i).

Using the other half of Minowski's inequality, similar arguments yield

$$E\left(\sum_{j=1}^{\infty}\prod_{i=1}^{j}\eta_{-i}\right)^\nu \geq \sum_{i=1}^{\infty}E\left(\prod_{i=1}^{j}\right)^\nu = \sum_{i=1}^{\infty}(E\eta_0^\nu)^j = \infty,$$

since by assumption $E\eta_0^\nu \geq 1$. Hence (ii) is also proven.

3 Further research

I wish to investigate the following problems:

Problem 1 How to estimate the parameters $(\omega, \alpha_1, \alpha_2, \beta)$ in the model?

Problem 2 Does GJR–GARCH give better fit for the IBM data then GARCH(1,1) model?

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