Using Rays of Light and a Mirror to Determine the Makeup of the Body of an Object

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1 The Problem

In physics, objects have a property known as an attenuation coefficient. This is a number with the dimension 1/length that measures the relative change in intensity of light that passes through a given portion of the object. If we denote f as the attenuation of the object, Δx as the length of the portion of the object through which the light passes, I_0 as the initial intensity, and ΔI as the change in intensity as it passes through the object, then we obtain the formula

$$\frac{\Delta I}{I_0} = f \Delta x \tag{1}$$

1.1 Direct Problem

If we know the attenuation coefficient of the object and the portion of the object through which a light passes, from (1) we can derive the relative change in intensity of the light. If we were to pass a light through an object that does not have a uniform attenuation coefficient (i.e. the attenuation coefficient at a given point in the object is a function of its position), then we could take the sum of all of the attenuation coefficients at each part of the object multiplied by the lengths of the portions of the respective parts through which the light passes. If we let there be n different "sub-objects" that represent the parts of the object with different attenuation coefficients, then (1) would transform into

$$\frac{\Delta I}{I_0} = \sum_{k=1}^{n} f_k \Delta x_k \tag{2}$$

So, let's say that we have a cross-section Ω of an object that does not have a uniform attenuation coefficient (and whose various attenuation coefficients are known) and it sits in a material z that has a uniform attenuation coefficient. Let's also say that there is a mirror on one side of Ω and we shoot rays of light through Ω from the side opposite the mirror. We will shoot rays from a source, they will travel through Ω and reflect against the mirror (assuming no absorption of the light by the mirror) and then travel back through Ω, where they will then be picked up and their intensities will be measured by a detector on the same side as the source. Knowing the attenuation coefficients of all of the subsections of Ω, we should be able predict what the relative changes in intensity will be using (2).

1.2 Inverse Problem

I propose the question of whether or not we can determine what the attenuation coefficients of the subsections of Ω (as defined above) by knowing just the relative changes in intensity of a set of rays of light that travel through Ω, reflect against a mirror, and then travel back through. If this is possible, then we can let the amount of subsections of Ω approach ∞ and thereby represent the attenuation coefficients continuously throughout Ω as a function of the position. Once we have a nearly continuous representation of the attenuation coefficients in Ω, we can then begin to assert what the actual make-up of the insides of Ω are.

1.3 Cases of Inquiry

1.3.1 Case 1

Suppose Ω to be in the shape of a rectangle (for the sake of simplicity). Let's then assume the subsections of Ω are equally-spaced diagonal partitions whose dividing lines make an angle α with the horizontal. Each partition of Ω would have its own attenuation coefficient and the partitions would thus work to discretize Ω.

1.3.2 Case 2

Suppose Ω to be in the shape of a rectangle (again, for the sake of simplicity). Let's then assume the subsections of Ω are boxes within Ω, thereby discretizing the area of Ω into rectangles. We will assume that there are n partitions along the horizontal and μ partitions along the vertical. We then assume that the attenuation coefficient is uniform within the area of each box. By discretizing Ω into enough rectangles (taking their length (h) and height (g) to be sufficiently small), we can approximate a continuous representation of the attenuation coefficient of Ω. Also, if we know the shape of Ω to be different than a rectangle, we can simply fill in the attenuation coefficients of whichever boxes we wish around the perimeter of the rectangle in such a way to leave the shape of Ω within the rectangle.

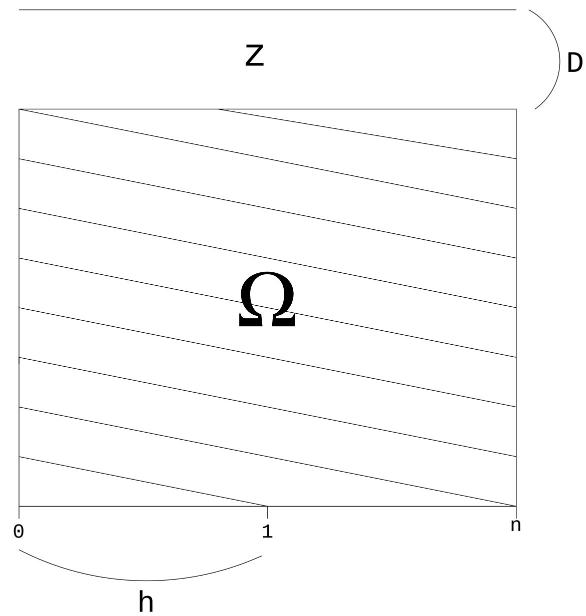


Figure 1: Case 1

2 The Method

When discretizing Ω, whether it be in accordance with Case 1 or Case 2, it is first necessary to place Ω on a Cartesian coordinate system. This is most sensibly done by placing the lower-left corner of Ω at the origin. The length and width (height) of Ω are given as L and W, respectively. We can divide L into n partitions, and denote the length (or step-size) of each of these partitions as h. Once the horizontal is partitioned, there will be points on the x-axis corresponding to where the dividing lines lie. I am choosing to call these points nodes. The rays of light which are material to the problem will enter and exit Ω only through these nodes. So, each ray of light can essentially be characterized by the node it enters and the node it exits. The left-most node will be denoted the 0^{th} node and the right-most node will be denoted the n^{th} node. So, the negative relative change of the intensity of the light entering through the j^{th} node and exiting from the k^{th} node will be denoted as $H_{j,k}$. So, if we denote the amount of partitions over which the ray spans (i.e. the difference in the nodes through which the ray enters and exits) as m, then a ray entering through the i^{th} node will exit through the $(i+m)^{th}$ node. Therefore, the negative relative change in intensity associated with such a ray would be $H_{i,i+m}$.

2.1 Case 1

Let's say we pass rays of light through Ω from our first case of the inverse problem in the way described above. Let's denote y_u (the u standing for "up") as the line that represents the ray of light before it hits the mirror. Let's similarly denote y_d (the d standing for "down") as the line that represents the ray of light after it hits the mirror. The following computation, as well as all

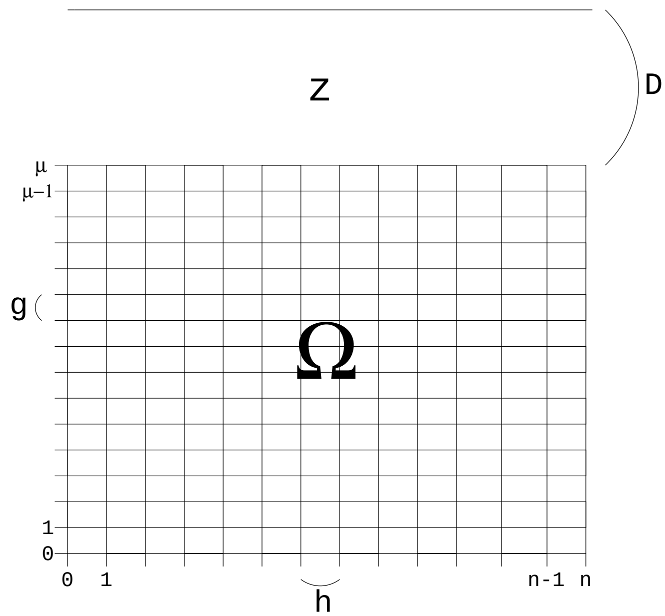


Figure 2: Case 2

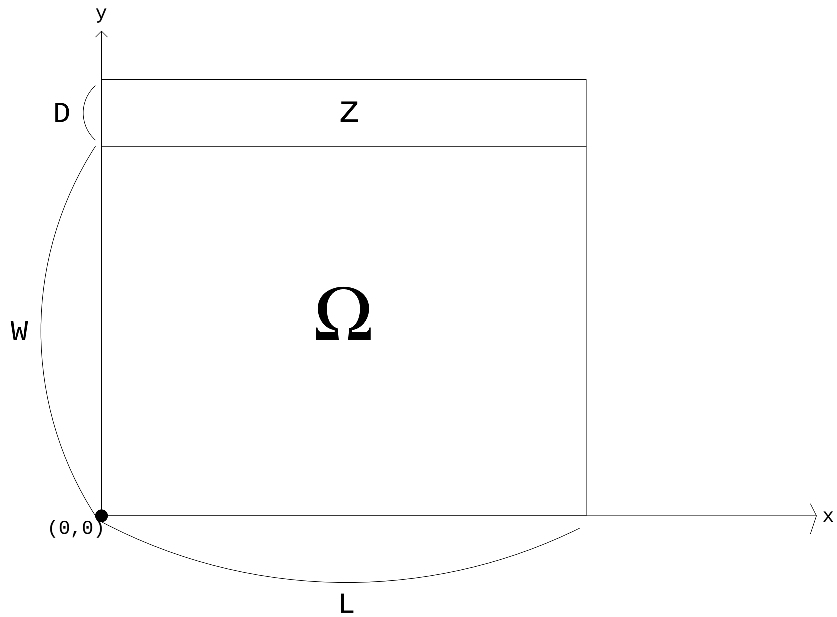
subsequent calculations and formulas from here on in are based in the principles of geometry and trigonometry. It can be shown that the distance the ray of light travels through z, the area in between the mirror and Ω, is $\frac{sd}{W+D}$ where s is the total distance traveled by the ray of light (from the source to the detector). It can also be shown that

$$s = \sqrt{(mh)^2 + (2(W+D))^2}$$

In fact, every distance computed dealing with this problem has in it an $\frac{s}{m}$ factor. So, rather than computing the actual distance traveled through each area, we can compute the distance coefficient (the distance divided by the factor $\frac{s}{m}$). In the case of the coefficient of the distance traveled through z, it would be $\frac{mD}{W+D}$ (i.e. $\frac{\frac{sd}{W+D}}{\frac{s}{m}}$. So, we will label this distance coefficient as τ and we get

$$\tau = \frac{mD}{W+D}$$

where τ is the distance coefficient through z. We can also compute the coefficient of the distance traveled through Ω by either y_u or y_d (it doesn't matter which one, both coefficients will be equal) to be $\frac{mW}{2(W+D)}$. We can compute that t_u, the coefficient of the distance traveled through the complete height of a partition (i.e. any partition besides possibly the last one, through whose height y_u may not completely pass), is



$$t_u = \frac{mh\tan\alpha}{2(W+D)+mh\tan\alpha}$$

Because we now know the coefficient of the distance traveled by y_u before z and the coefficient of the distance traveled by y_u through a complete partition, we are now able to calculate γ, which we'll use to denote the number of complete partitions through which y_u travels. We get

$$\gamma = \lfloor \frac{W(2(W+D)+mh\tan\alpha)}{2h(W+D)\tan\alpha} \rfloor$$

We now know how many partitions through which y_u passes completely and what the distance coefficient is for each partition. The distance coefficient for the last partition (if there is one through which y_u does not pass completely) would be $\frac{mW}{2(W+D)} - \gamma t_u$, which is just the coefficient of the total distance traveled by y_u subtracted by the number of complete partitions γ passed by y_u multiplied by the distance coefficient t_u of each of the partitions. So, for a ray of light (i, m) (where i is the node through which it enters and m is the number of partitions it spans horizontally) we are given a distance coefficient for the $(i+1)^{th}$ through the $(i+\gamma+1)^{th}$ partitions. Without doing any extra work, the ray also gives us a distance coefficient for the first through i^{th} partitions, which is just 0 as the ray does not pass through them. One can also repeat this work with y_d. The only difference is that the actual partitions through which y_d passes depend more so on α. There are two instances for which to account, one being when $\theta + \alpha < \frac{\pi}{2}$, and the other when $\theta + \alpha > \frac{\pi}{2}$, where θ is the angle that the ray of light makes with the normal to the horizontal. First, let's account for when $\theta + \alpha < \frac{\pi}{2}$. In this case, y_{d1} (as we'll call it) can hit many of the same partitions as does y_u. We know the distance y_{d1} travels through Ω and we can compute t_{d1}, the coefficient of the distance traveled by y_{d1} through a complete partition. We can derive that

$$t_{d1} = \frac{mh\tan\alpha}{2(W+D) - mh\tan\alpha}$$

And then, by using this, we can compute Γ_1, which I am denoting as the number of complete partitions through which y_{d1} passes. This yields us

$$\Gamma_1 = \lfloor \frac{W(2(W+D) - mh\tan\alpha)}{2h(W+D)\tan\alpha} \rfloor$$

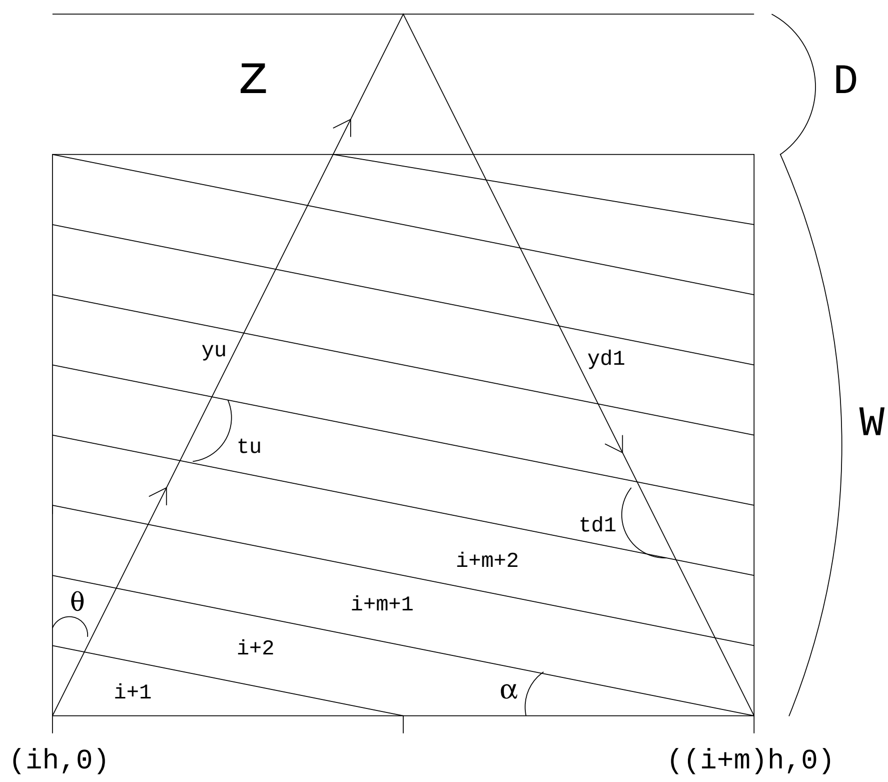


Figure 3: $\theta + \alpha < \frac{\pi}{2}$

So, we know the first partition (in ascending order) through which y_{d1} passes is the $(i+m+1)^{th}$ partition, and we know y_{d1} passes through Γ_1 partitions completely. We are therefore given distance coefficients of the ray (i, m) for the $(i+m+1)^{th}$ through the $(i+m+\Gamma_1)^{th}$ partitions. Then, we can also get the distance coefficient for the last partition through which y_d travels (i.e. the $(i+m+\Gamma_1+1)^{th}$ partition) by taking $\frac{mW}{2(W+D)} - \Gamma_1 t_{d1}$, similarly to how we did it with y_u. We also know that the distance coefficients of y_{d1} for the $(i+m+\Gamma_1+2)^{th}$ through the $(n+G)^{th}$ partition are all 0, as it does not pass through them. The quantity $n+G$ is simply the total number of partitions of Ω. Where

$$G = \lceil \frac{W}{h} \cot \alpha \rceil$$

The other instance for which we must account is that in which $\theta + \alpha > \frac{\pi}{2}$. In this case, y_{d2} (as we'll call it) can hit at most one of the same partitions as does y_u. We can compute t_{d2}, the coefficient of the distance traveled by y_{d2} through a complete partition to be

$$t_{d2} = \frac{mh \tan \alpha}{mh \tan \alpha - 2(W+D)}$$

Using the following, we can compute Γ_2, the number of complete partitions through which y_{d2} passes, to be

$$\Gamma_2 = \lfloor \frac{W(mh \tan \alpha - 2(W+D))}{2h(W+D) \tan \alpha} \rfloor$$

The partitions through which y_{d2} passes through completely are the $(i+m-\Gamma_2+1)^{th}$ through the $(i+m)^{th}$. The distance coefficient for the first partition through y_{d2} passes, namely the $(i+m-\Gamma_2)^{th}$, would be $\frac{mW}{2(W+D)} - \Gamma_2 t_{d2}$. We also know the distance coefficients from the $(i+m+1)^{th}$ through the $(n+G)^{th}$ partitions to be 0, as y_{d2} does not pass through them. Once we compute all of the distance coefficients for y_u, y_{d1}, and y_{d2}, we are given a vector of length $n+G$ corresponding to each of the three lines. The elements in each vector are the coefficients of the distance traveled through each partition by each respective line. In the instance where $\theta + \alpha < \frac{\pi}{2}$, we add the vectors y_u and y_{d1} to get a vector $\lambda_{i,m}$ (the subscripts are there because we get a vector for each ray (i,m)). In the instance where $\theta + \alpha > \frac{\pi}{2}$, we add the vectors y_u and y_{d2} to get $\lambda_{i,m}$. If we think of $\lambda_{i,m}$ as a row vector and f as a column vector also of length $n+G$ with its elements simply being the attenuation coefficients of the partitions 1 through $n+G$, then the dot product of the two vectors should be equal to $J_{i,m}$. I am denoting $J_{i,m}$ as $H_{i,i+m}$ divided by the aforementioned distance factor of $\frac{s}{m}$. We can express this as

$$J_{i,m} = \frac{mH_{i,m}}{\sqrt{(mh)^2 + (2(W+D))^2}}$$

We are left with the equation

$$\lambda_{i,m} * f = J_{i,m}$$

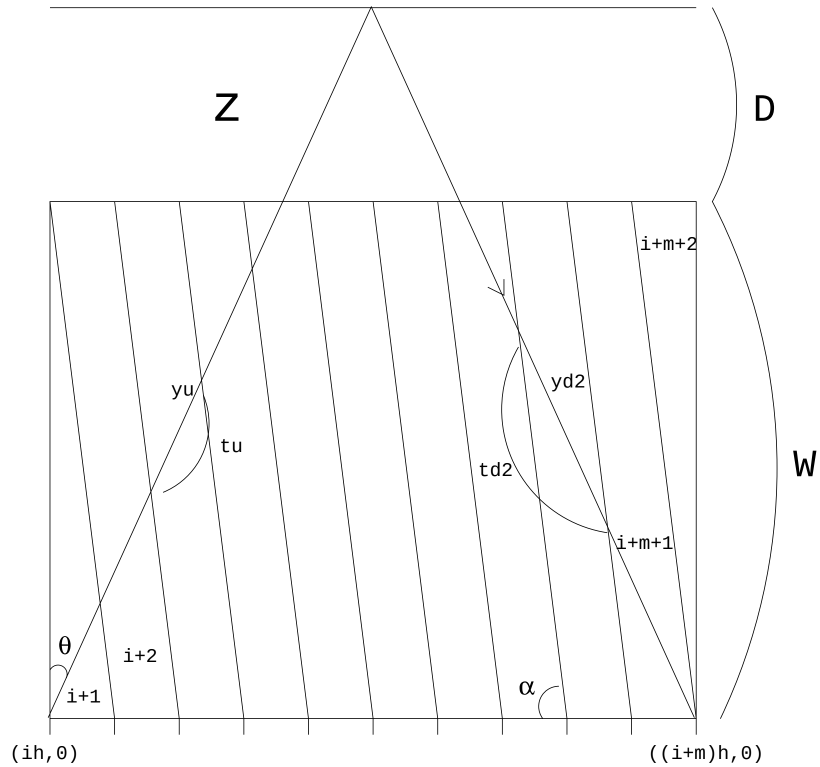


Figure 4: $\theta + \alpha < \frac{\pi}{2}$

This is true for $i = 0, ..., n-1$ and $m = 1, ..., n-i$. This yields us $\frac{1}{2}n^2 + \frac{1}{2}n$ equations in $n+G$ unknowns. This system of equations can be represented as

$$\Lambda f = j$$

where $\Lambda \in \mathbb{R}^{(\frac{1}{2}n^2+\frac{1}{2}n)*(n+G)}$, $f \in \mathbb{R}^{n+G}$, and $j \in \mathbb{R}^{\frac{1}{2}n^2+\frac{1}{2}n}$. To solve this system of equations, we can right-multiply both sides of the system by Λ^T to get

$$\Lambda^T \Lambda f = \Lambda^T j$$

To solve this system, we need would ideally only to right-multiply both sides by $(\Lambda^T \Lambda)^{-1}$ to obtain f. We'd get

$$f = (\Lambda^T \Lambda)^{-1} \Lambda^T j$$

Our solution f will be the best solution to match the either overdetermined or underdetermined (or, in the rare case in which $\frac{1}{2}n^2 + \frac{1}{2}n = n + G$, square) system of equations we are given by the rays we shoot.

2.2 Case 2

Let's say we pass $\frac{1}{2}n^2 + \frac{1}{2}n$ rays, each indexed as (i, m), through Ω as we did in the previous case. Let's take a look at what happens with the light before it hits the mirror (which I'll denote merely as y). The line intersects with many of the dividing lines of our sample of Ω. Since we have placed Ω in a Cartesian coordinate system, we can determine the intersection of y with all of these lines, both horizontal and vertical. First let's look at the vertical lines. Notice that y only hits a partial amount of these lines before it exits Ω. Let's denote the last vertical line it hits as the a^{th} line, where

$$a = \lfloor \frac{m\mu g}{2(\mu g + D)} \rfloor$$

The equation of the N^{th} vertical line is $x = (i + N)h$. The intersection of the N^{th} vertical line with y, which can be written as $y = \frac{2(\mu g+D)}{mh}(x - ih)$ occurs at $((i + N)h, \frac{2N(\mu g+D)}{m})$. Now, if we look at the horizontal lines, we see that y hits every horizontal line before it exits Ω. So, the last horizontal line it hits is the μ^{th} line. The equation of the M^{th} horizontal line is $y = Mg$. The intersection of the M^{th} horizontal line with y occurs at $((i + \frac{mMg}{2(\mu g+D)})h, Mg)$. We can let x_1 be the set of $(i+N)h$ for $N = 1, ..., a$, thereby representing the set of the x-coordinates of the intersections of y with the a vertical lines. We can also let y_1 be the set of $\frac{2N(\mu g+D)}{m}$ for $N = 1, ..., a$, thereby representing the corresponding set of y-coordinates of the intersections of y with the a vertical lines. Similarly, we can let x_2 be the set of $(i + \frac{mMg}{2(\mu g+D)})h$ for $M = 1, ..., \mu$, thereby representing the set of the x-coordinates of the intersections of y with the μ horizontal lines. We can also let y_2 be the set of Mg for $M = 1, ..., \mu$, thereby representing the corresponding set of y-coordinates of the intersections of y with the μ horizontal lines. Let's take X to be the union of x_1 and x_2 sorted in ascending order, and Y to be the union of y_1 and y_2 sorted in ascending order. The two-column matrix (X, Y) then represents every intersection of y with one of the dividing lines of our section of Ω. We can define a vector v to be the distance from an intersection of y with one of the dividing lines (i.e. a row of (X, Y)) from the point $(ih, 0)$, the starting point of the ray (i, m). If we append 0 to the beginning of v, and then we store the consecutive differences of v into a vector d, then d will be the vector of the distances that y travels through each box. So, we have

$$v_k = \sqrt{X_k^2 + Y_k^2}$$

for $k = 1, ...,$ number of rows in (X, Y), and

$$v_0 = 0$$

And then, we have

$$d_k = v_k - v_{k-1}$$

for $k = 1, ...,$ number of rows in (X, Y). Now, since we know the distances traveled by y in order of when the distance is traveled, we need to determine the box through which each distance was traveled. First, let's determine the coordinates of the box by a 2-tuple of the column in which the box lies followed by the row. The total numbers of columns and rows would simply be n and μ, respectively. The first column being the left-most column, the n^{th} column being the right-most column, the first row being the bottom-most row, and the μ^{th} row being the top-most row. With this notation, it can be found that the box hit immediately after y intersects with the N^{th} vertical line is the box $(i + N + 1, \epsilon_N + 1)$, where

$$\epsilon_N = \lfloor \frac{2N(\mu g + D)}{mg} \rfloor$$

The box hit immediately after y intersects with the M^{th} horizontal line is the box $(i+\eta_M+1, M+1)$, where

$$\eta_M = \lfloor \frac{mMg}{2(\mu g + D)} \rfloor$$

In the same manner as we formed the matrix (X, Y), we can form a two-column matrix (C, R), where C is the column-coordinate of each box hit and R is the row-coordinate of each box hit. In order to sort (C, R) into an order corresponding to the distances traveled through each box, we need only to order the rows of (C, R) according to the sum (i.e. $C_k + R_k$ for the k^{th} row) of the rows. If the rows are in such an order such that the sums of the rows go in ascending order, then the boxes will correspond to the distance y travels through them. Due to the symmetry of the problem, we can produce another two-column matrix consisting of a vector of column-coordinates and a vector of row-coordinates which corresponds to the boxes that are traveled through by the light on its way back from the mirror. We can keep R the same as the vector of row-coordinates, however, we will have to come up with a new vector of column-coordinates, CC. We can do this with the formula

$$CC_k = i + 2m + 1 - C_k$$

for every value of k corresponding to an element in C. So, we now know the distance traveled through each box by the ray (i, m). If the ray does not pass through a certain box, clearly the distance traveled through that box by the ray would be 0. Using MatLab, we are able to represent every ray we pass through Ω by a row in a matrix by using the SUB2IND function. The SUB2IND function just transforms a quantity that has two indices (e.g. the rays which are indexed by i and m) into a row vector with only one index. By doing this for every ray we shoot, we generate a $(\frac{1}{2}n^2 + \frac{1}{2}n) * \mu n$ matrix. Where each row represents a ray we shoot and each column corresponds to a rectangular partition of Ω (since we've partitioned it on the horizontal by n and on the vertical by μ, there are μn of these rectangular partitions). This matrix, which we can denote simply as A, will be multiplied by f, a column vector of the attenuation coefficients of each of the boxes, and h, the column vector of all of the aforementioned $H_{i,i+m}$ values. So, we get the system of equations

$$Af = h$$

where $A \in \mathbb{R}^{(\frac{1}{2}n^2+\frac{1}{2}n)*(\mu n)}$, $f \in \mathbb{R}^{\mu n}$, and $h \in \mathbb{R}^{\frac{1}{2}n^2+\frac{1}{2}n}$. The same mathematics as were applied in Case 1 can be applied here to yield us

$$f = (A^T A)^{-1} A^T h$$

where f, as previously, will be the best solution to our system of equations which implies the most sensible attenuation coefficients within Ω.

3 The Results

3.1 Case 1

In Case 1, we were solving the system of equations we got from partitioning Ω on an angle α by solving the normal equations. As would have been expected, the condition number of our matrix $\Lambda^T \Lambda$ was relatively small (less than the dimension of the problem) for $\alpha = \frac{\pi}{2}$. The problem was very well-conditioned and the dimension of the nullspace was 0, so there did exist a unique solution for f. As α decreased, the condition number of $\Lambda^T \Lambda$ fluctuated while increasing on the average. The nullspace remained 0 so the solution remained unique. As α approached 0, however, the condition number blew up and f became virtually impossible to recover. This can be explained because each ray being shot through Ω would travel the same distances through the same partitions because they become merely horizontal layers. It is this case, actually, that will help explain the nullspace of Case 2.

3.2 Case 2

Case 2 turned out to have a large nullspace. For the dimension of the problem, μn, the nullspace would have a dimension of $(\mu - 1)n$, which increases rapidly as μ and n increase. Since our representation of the attenuation coefficients becomes more accurate as μ and n both go to ∞ and

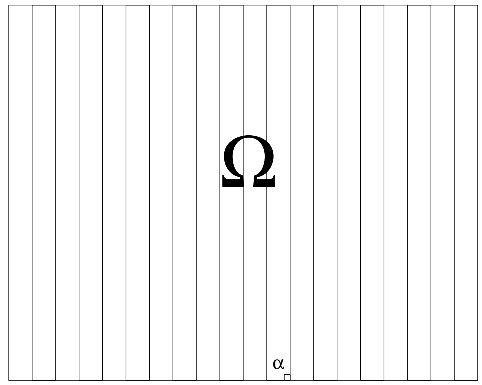


Figure 5: $\alpha = \frac{\pi}{2}$

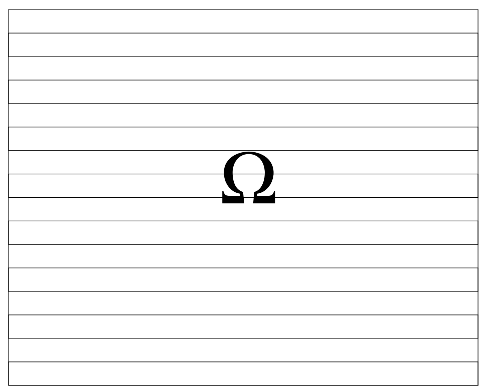


Figure 6: $\alpha = 0$

our nullspace gets relatively much larger as μ and n increase, the nullspace plays quite a significant role as our representation becomes more accurate. By using MatLab, we were able to produce a grayscale depiction of what arrangements of the attenuation coefficients would be in the nullspace (i.e. which arrangements we would not be able to recover by the proposed method of shooting $\frac{1}{2}n^2 + \frac{1}{2}n$ rays of light). By viewing these possible arrangements for various values of μ and n, one can deduce (or, more appropriately, intuit) why these arrangements would be in the nullspace. In all of these arrangements, the sums of the columns of boxes of Ω seemed to all add up to the same number. In other words, for many of the rays, when they go through these certain arrangements of boxes, will produce the same data. So, essentially, there will only be n elements of unique data. These elements will be the data reported when a ray only travels through one column, there being n of the $\frac{1}{2}n^2 + \frac{1}{2}n$ rays that do this. So, these n rays correspond to the n vectors that correspond to a non-zero singular value of A. (Whereas the $(\mu - 1)n$ vectors that span the nullspace of A correspond the the zero singular values of A). This occurrence was predicted by the results of Case 1, in which the system where each of the rows had the same attenuation coefficients could not be recovered (the condition number of the system went to ∞). In an attempt to regularize the system

$$A^TAf = A^Th \tag{3}$$

I added $\beta I_{\mu n}$ to A^TA on the left side of the equation, where $\beta > 0$ and I_a is the $a * a$ identity matrix. This gives us

$$(A^TA + \beta I_{\mu n})f = A^Th \tag{4}$$

I used MatLab to produce a random column vector of length μn to represent a solution f_0 to (3). I then set h equal to Af_0. Using this h, I utilized MatLab to find the least-squares solution f to (4). To find the f that best approximates f_0, I minimized the 2-norm of $f - f_0$ with respect to β and found that f is most accurate when $\beta \approx 10^{-5}$. When β becomes smaller than 10^{-5}, the 2-norm of $f - f_0$ tends towards the same value as if $\beta = 0$. So, if we add $10^{-5}I_{\mu n}$ to our left-hand operator, our solution is optimized.

4 The Future

Although f may have been optimized using the proposed method, I am certain that there is much more that can be done to obtain an even more accurate solution to the problem. Later research may consist of assuming Ω is periodic, rather than just a finite blob, as it were. In the results, it was noticed that there was an ideal multiple, β, of the identity matrix to add to our operator in order to best regularize the problem. It was never realized, however, why the actual ideal value of β we found would work the best. Further research can be done in an attempt to discover more truths about this particular inverse problem and possibly help explain why such a value of β would work best. Of course, since all of this research was done in the better part of only six weeks, it was not as thorough as I would have liked. If more time were allotted to the problem, I would have definitely liked to have seen what would happen if we set certain boxes around the perimeter of Ω to the same attenuation coefficient as the surrounding material, z, and examined the case in which we tried to determine the attenuation coefficient through a cross-section that is not so "pretty". It may also have been interesting, if not of the utmost difficulty, to look into partitioning Ω with lines that weren't only parallel to the horizontal and vertical (e.g. diagonal lines, circles, parabolas, etc.) Even further inquiry can take into account Snell's Law of Refraction ($n_1 \sin\theta_1 = n_2 \sin\theta_2$) when the light travels through Ω and possibly determine a correlation between an object's refractive index and its attenuation coefficient (perhaps one depends on the other, which would make the problem quite a bit more complicated). All of these ideas and more would make interesting research for the future of this problem; I only hope that I can some day perform the actual research (or, at least, hear the results of someone who has performed it).