Inverse Problems REU: Week 1

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June 4, 2004

1   Introduction

The purpose of these notes is to give some clarity to the goals of the lab sessions and to establish the relationship of the lab sessions to the afternoon lectures.

2   Integral Equations

During the first week of this REU, we have discussed solving integral equations. As in the text, we consider several types of integral equation.

2.1   Fredholm Integral Equations of the First Kind

In lecture on Tuesday, we talked about the integral equation:

$$y(s) = \int_0^1 K(s,t)x(t)dt \tag{2.1}$$

In general, $K(s,t)$ is called the *kernel* of the equation. Often equation (2.1) is used as a solution of a differential equation. That is, given a function x we can find a kernel K such that the integral satisfies a differential equation. In our context, we will consider the inverse problem. That is, given some set of measurements y, what kind of function x best represents the solution?

To compute such a solution, we must discretize the integral given in equation (2.1) and solve a linear system. Specifically, we saw that discretizing the independent and dependent variables with

$$s_j = \frac{j-1}{N} \qquad\qquad t_l = \frac{l-1}{N},$$
$$y(s_j) = y_j \qquad\qquad x(t_l) = x_l,$$
$$K(s_j, t_l) = K_{jl},$$

and applying the trapezoidal rule with

$$A_{jl} = \begin{cases} \dfrac{1}{2n}K_{jl} & l = 1 \text{ or } l = N+1, \\[2mm] \dfrac{1}{N}K_{jl} & \text{otherwise,} \end{cases}$$

we must solve the linear system

$$Ax = y.$$

EXERCISES

2.1 Consider the system describing **Geological Prospecting** given in Groetsch (take $\gamma = 1$). In this case, we consider the kernel

$$K(s,t) = [(s-t)^2 + 1]^{(-3/2)}$$

For $0 < t < 1$ and $0 < s < 1$, write a Matlab function to compute $K(s,t)$ given a discretization as described above.

2.2 What is the matrix A that corresponds to the linear system $Ax = y$? Compute the SVD and plot the singular values. *Hint*: Use a semilog plot!

2.3 What is the relationship between the condition number and the ratio of the largest to the smallest singular value?

2.4 How does the matrix change if we change the discretization to write $s_j = (j-1)/N$ and $t_l = (l-1)/M$ for $M \neq N$? Examine the cases $M < N$ and $M > N$ separately.

2.2 Volterra Integral Equations

An equally useful type of integral equation is given by the equation

$$y(s) = \int_0^s K(s-t)x(t)dt \tag{2.2}$$

For example, the system

$$\begin{aligned} y'(s) + ay(s) &= x(s), \\ y(0) &= 0, \end{aligned} \tag{2.3}$$

has solution

$$y(s) = \int_0^s e^{-a(s-t)}x(t)dt. \tag{2.4}$$

In the solution given by equation (2.4), the kernel is the function

$$K(\tau) = e^{-a\tau}. \tag{2.5}$$

EXERCISES

2

2.5 Verify the the integral equation (2.4) is the solution of the system (2.3).

2.6 Once again using the trapezoidal rule, find the matrix that solves the linear system $Ax = y$ corresponding to the integral equation (2.4) for $0 < s < 1$.

2.7 Compute the SVD of the matrix A and plot the singular values. *Hint*: Use a semilog plot!

2.3   Backward Heat Equation

During lecture on Wednesday, we considered the heat equation

$$
\begin{aligned}
u_t &= u_{xx}, \\
u(0, t) = u(\pi, t) &= 0, \\
u(x, 0) &= f(x).
\end{aligned}
\tag{2.6}
$$

We saw that system (2.6) is well posed as a forward problem with solution

$$
u(x, t) = \sum_{n=0}^{\infty} f_n e^{-n^2 t} \sin nx.
\tag{2.7}
$$

Then, we formulated the backward problem as follows. Given a profile $g(x)$ at time $t = 1$, what was the original profile? That is, given $g(x) = u(x, 1)$, can we recover $f(x) = u(x, 0)$? Explicitly, we see that given some Fourier representation of $f(x)$ and $g(x)$, we can write

$$
u(x, 1) = g(x) = \sum_{n=0}^{\infty} g_n \sin nx,
$$

$$
u(x, 0) = f(x) = \sum_{n=0}^{\infty} f_n \sin nx.
$$

From the solution (2.7) we see that

$$
g(x) = \sum_{n=0}^{\infty} f_n e^{-n^2} \sin nx,
$$

so that

$$
g_n = f_n e^{-n^2}.
$$

Writing the matrix A with

$$
A_{jl} = \begin{cases} e^{-j^2} & j = l, \\ 0 & j \neq l, \end{cases}
$$

we see that the linear system to solve is

$$
Af = g.
$$

EXERCISES

2.8 In the problem posed above, the matrix A is infinite dimensional. Write a Matlab function to form A truncated to represent an $m \times m$ system.

2.9 Compute the SVD of A. Plot the singular values. For increasing values of m, use Matlab to find the condition number of A. Plot the condition number versus the size of the system. *Hint*: A semilog plot may be useful.

2.4 Backward Heat Equation, Part II

For $t > 0$, define the function $G(x, t)$ such that

$$G(x,t) = \frac{1}{\sqrt{4\pi t}} \sum_{n=-\infty}^{\infty} e^{-(x-n)^2/4t} \tag{2.8}$$

Then, the solution of the system (2.6) is given by

$$u(x,t) = \int_0^1 K(x,y,t)f(y)dy \tag{2.9}$$

where

$$K(x,y,t) = \frac{1}{2}\left[G\left(\frac{x-y}{2}, \frac{t}{4}\right) - G\left(\frac{x+y}{2}, \frac{t}{4}\right) \right] \tag{2.10}$$

EXERCISES

2.10 Verify that the integral equation (2.9) with kernel given by (2.10) is a solution of the system (2.6).

2.11 Truncating and using a trapezoidal rule as in previous exercises, what is the matrix A that allows the integral equation (2.9) to be represented as a linear system of $2m + 1$ equations?

2.12 Write a Matlab function that computes A given m.

2.13 Compute the SVD of A. Plot the singular values. For increasing values of m, use Matlab to find the condition number of A. Plot the condition number versus the size of the system. *Hint*: Again, a semilog plot may be useful.