REPRESENTATION THEORY

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CONTENTS

FOREWORD

These are the notes for the graduate-level representation theory course that I taught at the University of Utah in the spring semester of 2024. The topics presented here correspond to roughly thirteen weeks (that is, about 40 hours) of lectures.

The course consists of two parts that differ both in content and in style. The first part of the course — consisting of Chapters I and II — could be considered a standard introduction to representation theory: Chapter I covers the representation theory of finite groups, while Chapter II treats the case of compact groups, culminating with the Peter–Weyl theorem. The topics and the exposition in this part of the course are more-or-less traditional. The one choice that is perhaps unusual stems from the effort to keep the discussion self-contained: Chapter II features a considerable amount of analysis, including a proof of the existence of the Haar measure (following Rudin). There are a number of problems (of varying difficulty) scattered across the text; they are meant to provide examples and fill in any gaps left in the proofs. For ease of reference, some standard results from analysis are listed in Appendix A.

The second part of the course, which corresponds to Chapter III, is of a much more experimental nature. My goal here was to introduce students to the representation theory of compact Lie groups, and in particular, to explain the Theorem of the highest weight. Of course, presenting this material over just six weeks of lectures requires one to make certain compromises. My approach was to get to representation-theoretic results as fast as possible, at the expense of much of the background material. This was by no means an easy decision, as it meant omitting (or skimming over) a large part of the Lie group–Lie algebra correspondence and the structure theory of semisimple Lie algebras. As a partial substitute, all the crucial results are stated within the text, and ample references are provided in Appendix B. The decision to compress all these topics into a three-week crash course is certainly unorthodox, but it leaves just enough time for a nice overview of highest weight theory. The exposition here follows Hall.

Big thanks are due to all the students in this class — I hope you find these notes useful.

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CHAPTER I: REPRESENTATION THEORY OF FINITE GROUPS

1. Group representations

Let G be a group. A **representation** of G on a complex vector space V is a homomorphism $\pi : G \to \mathrm{GL}(V)$. We write (π, V) for the corresponding representation. In this course we will deal only with complex representations, i.e. V is always a complex vector space.

Suppose (π, V) and (ρ, W) are representations of G. We say that a linear map $A : V \to W$ is an **intertwining operator** if

$$A\pi(g) = \rho(g)A$$

for all $g \in G$. The set of all intertwining operators from (π, V) to (ρ, W) is denoted $\mathrm{Hom}_G(V, W)$ or $\mathrm{Hom}_G(\pi, \rho)$. One checks that this is a vector space under the usual (pointwise) operations.

We have thus described the objects and the morphisms of the category $\mathrm{Rep}(G)$. Isomorphic objects in this category are said to be **equivalent**. Given representations (π, V) to (ρ, W), we can always consider their (external) **direct sum**. This is the representation $\pi \oplus \rho$ on $V \oplus W$ defined by

$$(\pi \oplus \rho)(g)(v, w) = (\pi(g)v, \rho(g)w).$$

Let (π, V) be a representation. We say that a subspace $W \subseteq V$ is a **subrepresentation** of V if it is G-invariant: $\pi(g)W \subseteq W$ for each $g \in G$. Restricting $\pi(g)$ to W for each $g \in G$, we get a new representation (σ, W) of G. We say that a non-zero representation (π, V) of G is **irreducible** if it does not have subrepresentations other than 0 and V. The set of (equivalence classes of) all irreducible representations of G will be denoted $\hat{G}$ or $\mathrm{Irr}\, G$. If $W \subseteq V$ is a subrepresentation, then the quotient V/W is also naturally a representation of G, with the action given by

$$\pi(g)(v + W) = \pi(g)v + W.$$

If $A \in \mathrm{Hom}_G(V, W)$ is an intertwining, then the kernel of A (denoted $\ker A$) is a subrepresentation of V. Similarly, the image (denoted $\mathrm{im}\, A$) is a subrepresentation of W. We have the usual isomorphism theorems, for instance

$$V/\ker A \cong \mathrm{im}\, A.$$

We say that a representation (π, V) of G is **semisimple** (or **completely reducible**) if it satisfies the following property: if W is a subrepresentation of V, then there exists another subrepresentation of U of V such that $V = W \oplus U$. See §1.1 for more on semisiplicity.

Lemma 1.1. *Let (π, V) and (ρ, W) be representations of G.*

(i) If V is irreducible, then any non-zero intertwiner $A \in \mathrm{Hom}_G(V, W)$ is injective.

(ii) If W is irreducible, then any non-zero intertwiner $A \in \mathrm{Hom}_G(V, W)$ is surjective.

(iii) If both V and W are irreducible, then any non-zero intertwiner $A \in \mathrm{Hom}_G(V, W)$ is an isomorphism.

Proof. (i) Suppose $A \in \mathrm{Hom}_G(V, W)$ is non-zero. Then $\ker A$ is a subrepresentation of V. However, V is irreducible, so the only possible subrepresentations are 0 and V. Thus $\ker A$ is equal to 0 or V. Since we are assuming $A \neq 0$, the kernel cannot be equal to all of V. We conclude $\ker A = 0$, so A is injective.

(ii) Similar, by considering $\mathrm{im}\, A$.

(iii) Follows from (i) and (ii). □

In particular, if both V and W are irreducible, $\mathrm{Hom}_G(V, W) = 0$ unless $V \cong W$. In that case, we have the following result:

Theorem 1.2 (Schur's Lemma). *Let V be a finite-dimensional irreducible representation. Then $\mathrm{Hom}_G(V, V) = \mathbb{C}$.*

Proof. Let $A \in \mathrm{Hom}_G(V, V)$. Then A has an eigenvalue, say λ. Now $A - \lambda \cdot I$ is an element of $\mathrm{Hom}_G(V, V)$ with a non-trivial kernel. Since V is irreducible, Proposition 1.1 shows that $A - \lambda \cdot I = 0$. Thus $A = \lambda \cdot I$, which we needed to prove. □

We end this section with two general-purpose definitions.

We say that a representation (π, V) of G is **unitary** if there exists a G-invariant inner product on G, i.e. an inner product with respect to which every $\pi(g)$ is unitary.

For any representation (π, V) of G, we can define the **contragredient** of π, denoted by $\pi^\vee$. This is a representation of G on V^*, the dual space of V (that is, the space of all linear maps $V \to \mathbb{C}$) defined by

$$(\pi^\vee(g)f)v = f(\pi(g^{-1})v), \quad \text{for all } v \in V, f \in V^*.$$

1.1. **Semisimplicity.** Here we state some standard results about semisimplicity. The proofs are left as exercises.

Proposition 1.3. *Let (π, V) be a semisimple representation and let W be a subrepresentation of V. Then W and V/W are also semisimple.*

Problem 1. Prove Proposition 1.3.

Proposition 1.4. *Let (π, V) be a non-zero semisimple representation. Then (π, V) contains an irreducible representation.*

Problem 2. Prove Proposition 1.4.

Theorem 1.5. *Let (π, V) be a representation. The following are equivalent:*

(i) V is semisimple;

(ii) V is equal to the sum of (all of) its irreducible submodules;

(iii) V is equal to a direct *sum of irreducible submodules.*

Problem 3. Prove Theorem 1.5.

1.2. **Examples.**

Example 1.6. Let $G = \mathrm{GL}_n(\mathbb{C})$. The *standard representation* of $\mathrm{GL}_n(\mathbb{C})$ is given by matrix multiplication: $\pi(g)v = g \cdot v$ for $g \in G, v \in \mathbb{C}^n$.

Problem 4. Prove that the standard representation of $\mathrm{GL}_n(\mathbb{C})$ is irreducible.

Example 1.7. Another representation of $G = \mathrm{GL}_n(\mathbb{C})$ is the determinant: G acts on the 1-dimensional space $\mathbb{C}$ by $\det(g)$.

Example 1.8. Let $G = S_n$ (the symmetric group on n letters). Let $\{e_1, \ldots, e_n\}$ be the standard basis for $V = \mathbb{C}^n$. Then there is a natural representation π of G on V given by

$$\pi(\sigma)e_i = e_{\sigma(i)}.$$

Problem 5. Keep the notation from Example 1.8. Show that

$$U = \{\sum_{i=1}^n c_i \cdot e_i : \sum_{i=1}^n c_i = 0\} \quad \text{and} \quad W = \mathbb{C} \cdot x, \quad \text{where } x = \sum_{i=1}^n e_i$$

are subrepresentations of G, and that $V = U \oplus W$. Moreover, show that U is irreducible.

2. Complete reducibility

Henceforth, until the end of this chapter, G will be a finite group. We let $|G|$ denote the number of elements in G.

Lemma 2.1. *Let V be a representation of G. For each $v \in V$ there exists a finite-dimensional subrepresentation of V containing v.*

Proof. Let $v \in V$. The set $\{\pi(g)v : g \in G\}$ is a finite subset of V, and its span is clearly G-invariant. It is thus a finite-dimensional subrepresentation, and it contains v because $\pi(e)v = Iv = v$. $\square$

Corollary 2.2. *Let (π, V) be a non-zero representation of G. Then V contains an irreducible subrepresentation.*

Proof. Lemma 2.1 shows that there exists a non-zero finite-dimensional subrepresentation W of V. If W is irreducible, we are done. Otherwise, W contains a non-zero subrepresentation of strictly smaller dimension. Proceeding inductively, we arrive at an irreducible representation of V. $\square$

Problem 6. Find an example showing that Lemma 2.1 and Corollary 2.2 can fail if G is infinite.

Proposition 2.3. *Every irreducible representation of G is finite-dimensional.*

Proof. Let (π, V) be an irreducible representation of G. Take $0 \neq v \in V$. By the Lemma 2.1, there exists a finite-dimensional subrepresentation of V containing v. Since V is irreducible, this subrepresentation must be equal to all of V (it cannot be 0 because it contains $v \neq 0$). Thus V is finite-dimensional. $\square$

Theorem 2.4 (Maschke)**.** *Every representation of G is semisimple.*

Proof. Let (π, V) be a representation of G and let $W \subseteq V$ be a subrepresentation. Let $A : V \to W$ be a projector ($A^2 = A$) onto W (not necessarily an intertwining operator). Consider the linear map A^G defined by

$$A^G = \frac{1}{|G|} \sum_{g \in G} \pi(g^{-1}) A \pi(g).$$

Cleary, A^G is a linear map from V to W. We claim that it is moreover an intertwining operator for π. Indeed, for any $g \in G$

$$A^G \pi(g) = \frac{1}{|G|} \sum_{h \in G} \pi(h^{-1}) A \pi(hg) = \frac{1}{|G|} \sum_{h \in G} \pi(g) \pi((hg)^{-1}) A \pi(hg) = \pi(g) A^G,$$

by a simple change of variables. Furthermore, for any $w \in W$ we have

$$A^G w = \frac{1}{|G|} \sum_{h \in G} \pi(h^{-1}) A \pi(h) w = \frac{1}{|G|} \sum_{h \in G} \pi(h^{-1}) \pi(h) w = \frac{1}{|G|} \sum_{h \in G} w = w.$$

This shows that A^G is also a projector onto W. We thus have

$$V = \text{im } A^G \oplus \ker A^G = W \oplus \ker A^G.$$

In other words, W has a G-invariant direct complement, which is what we needed to prove. □

The finiteness assumption is important:

Example 2.5. Let $G = \mathbb{R}$ and consider the representation of G on $\mathbb{R}^2$ given by

$$\pi(x) = \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}$$

This is indeed a representation:

$$\pi(x)\pi(y) = \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & y \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & x+y \\ 0 & 1 \end{pmatrix} = \pi(xy).$$

Notice that it contains the trivial representation as a subrepresentation, spanned by the vector $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$. The corresponding quotient is also (1-dimensional and) trivial. However, π cannot be decomposed into a direct sum of two 1-dimensional subrepresentations. This would imply that $\mathbb{R}^2$ has a basis consisting of eigenvectors for the matrix $\pi(x)$. In other words, $\pi(x)$ would need to be diagonalizable, which it is not (the matrix is already in its Jordan normal form).

Corollary 2.6. *Let (π, V) be a finite-dimensional representation of G. Then π decomposes as a direct sum*

$$\bigoplus_{i=1}^{n} V_i^{\oplus n_i}$$

where the V_i are mutually non-isomorphic and n_i are positive integers. This decomposition is unique, in the sense that the numbers n_i attached to the isomorphisms classes of V_i do not depend on the decomposition.

Problem 7. Prove Corollary 2.6.

The averaging method used above plays a crucial role in representation theory of finite groups. We will need a slightly more general statement:

Theorem 2.7. *Let (π, V) and (ρ, W) be two representations of G. For any linear map $A : V \to W$ let*

$$A^G = \frac{1}{|G|} \sum_{g \in G} \rho(g^{-1}) A \pi(g)$$

Then $A \mapsto A^G$ is a linear projector from $L(V, W)$ (the space of linear maps $V \to W$) to $\mathrm{Hom}_G(V, W)$.

Problem 8. Prove Theorem 2.7.

Here is another application of the averaging method:

Proposition 2.8. *Let (π, V) be a representation of G. Then there exists a G-invariant inner product on V. In other words, there exists an inner product with respect to which π is unitary.*

Proof. Let $(\cdot|\cdot)$ be an arbitrary inner product on V. For any $v, w \in V$ set

$$\langle v, w \rangle = \frac{1}{|G|} \sum_{g \in G} (\pi(g)v | \pi(g)w).$$

We claim that $\langle \cdot, \cdot \rangle$ is a G-invariant inner product on V. One verifies immediately that this form is sesquilinear and positive-definite. To see that it is also G-invariant, we compute

$$\langle \pi(h)v, \pi(h)w \rangle = \frac{1}{|G|} \sum_{g \in G} (\pi(g)\pi(h)v | \pi(g)\pi(h)w) = \frac{1}{|G|} \sum_{g \in G} (\pi(gh)v | \pi(gh)w) = \langle v, w \rangle,$$

where the last equality follows from a simple change of variables. We have thus constructed a G-invariant inner product on V. □

3. The group algebra

In this section we analyze the space $\mathbb{C}[G]$ of all complex-valued functions on G. The space $\mathbb{C}[G]$ comes equipped with a natural representation of G, the (right) **regular representation**:

$$[R(g)\phi](x) = \phi(xg), \quad \text{for } \phi \in \mathbb{C}[G], x, g \in G.$$

One checks immediately that the representation R is unitary with respect to the following inner product

$$\langle \phi, \psi \rangle = \frac{1}{|G|} \sum_{g \in G} \phi(g) \overline{\psi(g)}.$$

The following construction explains why we are interested in $\mathbb{C}[G]$. Let (π, V) be an irreducible representation of G. Fix a non-zero element $f \in V^*$. For any $v \in V$ and $g \in G$, define $\phi_{f,v} \in \mathbb{C}[G]$

$$\phi_{f,v}(g) = f(\pi(g)v), \quad \text{for } g \in G.$$

We claim that the map $A_f : v \mapsto \phi_{f,v}$ is a non-zero element of $\mathrm{Hom}_G(\pi, R)$. Indeed, for any $x \in G$, we have

$$\phi_{f,\pi(g)v}(x) = f(\pi(x)\pi(g)v) = f(\pi(xg)v) = [R(g)\phi_{f,v}](x),$$

which shows that A_f is an intertwining map. Moreover, since $f \neq 0$, there exists an element $v \in V$ such that $f(v) \neq 0$. In other words, $\phi_{f,v}(1) \neq 0$, showing that the map A_f is non-zero. Note that A_f is necessarily injective because π is irreducible.

To summarize, we have shown that $\mathbb{C}[G]$ contains every irreducible representation of G. This means that we should have a good grasp of the representation theory of G once we analyze the regular representation on the space $\mathbb{C}[G]$. The rest of this section is devoted to this task.

Guided by the above sketch, we consider the space of **matrix coefficients** attached to each irreducible representation (π, V) of G. Choosing a basis $\{e_1, \dots, e_n\}$ for V (where $n = \dim \pi$), let $M(\pi)$ denote the subspace of $\mathbb{C}[G]$ spanned by the entries $\pi(g)_{ij}$, $1 \leq i, j \leq n$, of the matrix of $\pi(g)$ with respect to this basis. One verifies immediately that the space $M(\pi)$ is independent of the choice of basis; moreover, it depends only on the equivalence class of π (not the particular representative).

Proposition 3.1. *$M(\pi)$ is a G-invariant subspace of $\mathbb{C}[G]$.*

Proof. For any matrix coefficient $\pi(g)_{ij}$, we have

$$R(h)\pi(g)_{ij} = \pi(gh)_{ij} = \sum_{k=1}^{n} \pi(g)_{ik}\pi(h)_{kj},$$

showing that $R(h)\pi(g)_{ij}$ is a linear combination of the functions $\pi(g)_{ik}$. $\square$

Remark 3.2. Alternatively, one can prove this by considering the intertwining maps A_f constructed above. Taking a basis $\{f_1, \dots, f_n\}$ for V^*, we see that $M(\pi)$ is the sum of G-invariant subspaces $\operatorname{im} A_{f_i}$, $i = 1, \dots, n$. This implies $M(\pi)$ is G-invariant. This also shows that $M(\pi)$ is π-**isotypic**: π is the only irreducible representation that appears (possibly with multiplicity) in $M(\pi)$. (This is because $\operatorname{im} A_{f_i}$ is isomorphic to π.)

Recall Theorem 2.7: for any two representations (π, V) and (ρ, W) of G we have a projector from $L(V, W)$ to $\mathrm{Hom}_G(V, W)$ given by $A \mapsto A^G$,

$$A^G = \frac{1}{|G|} \sum_{g \in G} \rho(g^{-1}) A \pi(g). \tag{1}$$

Proposition 3.3. *Let (π, V) and (ρ, W) be non-equivalent irreducible representations of G. Then $M(\pi) \perp M(\rho)$.*

Proof. Recall that each representation of G comes equipped with a G-invariant inner product. Let $\{e_1, \dots, e_n\}$ (resp. $\{f_1, \dots, f_m\}$) be an orthonormal basis for V (resp. W) with respect to this inner product. With this choice of basis, the matrix of $\pi(g)$ (resp. $\rho(g)$) is unitary.

Since π and ρ are non-isomorphic, the intertwining map A^G defined by (1) is necessarily 0. Writing this in matrix form and focusing on a particular coefficient in the matrix, we have:

$$\frac{1}{|G|} \sum_{g \in G} \sum_{q=1}^{m} \sum_{r=1}^{n} \rho(g^{-1})_{pq} A_{qr} \pi(g)_{rs} = 0,$$

for any $m \times n$ matrix A, and any $p = 1, \dots, m$, $s = 1, \dots, n$. Fixing r and q, let A be the matrix which has zeroes everywhere, except at the (q, r) entry, where we set $A_{qr} = 1$. Then the above reads

$$\frac{1}{|G|} \sum_{g \in G} \rho(g^{-1})_{pq} \pi(g)_{rs} = 0.$$

This equation holds for all p, q, r, s. Finally, remembering that $\rho(g)$ is a unitary matrix (so $\rho(g^{-1}) = \rho(g)^{-1} = \rho(g)^*$), we get

$$\frac{1}{|G|} \sum_{g \in G} \overline{\rho(g)}_{qp} \pi(g)_{rs} = 0,$$

which shows that every matrix coefficient of π is orthogonal to every matrix coefficient of ρ. $\square$

To analyze the space $M(\pi)$, we employ a similar strategy. Since (π, V) is irreducible, Schur's Lemma shows that every element of $\mathrm{Hom}_G(\pi, \pi)$ is a scalar multiple of the identity. Formula (1) thus becomes

$$\frac{1}{|G|} \sum_{g \in G} \pi(g^{-1}) A \pi(g) = \lambda I, \quad \text{for some } \lambda \in \mathbb{C}.$$

Applying the trace, we get

$$\lambda \cdot \dim(\pi) = \frac{1}{|G|} \sum_{g \in G} \mathrm{tr}(\pi(g^{-1}) A \pi(g)) = \frac{1}{|G|} \sum_{g \in G} \mathrm{tr}(A\, \pi(g) \pi(g^{-1})) = \frac{1}{|G|} \sum_{g \in G} \mathrm{tr}(A) = \mathrm{tr}(A).$$

From this, we read $\lambda = \dfrac{\mathrm{tr}(A)}{\dim(\pi)}$. To summarize, we have

$$\frac{1}{|G|} \sum_{g \in G} \pi(g^{-1}) A \pi(g) = \frac{\mathrm{tr}(A)}{\dim(\pi)} I \tag{2}$$

for any linear map $A : V \to V$.

We are now ready to repeat the steps of Proposition 3.3. Fixing an orthonormal basis $\{e_1, \dots, e_n\}$ for V and focusing on the matrix entry (p, s) in Equation (2), we get

$$\frac{1}{|G|} \sum_{g \in G} \sum_{q=1}^{n} \sum_{r=1}^{n} \pi(g^{-1})_{pq} A_{qr} \pi(g)_{rs} = \frac{\mathrm{tr}(A)}{\dim(\pi)} \cdot \delta_{ps},$$

where δ_{ps} is the Kronecker delta symbol:

$$\delta_{ps} = \begin{cases} 1, & \text{if } p = s; \\ 0, & \text{if } p \neq s. \end{cases}$$

(Indeed, the matrix on the right-hand side has a non-zero entry at position (p, s) only if $p = s$, that is, only on the diagonal.) Again, the above holds for any $n \times n$ matrix A. Fixing q and r, we let A be the matrix which has 1 at the (q, r) position and zeroes elsewhere. Note that $\operatorname{tr}(A) = \delta_{qr}$ for such a matrix. We get

$$\frac{1}{|G|} \sum_{g \in G} \pi(g^{-1})_{pq} \pi(g)_{rs} = \frac{1}{\dim(\pi)} \cdot \delta_{qr} \delta_{ps}.$$

Finally, recall that $\pi(g)$ is a unitary matrix, so the above becomes

$$\frac{1}{|G|} \sum_{g \in G} \overline{\pi(g)}_{qp} \pi(g)_{rs} = \frac{1}{\dim(\pi)} \cdot \delta_{qr} \delta_{ps}, \quad \text{for any } p, q, r, s.$$

These are the celebrated **Schur orthogonality relations**: the result shows that the matrix coefficient $\pi(g)_{rs}$ is orthogonal to $\pi(g)_{qp}$ unless $p = s$ and $r = q$. On the other hand, applying the formula with $p = s$ and $r = q$ shows that the matrix coefficient $\pi(g)_{qr}$ is non-zero. We have thus proved the following:

Proposition 3.4. *The dimension of* $M(\pi)$ *is* $\dim(\pi)^2$.

To complete our analysis of $\mathbb{C}[G]$, it remains to prove

Theorem 3.5.

$$\mathbb{C}[G] = \bigoplus_{\pi \in \hat{G}} M(\pi).$$

Proof. Let $M = \bigoplus_{\pi \in \hat{G}} M(\pi)$ and let $M^\perp$ be its orthogonal complement in $\mathbb{C}[G]$. Note that $M^\perp$ is automatically a subrepresentation of $\mathbb{C}[G]$ (because the inner product used to define orthogonality is G-invariant). Assume $M^\perp \neq 0$. Then it contains an irreducible subrepresentation, say π. Let $\{f_1, \ldots, f_n\}$ be a basis for π. We then have, for any $i = 1, \ldots, n$:

$$f_i(g) = [R(g) f_i](1) = [\pi(g) f_i](1) = \sum_{j=1}^{n} \pi(g)_{ji} f_j(1).$$

Indeed, the first equality follows from the definition of the regular representation; the second holds because R acts like π on this subspace; and the third equality is simply the definition of matrix coefficients with respect to the basis $\{f_1, \ldots, f_n\}$. This shows that f_i is a linear combination of π_{ji}'s for $j = 1, \ldots, n$. Therefore, $f_i \in M(\pi)$, which contradicts our assumption $f_i \in M^\perp$. We conclude $M^\perp = 0$; this is what we needed to show. $\square$

Corollary 3.6.

$$\sum_{\pi \in \hat{G}} (\dim \pi)^2 = |G|.$$

Proof. Indeed, this follows immediately from Theorem 3.5 and Proposition 3.4 combined with the fact that $\dim \mathbb{C}[G] = |G|$. □

Problem 9. Describe all the irreducible representations of the symmetric group on three letters, S_3. Hint: Problem 5 already gives you two irreducible representations of S_3; Corollary 3.6 will tell you what is missing.

Problem 10. Throughout this section, we have been working with the right regular representation R. However, one can also define the *left regular representation* L on $\mathbb{C}[G]$ by setting

$$L(g)\phi(x) = \phi(g^{-1}x).$$

Prove that $L \cong R$ (construct an explicit isomorphism).

As suggested by the title of this section, $\mathbb{C}[G]$ is not only a vector space, but an algebra. Multiplication on $\mathbb{C}[G]$ is defined by convolution:

$$(\phi \star \psi)(g) = \sum_{h \in G} \phi(h)\psi(h^{-1}g).$$

The algebra comes with a unit,

$$\delta_e(g) = \begin{cases} 1, & \text{if } g = e; \\ 0, & \text{otherwise.} \end{cases}$$

Of course, convolution is just a fancy way of saying the following: any function $\phi \in \mathbb{C}[G]$ can be represented by a formal linear combination of elements in G,

$$\phi = \sum_{g \in G} \phi(g) \cdot g.$$

Then the linear combination representing $\phi \star \psi$ is just the product of the formal linear combinations for ϕ and ψ.

Suppose (π, V) is a representation of G. For any $\phi \in \mathbb{C}[G]$, one can define a linear map $\pi(\phi)$ on V by

$$\pi(\phi)v = \sum_{g \in G} \phi(g)\pi(g)v.$$

In this way, V becomes a $\mathbb{C}[G]$-module. The category $\mathrm{Rep}(G)$ is equivalent to the category of $\mathbb{C}[G]$-modules. Adopting the module-theoretic point of view will often be convenient, as exemplified by the following discussion.

Let (π, V) be an irreducible representation of G. Recall that V^* is equipped with the natural action of G through the contragredient of π (denoted $\pi^\vee$). In particular, $V \otimes V^*$ is a representation of $G \times G$. On the other hand, $\mathbb{C}[G]$ is also a representation for $G \times G$, if we act from both the left and the right.

Proposition 3.7. *Let (π, V) be an irreducible representation of G, and let $M(\pi)$ denote the π-isotypic subspace of the right regular representation $(R, \mathbb{C}[G])$. Then $M(\pi)$ is a $G \times G$-invariant subspace of $\mathbb{C}[G]$, and we have*

$$M(\pi) \cong V \otimes V^*$$

as representations of $G \times G$.

Proof. Let

$$N(\pi) = \bigcap_{F \in \mathrm{Hom}_G(R,\pi)} \ker F.$$

One verifies immediately that this is a subrepresentation of R. Therefore, the orthogonal complement of $N(\pi)$ is also a subrepresentation. It is not difficult that it contains every subrepresentation of R isomorphic to π, and nothing else (see Problem 11). Thus

$$\mathbb{C}[G] \cong N(\pi) \oplus M(\pi).$$

Now consider the mapping $\mathbb{C}[G] \to \mathrm{End}_{\mathbb{C}}(V) = V \otimes V^*$ given by

$$f \mapsto \pi(f).$$

This is a non-zero $G \times G$-intertwining (see Problem 12). Since $V \otimes V^*$ is irreducible, this map is surjective. It remains to show that its kernel is equal to $N(\pi)$.

Let $F \in \mathrm{Hom}_G(R, \pi)$. Then $F(f \star \phi) = \pi(f)F(\phi)$. If $\pi(f) = 0$, then $f \star \phi$ is in $\ker F$ for all ϕ. In particular, we may take $\phi = \delta_e$ to get $f = f \star \delta_e \in \ker F$. Thus $\pi(f) = 0$ implies $f \in N(\pi)$.

Conversely, for any $v \in V$, we may define an element F_v of $\mathrm{Hom}_G(R, \pi)$ by setting $F_v : f \mapsto \pi(f)v$. This shows that $f \in N(\pi)$ implies $\pi(f) = 0$. □

Problem 11. Show that $\mathbb{C}[G] \cong N(\pi) \oplus M(\pi)$ for any irreducible π.

Problem 12. Show that $f \mapsto \pi(f)$ is a $G \times G$-intertwining $\mathbb{C}[G] \to V \otimes V^*$.

4. Characters and class functions

In this section, we consider only finite-dimensional representations. Let (π, V) be a representation of G. The **character** of π is a function $\chi_\pi \in \mathbb{C}[G]$ given by

$$\chi_\pi(g) = \mathrm{tr}(\pi(g)).$$

Lemma 4.1. *The character of a representation satisfies the following properties:*

(i) $\chi_\pi(e) = \dim \pi$*;*

(ii) $\chi_\pi(g^{-1}) = \overline{\chi_\pi(g)}$*, for all* $g \in G$*;*

(iii) $\chi_\pi(gh) = \chi_\pi(hg)$*, for all* $h, g \in G$*;*

(iv) $\chi_{\pi+\rho} = \chi_\pi + \chi_\rho$*;*

(v) $\chi_{\pi^\vee} = \overline{\chi_\pi}$*.*

Problem 13. Prove Lemma 4.1.

4.1. **Multiplicity.** Characters offer an elegant way to compute dimensions of Hom-spaces.

Proposition 4.2. *Let π and ρ be irreducible representations of G. Then*

$$\langle \chi_\pi, \chi_\rho \rangle = \begin{cases} 1, & \text{if } \pi \cong \rho; \\ 0, & \text{if } \pi \ncong \rho. \end{cases}$$

Proof. The character of a representation can be expressed as a sum of matrix coefficients:

$$\chi_\pi(g) = \sum_{i=1}^{\dim \pi} \pi(g)_{ii}$$

Now

$$\langle \chi_\pi, \chi_\rho \rangle = \frac{1}{|G|} \sum_{g \in G} \left(\sum_{i=1}^{\dim \pi} \pi(g)_{ii} \right) \cdot \left(\sum_{j=1}^{\dim \rho} \overline{\rho(g)_{jj}} \right) = \sum_{i=1}^{\dim \pi} \sum_{j=1}^{\dim \rho} \frac{1}{|G|} \sum_{g \in G} \pi(g)_{ii} \cdot \overline{\rho(g)_{jj}},$$

and the result follows from the Schur orthogonality relations. □

Proposition 4.3. *Let (π, V) be a representation of G which decomposes as a direct sum of irreducible representations:*

$$V = V_1 \oplus \cdots \oplus V_n.$$

Let ρ be an irreducible representation of G. Then $\langle \chi_\rho, \chi_\pi \rangle = \#\{i = 1, \ldots, n : V_i \cong \rho\}$.

Proof. By (iv) of Lemma 4.1, $\chi_\pi = \sum_{i=1}^n \chi_{V_i}$. Therefore

$$\langle \chi_\rho, \chi_\pi \rangle = \sum_{i=1}^n \langle \chi_\rho, \chi_{V_i} \rangle,$$

and the claim follows from Proposition 4.2. □

Remark 4.4. Notice that this result provides us with another proof of uniqueness of decomposition in Corollary 2.6. The number of times an irreducible representation $\alpha \in \hat{G}$ appears V is equal to $\langle \chi_\alpha, \chi_\pi \rangle$, and is thus independent of the way in which we decompose V into a direct sum of irreducibles. This number is denoted by $m(\pi, \alpha)$ and is called the **multiplicity** of α in π. Using this notation, we have

$$\pi = \bigoplus_{\alpha \in \hat{G}} m(\pi, \alpha) \alpha \quad \text{and} \quad \chi_\pi = \sum_{\alpha \in \hat{G}} m(\pi, \alpha) \chi_\alpha$$

for any representation π of G.

We list two immediate consequences of Proposition 4.3:

Proposition 4.5. *Let π and ρ be representations of G. Then $\pi \cong \rho$ if and only if $\chi_\pi = \chi_\rho$.*

Problem 14. Prove Proposition 4.5.

Proposition 4.6. *Let π be a representation of G. Then $\langle \chi_\pi, \chi_\pi \rangle$ is a positive integer, which is equal to 1 if and only and only if π is irreducible.*

Problem 15. Prove Proposition 4.6.

The last proposition is a special case of the following result:

Theorem 4.7. *Let (π, V) and (ρ, W) be representations of G. Then*

$$\dim \operatorname{Hom}(\pi, \rho) = \dim \operatorname{Hom}(\rho, \pi) = \langle \chi_\pi, \chi_\rho \rangle = \sum_{\alpha \in \hat{G}} m(\pi, \alpha) m(\rho, \alpha).$$

Proof. Let $V = \oplus_{i=1}^n V_i$ and $W = \oplus_{j=1}^m W_j$ be decompositions of V and W into a direct sum of irreducibles. Then

$$\dim \operatorname{Hom}_G(V, W) = \sum_{i=1}^n \dim \operatorname{Hom}_G(V_i, W) = \sum_{i=1}^n \sum_{j=1}^m \dim \operatorname{Hom}_G(V_i, W_j).$$

Similarly,

$$\langle \chi_V, \chi_W \rangle = \sum_{i=1}^n \langle \chi_{V_i}, \chi_W \rangle = \sum_{i=1}^n \sum_{j=1}^m \langle \chi_{V_i}, \chi_{W_j} \rangle.$$

By Proposition 4.2,

$$\langle \chi_{V_i}, \chi_{W_j} \rangle = \begin{cases} 1, & \text{if } V_i \cong W_j \\ 0, & \text{if } V_i \ncong W_j \end{cases}$$

which is equal to $\dim \operatorname{Hom}_G(V_i, W_j)$ by Schur's Lemma. This shows that the two sums above are equal: $\dim \operatorname{Hom}_G(V, W) = \langle \chi_V, \chi_W \rangle$. Next,

$$\dim \operatorname{Hom}_G(V, W) = \langle \chi_V, \chi_W \rangle = \langle \chi_W, \chi_V \rangle = \dim \operatorname{Hom}_G(W, V).$$

Finally, by Remark 4.4 we have

$$\chi_\pi = \sum_{\alpha \in \hat{G}} m(\pi, \alpha) \chi_\alpha \quad \text{and} \quad \chi_\rho = \sum_{\alpha \in \hat{G}} m(\rho, \alpha) \chi_\alpha.$$

Now $\langle \chi_\pi, \chi_\rho \rangle = \sum_{\alpha \in \hat{G}} m(\pi, \alpha) m(\rho, \alpha)$ follows from Proposition 4.2. $\square$

4.2. **Central functions.** Recall part (iii) of Lemma 4.1: χ_π satisfies $\chi_\pi(hg) = \chi_\pi(gh)$ for all $g, h \in G$. We say that a function $\phi \in \mathbb{C}[G]$ is a **class function** if it satisfies $\phi(gh) = \phi(hg)$ for all $h, g \in G$. Let $\mathbb{C}_{\text{ab}}[G]$ denote the set of all such functions; clearly, this is a subspace of $\mathbb{C}[G]$. The following problem explains the name:

Problem 16. Prove that ϕ is central if and only if it is constant on conjugacy classes of G.

Proposition 4.8. *Let (π, V) be an irreducible representation of G and let $\phi \in \mathbb{C}_{ab}[G]$ be a class function. Then $\pi(\phi)$ is a scalar operator:*

$$\pi(\phi) = \frac{|G|}{\dim \pi} \langle \chi_\pi, \overline{\phi} \rangle \cdot I.$$

Proof. For any $g \in G$, we have

$$\pi(\phi)\pi(g) = \sum_{h \in G} \phi(h)\pi(hg) = \sum_{h \in G} \phi(h)\pi(gg^{-1}hg) = \pi(g) \sum_{h \in G} \phi(h)\pi(g^{-1}hg).$$

Since ϕ is a class function, we have $\phi(h) = \phi(g^{-1}hg)$. The right-hand side can thus be written $\pi(g)\sum_{h\in G} \phi(g^{-1}hg)\pi(g^{-1}hg)$, which is equal to $\pi(g)\pi(\phi)$ because $h \mapsto g^{-1}hg$ is a bijection. This shows

$$\pi(\phi)\pi(g) = \pi(g)\pi(\phi), \quad \text{for any } g \in G.$$

In other words, $\pi(\phi)$ is an element of $\mathrm{Hom}_G(\pi, \pi)$. By Schur's lemma, this implies

$$\pi(\phi) = \lambda \cdot I, \quad \text{for some } \lambda \in \mathbb{C}.$$

Taking the trace, we get

$$\dim(\pi) \cdot \lambda = \mathrm{tr}\pi(\phi) = \sum_{g \in G} \phi(g)\mathrm{tr}\,\pi(g) = \sum_{g \in G} \phi(g)\chi_\pi(g) = |G| \cdot \langle \chi_\pi, \overline{\phi} \rangle.$$

This proves the proposition. □

We are now ready to prove the main result of this section:

Theorem 4.9. $\{\chi_\alpha : \alpha \in \hat{G}\}$ *is an orthonormal basis for* $\mathbb{C}_{ab}[G]$.

Proof. By part (iii) of Lemma 4.1 we know that each χ_α is a class function. Proposition 4.2 shows that the set $\{\chi_\alpha : \alpha \in \hat{G}\}$ is orthonormal. It remains to show that it spans $\mathbb{C}_{\mathrm{ab}}[G]$.

Suppose $\phi \in \mathbb{C}_{\mathrm{ab}}[G]$ is orthogonal to χ_α for each $\alpha \in \hat{G}$: $\langle \chi_\alpha, \phi \rangle = 0$. By the above proposition, this implies $\pi(\overline{\phi}) = 0$ for any irreducible π. But every representation of G is a direct sum of irreducibles, so we get $\pi(\overline{\phi}) = 0$ for any representation of G.

In particular, this is also true for the left regular representation of G. Let $\delta_g \in \mathbb{C}[G]$ be the function which is equal to 1 at $g \in G$, and 0 elsewhere. Clearly, $\{\delta_g : g \in G\}$ is a basis for G. Apply $L(\overline{\phi}) = 0$ to δ_e:

$$0 = L(\overline{\phi})\delta_e = \sum_{g \in G} \overline{\phi}(g)L(g)\delta_e = \sum_{g \in G} \overline{\phi}(g)\delta_g.$$

Since δ_g are independent, this is possible only if $\phi(g) = 0$ for every g. We have thus shown that the orthogonal complement of the set $\{\chi_\alpha : \alpha \in \hat{G}\}$ is 0, which is what we needed to prove. □

Corollary 4.10. *The number of isomorphism classes of irreducible representations of G is equal to the number of conjugacy classes in G.*

Example 4.11. We know that conjugacy classes in S_n are in bijection with partitions of n. For $n = 3$, we have

$$3 = 2 + 1 = 1 + 1 + 1,$$

which shows that S_3 has three irreducible representations.

Problem 17. For each irreducible representation π of S_3, compute its character χ_π. Find the transition matrix between the two natural bases for $\mathbb{C}_{\text{ab}}(S_3)$, namely the basis given by the characters, and the basis given by conjugacy classes.

5. Abelian groups

In this section, we work with finite Abelian groups.

Proposition 5.1. *Let G be a finite Abelian group. Then $|\hat{G}| = |G|$. Furthermore, every irreducible representation of G is one-dimensional.*

Proof. Since G is Abelian, every conjugacy class is a singleton. Now $|\hat{G}| = |G|$ follows from Corollary 4.10. We also know

$$|G| = \sum_{\pi \in \hat{G}} (\dim \pi)^2$$

by Corollary 3.6. But the right-hand side has $|G|$ terms, all of which are positive integers. Therefore the equality is only possible if $\dim \pi = 1$ for each π. $\square$

Of course, a one-dimensional representation is just a group homomorphism $\psi : G \to \mathbb{C}^\times$, i.e. a character[1] of G. Moreover, we have (using multiplicative notation in the group G) $g^{|G|} = e$ for any g. This shows

$$\psi(g)^{|G|} = \psi(g^{|G|}) = \psi(e) = 1, \quad \text{for every } g \in G.$$

In other words, $\psi(g)$ is a root of unity for each g.

Since G is Abelian, we have $\mathbb{C}_{\text{ab}}[G] = \mathbb{C}[G]$. Thus, by Theorem 4.9, the characters of G form an orthonormal basis for $\mathbb{C}[G]$. We can write any $f \in \mathbb{C}[G]$ in this basis:

$$f = \sum_{\phi \in \hat{G}} \langle f, \phi \rangle \phi.$$

Taking the inner product with f (on both sides), we get

Proposition 5.2 (Bessel's equality)**.**

$$||f|| = \sum_{\phi \in \hat{G}} |\langle f, \phi \rangle|^2.$$

Note that $\hat{G}$ (the set of all characters of G) comes endowed with a group structure: the group operation is simply pointwise multiplication. Since $|\hat{G}| = |G|$, it is natural to ask whether the two groups are isomorphic.

Proposition 5.3. *$\hat{G}$ is isomorphic G.*

[1] For Abelian groups, the character of a representation is equal to the representation itself. For non-Abelian groups, the character of a representation is **not** a homomorphism $G \to \mathbb{C}^\times$.

Sketch of proof. Using the classification theorem for finite Abelian groups, we see that it suffices to prove the Proposition in case when G is cyclic, i.e. $G = \mathbb{Z}/n\mathbb{Z}$. Note that $\widehat{\mathbb{Z}/n\mathbb{Z}}$ is isomorphic to the group of n-th roots of unity: to specify an element $\phi \in \widehat{\mathbb{Z}/n\mathbb{Z}}$, it suffices to choose $\phi(1)$, which is necessarily a root of 1. However, the group of n-th roots of unity is itself isomorphic to $\mathbb{Z}$. $\square$

Note that this isomorphism is non-canonical. It is more interesting to note the following:

Theorem 5.4. *There is a canonical isomorphism $G \cong \hat{\hat{G}}$.*

Proof. The isomorphism is given by $g \mapsto \mathrm{ev}_g$, where

$$\mathrm{ev}_g(\phi) = \phi(g)$$

for every $g \in G, \phi \in \hat{G}$. One checks that this is a group homomorphism. Let us prove that it is also injective.

Suppose $\mathrm{ev}_g(\phi) = 1$ or, equivalently, $\phi(g) = 1$ for every $\phi \in \hat{G}$. We need to prove $g = e$. Any function in $\mathbb{C}[G]$ can be written as a linear combination of characters. In particular,

$$\delta_e = \sum_{\phi \in \hat{G}} c_\phi \phi$$

Now apply $L(g)$ (left translation by g^{-1}) to both sides of the equation by g. Since $\phi(g) = 1$, the right-hand side remains unaffected, while the left becomes δ_g. Thus

$$\delta_g = \sum_{\phi \in \hat{G}} c_\phi \phi = \delta_e.$$

Since $\delta_g = \delta_e$, we conclude $g = e$. This shows injectivity; surjectivity is now automatic because $|\hat{\hat{G}}| = |G|$. $\square$

Although G and $\hat{G}$ are not canonically isomorphic, there is a (more-or-less) canonical way to relate the functions on these two groups: the Fourier transform. Taking $f \in \mathbb{C}[G]$, we define its Fourier transform as the function $\hat{f}$ on $\hat{G}$ given by

$$\hat{f}(\phi) = \frac{1}{|G|} \sum_{g \in G} f(g)\overline{\phi(g)}.$$

Of course, one can go in the opposite direction in a similar manner: one checks that the following formula holds for any $f \in \mathbb{C}[G]$:

$$f = \sum_{\phi \in \hat{G}} \hat{f}(\phi)\phi.$$

Taking the L^2-norm of both sides (and using the fact that ϕ's are orthonormal), we get the **Plancherel formula**:

$$||f||_2 = ||\hat{f}||_2.$$

6. Induced representations

In this section we introduce and explore the induction functor. G is still a finite group.

Let (π, V) be a representation of G. For any subgroup H of G, we may consider the **restriction** of π to H. The corresponding representation is denoted $\mathrm{Res}_H^G(\pi)$ (or just $\mathrm{Res}(G)$, when H is implied); we will also occasionally use $\pi|_H$, or even just π to denote the restricted representation. It is immediate that Res_H^G is an exact functor from $\mathrm{Rep}(G)$ to $\mathrm{Rep}(H)$.

We now construct a functor in the opposite direction. Let H be a subgroup of G and let (ρ, W) be a representation of H. Consider the space of functions

$$\mathrm{Ind}_H^G W := \{f \colon G \to W : f(hg) = \rho(h)f(g) \text{ for all } h \in H, g \in G\}.$$

On this space, we define the **induced representation** $\mathrm{Ind}_H^G \rho$ of G by

$$[(\mathrm{Ind}_H^G \rho)(g)f](x) = f(xg), \quad \text{for any } f \in \mathrm{Ind}_H^G W \text{ and } g, x \in G,$$

that is, G acts on $\mathrm{Ind}_H^G W$ by right translation.

Example 6.1. Here are two (extreme) examples:

- $\mathrm{Ind}_G^G \pi \cong \pi$, for any representation π of G.
- $\mathrm{Ind}_1^G 1 \cong R$, the right regular representation.

Like restriction, induction is an exact functor:

Problem 18. Show that the functor Ind_H^G is exact.

Note that any function $f \in \mathrm{Ind}_H^G W$ is uniquely determined by its values on any set of coset representatives $H\backslash G$. This immediately gives us

Proposition 6.2. *Let $[G : H]$ denote the index of H in G. Then*

$$\dim \mathrm{Ind}_H^G W = [G : H] \cdot \dim W.$$

Here is the main result of this section:

Theorem 6.3 (Frobenius reciprocity). *Let H be a subgroup of G; let (π, V) a representation of G, and let ρ be a representation of H. Then there is a natural isomorphism of vector spaces*

$$\mathrm{Hom}_G(\pi, \mathrm{Ind}_H^G \rho) \cong \mathrm{Hom}_H(\pi|_H, \rho).$$

In other words, Ind_H^G is the right adjoint to Res_H^G.

Proof. To simplify notation, we will let σ denote $\mathrm{Ind}_H^G \rho$ throughout this proof. We construct a map Φ from $\mathrm{Hom}_G(\pi, \sigma)$ to $\mathrm{Hom}_H(\pi|_H, \rho)$. Let $A \in \mathrm{Hom}_G(\pi, \sigma)$. For any $v \in V$, Av is thus a function $G \to W$. Evaluating at the identity, we get a vector in W. We thus define $\Phi(A) \in \mathrm{Hom}_H(\pi|_H, \rho)$ by

$$\Phi(A)v = Av(e).$$

It is clear that $\Phi(A)$ is linear; we check that it is also an H-intertwiner:

$$\Phi(A)(\pi(h)v) = [A(\pi(h)v)](e) = [\sigma(h)Av](e) = A(v)(h) = \rho(h)[Av(e)] = \rho(h)[\Phi(A)v].$$

Indeed,

- the first equality holds by definition of Φ;
- the second equality comes from the fact that ϕ is an intertwiner between π and σ;
- the third equality follows by definition of σ (recall that G acts on $\mathrm{Ind}_H^G\rho$ by right translation);
- the fourth equality holds because Av is an element of $\mathrm{Ind}_H^G W$;
- the last equality is again just the definition of Φ.

Thus $\Phi(A)(\pi(h)v) = \rho(h)\Phi(A)v$, which means that $\Phi(A)$ is an element of $\mathrm{Hom}_H(\pi|_H, \rho)$. We have thus constructed our map Φ; note that Φ is clearly linear.

We show that Φ is a bijection by constructing an inverse, Ψ. For any $B \in \mathrm{Hom}_H(\pi|_H, \rho)$ we define $\Psi(B) \in \mathrm{Hom}_G(\pi, \mathrm{Ind}_H^G\rho)$ as follows. For any $v \in V$, $\Psi(B)v$ is supposed to be a function $G \to W$. We set

$$[\Psi(B)v](g) = B(\pi(g)v).$$

Notice that

$$[\Psi(B)v](hg) = B(\pi(hg)v) = B(\pi(h)\pi(g)v) = \rho(h)B(\pi(g)v)$$

where the last inequality holds because B is an element of $\mathrm{Hom}_H(\pi|_H, \rho)$. This shows that $\Psi(B)v$ is indeed a function from $\mathrm{Ind}_H^G\rho$. Furthermore, it is clear that $\Psi(B)$ is a linear map. It remains to show that it is also a G-intertwiner.

To that end, we compute the function $\Psi(B)(\pi(g)v)$. For any $x \in G$, we have

$$[\Psi(B)(\pi(g)v)](x) = B(\pi(x)\pi(g)v) = B(\pi(xg)v).$$

On the other hand, $\sigma = \mathrm{Ind}_H^G\rho$ is simply right translation, so we have

$$[\sigma(g)\Psi(B)v](x) = [\Psi(B)v](xg) = B(\pi(xg)v),$$

by definition of Ψ. We have thus shown $[\Psi(B)(\pi(g)v)](x) = [\sigma(g)\Psi(B)v](x)$, which means that $\Psi(B)$ is an element of $\mathrm{Hom}_G(\pi, \sigma)$. We have thus constructed our map Ψ; note that Ψ is clearly linear.

It remains to show that Ψ is the inverse of Φ. For any $v \in V$, and $x \in G$, we have

$$[\Psi(\Phi(A))v](x) = \Phi(A)(\pi(x)v) = A(\pi(x)v)(e) = [\sigma(x)Av](e) = Av(x)$$

This shows $\Psi(\Phi(A))v = Av$ for any $v \in V$, that is, $(\Psi \circ \Phi)A = A$. Hence $\Psi \circ \Phi = \mathrm{id}$.

In the other direction, we have

$$\Phi(\Psi(B))v = [\Psi(B)v](e) = B(\pi(e)v) = B(v)$$

for any B, v. This shows $\Phi \circ \Psi = \mathrm{id}$.

We have thus shown that Φ is bijective. The proof of naturality is left as an exercise. $\square$

Problem 19. Prove that Φ is natural in both variables. What this means: there are two functors from Rep_G to the category Vec of vector spaces over $\mathbb{C}$:

$$\mathrm{Hom}_G(\cdot, \mathrm{Ind}_H^G \rho) \quad \text{and} \quad \mathrm{Hom}_H(\mathrm{Res}_H^G(\cdot), \rho);$$

similarly, we have two functors from Rep_H to Vec:

$$\mathrm{Hom}_G(\pi, \mathrm{Ind}_H^G(\cdot)) \quad \text{and} \quad \mathrm{Hom}_H(\pi|_H, \cdot).$$

In both cases, Φ provides a transformation from one functor to the other. You need to prove that this transformation is natural.

Corollary 6.4. *Let H be a subgroup of G. Let π (resp. ρ) be an irreducible representation of G (resp. H). Then there is an equality of multiplicities*

$$m(\mathrm{Ind}_H^G \rho, \pi) = m(\pi|_H, \rho).$$

Proof. Indeed,

$$m(\mathrm{Ind}_H^G \rho, \pi) = \dim \mathrm{Hom}_G(\pi, \mathrm{Ind}_H^G \rho) = \dim \mathrm{Hom}_H(\pi|_H, \rho) = m(\pi|_H, \rho). \qquad \square$$

Frobenius reciprocity also provides us with an easy proof of the following useful fact:

Proposition 6.5 (**Induction in stages**)**.** *Let $H \leq K \leq G$ be subgroups. Then*

$$\mathrm{Ind}_K^G \circ \mathrm{Ind}_H^K \cong \mathrm{Ind}_H^G.$$

Proof. Clearly, the analogous relation is true for restriction:

$$\mathrm{Res}_H^K \circ \mathrm{Res}_K^G \cong \mathrm{Res}_H^G.$$

The claim now follows because Ind is adjoint to Res. $\square$

6.1. **Module-theoretic interpretation.** The $\mathbb{C}[G]$-module perspective offers an elegant way to view the functor Ind_H^G. Let W be an H-module, and let V be a $\mathbb{C}[G]$-module. Let us view $\mathbb{C}[G]$ as a $(\mathbb{C}[H], \mathbb{C}[G])$-bimodule. Then there is a natural way to interpret induction and restriction:

Problem 20. Prove that

$$\mathrm{Ind}_H^G W \cong \mathrm{Hom}_{\mathbb{C}[H]}(\mathbb{C}[G], W)$$

as $\mathbb{C}[G]$-modules, and

$$\mathrm{Res}_H^G V \cong \mathbb{C}[G] \otimes_{\mathbb{C}[G]} V$$

as $\mathbb{C}[H]$-modules.

We can now view Frobenius reciprocity as an instance of the usual tensor–hom adjunction:

$$\mathrm{Hom}_{\mathbb{C}[H]}(\mathbb{C}[G] \otimes_{\mathbb{C}[G]} V, W) \cong \mathrm{Hom}_{\mathbb{C}[G]}(V, \mathrm{Hom}_{\mathbb{C}[H]}(\mathbb{C}[G], W)).$$

However, we can also view $\mathbb{C}[G]$ as a $(\mathbb{C}[G], \mathbb{C}[H])$-bimodule (note the change in order). We then get another way to describe induction and restriction:

Problem 21. Prove that

$$\mathrm{Ind}_H^G W \cong \mathbb{C}[G] \otimes_{\mathbb{C}[H]} W$$

as $\mathbb{C}[G]$-modules, and

$$\mathrm{Res}_H^G V \cong \mathrm{Hom}_{\mathbb{C}[G]}(\mathbb{C}[G], V)$$

as $\mathbb{C}[H]$-modules.

Applying the tensor–hom adjunction, we get

$$\mathrm{Hom}_{\mathbb{C}[G]}(\mathbb{C}[G] \otimes_{\mathbb{C}[H]} W, V) = \mathrm{Hom}_{\mathbb{C}[H]}(W, \mathrm{Hom}_{\mathbb{C}[G]}(\mathbb{C}[G], V)).$$

Of course, this means that Ind_H^G is also a left adjoint to Res_H^G!

Some results about induction have a natural interpretation in this seting. For example, induction in stages boils down to associativity of tensor products:

$$\mathbb{C}[G] \otimes_{\mathbb{C}[K]} (\mathbb{C}[K] \otimes_{\mathbb{C}[H]} W) = \mathbb{C}[G] \otimes_{\mathbb{C}[H]} W.$$

6.2. **The other adjunction.** The second tensor–hom adjunction discussed above tells us that there should exist a natural isomorphism

$$\mathrm{Hom}_G(\mathrm{Ind}_H^G \rho, \pi) \cong \mathrm{Hom}_H(\rho, \pi|_H),$$

where (π, V) is a representation of G and (ρ, W) a representation of H. What does this isomorphism look like from a representation-theoretic perspective?

Problem 22. For any $w \in W$, define an element f_w of $\mathrm{Ind}_H^G W$ by

$$f_w(h) = \rho(h)w, \text{ for } h \in H \quad \text{and} \quad f(g) = 0 \text{ for } g \notin H.$$

Given $A \in \mathrm{Hom}_G(\mathrm{Ind}_H^G \rho, \pi)$, set

$$\Phi(A)w = Af_w$$

for any $w \in W$.

For $B \in \mathrm{Hom}_H(\rho, \pi|_H)$ and $f \in \mathrm{Ind}_H^G \rho$, let

$$\Psi(B)f = \sum_{g \in H \backslash G} \pi(g^{-1})B(f(g)).$$

Here $H\backslash G$ denotes any set of coset representatives. (Prove that the above definition does not depend on the chosen set of coset representatives!)

Prove that Φ and Ψ are the isomorphisms that give us the second adjunction:

$$\mathrm{Hom}_G(\mathrm{Ind}_H^G \rho, \pi) \cong \mathrm{Hom}_H(\rho, \pi|_H).$$

6.3. **The character of an induced representation.** Let H be a subgroup of G and let (ρ, W) be a finite-dimensional representation of H. Let $(\pi, V) = (\mathrm{Ind}_H^G \rho, \mathrm{Ind}_H^G W)$. We would like to compute the character χ_π in terms of χ_ρ.

Recall the element $f_w \in V$ defined in §6.2. Notice that the map $w \mapsto f_w$ is an injective H-equivariant map $\rho \mapsto \pi$. This shows that π has a subrepresentation isomorphic to ρ. We abuse notation and denote this subspace by W.

Now let $g_1, \ldots, g_n$ be a set of coset representatives for $H\backslash G$. We claim that, as vector spaces,

$$V = \bigoplus_{i=1}^{n} W_i, \quad \text{where } W_i = \pi(g_i^{-1})W.$$

To show this, let $f_{g,w} \in V$ be the function which satisfies $f_{g,w}(g) = w$, and is zero outside of Hg. One checks that $f_{g,w} = \pi(g^{-1})f_w$. Now take any function $f \in V$ and set $w_i = f(g_i)$ for $i = 1, \ldots, n$. Then it is clear that

$$f = \sum_{i=1}^{n} f_{g_i,w_i} = \sum_{i=1}^{n} \pi(g_i^{-1}) f_{w_i}.$$

This shows that V is equal to the sum of $\pi(g_i^{-1})W$'s, and the sum is clearly direct: W_i is precisely the subspace consisting of functions supported on Hg_i.

Next, let $x \in G$. Since right multiplication by x permutes the set $H\backslash G$, it follows that $\pi(x)$ permutes the spaces W_i, $i = 1, \ldots, n$. Recall that we are trying to compute $\operatorname{tr} \pi(x)$, so we need to compute the sum of the diagonal matrix coefficients. But, because $\pi(x)$ permutes the subspaces W_i, we see that the diagonal matrix coefficient corresponding to a basis element from W_i can be non-zero only if $\pi(x)W_i = W_i$, or in other words (remembering that $W_i = \pi(g_i^{-1})W$),

$$\pi(g_i x g_i^{-1})W = W.$$

Of course, this is equivalent to the requirement $g_i x g_i^{-1} \in H$.

Finally, we need to compute the trace of $\pi(x)$ on W_i when $g_i x g_i^{-1} \in H$. We have

$$\operatorname{tr}(\pi(x)|_{W_i}) = \operatorname{tr}(\pi(g_i^{-1})\pi(g_i x g_i^{-1})\pi_i(g_i)|_{W_i}) = \operatorname{tr}(\pi(g_i x g_i^{-1})|_W) = \chi_\rho(g_i x g_i^{-1}).$$

Indeed, the second equality holds because $\pi(g_i)$ is a linear isomorphism $W_i \to W$, and the third equality comes from the fact that W (as a subrepresentation of π) is isomorphic to ρ.

Collecting the results, we have

$$\chi_\pi(x) = \sum_{\substack{i=1 \\ g_i x g_i^{-1} \in H}}^{n} \chi_\rho(g_i x g_i^{-1}).$$

If we extend χ_ρ by zero outside of H, we have a nice way of writing this:

Theorem 6.6. *Let $\pi = \mathrm{Ind}_H^G \rho$. Then*

$$\chi_\pi(x) = \sum_{g \in H\backslash G} \chi_\rho(gxg^{-1}) = \frac{1}{|H|} \sum_{g \in G} \chi_\rho(gxg^{-1}) \quad \textit{for any } x \in G.$$

One checks that the sum in the middle does not depend on the choice of representatives. The theorem shows that the support of χ_π is contained in the union of conjugacy classes which intersect H. The result is particularly nice when H is normal: in that case, the support of χ_π is contained in H.

7. The Mackey machine

In this section, we analyze restrictions of induced representations: $\mathrm{Res}^G_K \mathrm{Ind}^G_H$. Note that we need to understand this functor whenever we want to study the Hom-space between two induced representations.

Let H be a subgroup of G and let (ρ, W) be a representation of H. We consider the representation (ρ^g, W) of the group $g^{-1}Hg$ given by

$$\rho^g(x) = \rho(gxg^{-1}).$$

Proposition 7.1. $\mathrm{Ind}^G_H \rho \cong \mathrm{Ind}^G_{g^{-1}Hg} \rho^g$.

Proof. One checks that $f \mapsto f(g\,\cdot)$ (left translation by g) is a G-equivariant map between $\mathrm{Ind}^G_H \rho$ and $\mathrm{Ind}^G_{g^{-1}Hg} \rho^g$. It is invertible, and therefore an isomorphism. $\square$

Now let H and K be subgroups of G. For any $g \in G$, we consider the subgroup $K_g = K \cap g^{-1}Hg$. If ρ is a representation of H, we define a representation ρ_g of K_g by

$$\rho_g(k) = \rho(gkg^{-1}), \quad \text{for } k \in K_g = K \cap g^{-1}Hg.$$

We are interested in the representation $\mathrm{Ind}^K_{K_g} \rho_g$. Let us define an equivalence relation on G as follows: $g \sim g'$ if they belong to the same double coset in $H \backslash G / K$. Equivalently, $g \sim g'$ if there exist $h \in H$ and $k \in K$ such that

$$g' = hgk.$$

The following lemma shows that $\mathrm{Ind}^K_{K_g} \rho_g$ depends only on the coset containing g, not the representative itself:

Lemma 7.2. *If* $g \sim g'$, *then* $\mathrm{Ind}^K_{K_g} \rho_g \cong \mathrm{Ind}^K_{K_{g'}} \rho_{g'}$.

Proof. Let $g' = hgk$ for some $h \in H$ and $k \in K$. Then $K_{g'} = k^{-1}K_g k$:

$$K \cap hgk^{-1}Hhgk = K \cap k^{-1}g^{-1}Hgk = k^{-1}(K \cap g^{-1}Hg)k = k^{-1}K_g k.$$

Next, for any $x \in K_{g'}$, we have

$$\rho_{g'}(x) = \rho(g'xg'^{-1}) = \rho(hgkxk^{-1}g^{-1}h^{-1}) = \rho(h)\rho(gkxk^{-1}g^{-1})\rho(h^{-1}).$$

But kxk^{-1} is an element of K_g, so we can interpret this as

$$\rho(h)\rho_g(kxk^{-1})\rho(h^{-1}) = \rho(h)\rho_g^k(x)\rho(h^{-1}).$$

Thus $\rho_{g'}(x) = \rho(h)\rho_g^k(x)\rho(h^{-1})$, which shows that $\rho(h)$ is an intertwiner between $\rho_{g'}$ and ρ_g^k (representations of $K_{g'}$). Since $\rho(h)$ is invertible, this means $\rho_{g'} \cong \rho_g^k$. Now

$$\mathrm{Ind}_{K_{g'}}^K \rho_{g'} \cong \mathrm{Ind}_{K_{g'}}^K \rho_g^k \cong \mathrm{Ind}_{K_g}^K \rho_g,$$

where the second isomorphism follows from Proposition 7.1. □

Finally, we come to the main result of this section:

Theorem 7.3 (Mackey)**.** *Let H and K be subgroups of G, and let (ρ, W) be a representation of H. Fix a set of representatives $a_1, \dots, a_n$ for $H\backslash G/K$. For every $i = 1, \dots, n$ let $K_i = K \cap {a_i}^{-1} H a_i$ and let $\rho_i = \rho_{a_i}$ be the corresponding representation of K_i. Then*

$$\mathrm{Res}_K^G \mathrm{Ind}_H^G \rho = \bigoplus_{i=1}^n \mathrm{Ind}_{K_i}^K \rho_i.$$

Proof. To simplify notation, let $(\pi, V) = (\mathrm{Ind}_H^G \rho, \mathrm{Ind}_H^G W)$ and $\pi_i = \mathrm{Ind}_{K_i}^K \rho_i$ for each $i = 1, \dots, n$. Furthermore, let V_i be the subspace of V consisting of functions supported on Ha_iK. It is immediate that V_i is a K-invariant subspace of V. Since G is partitioned into double cosets, we have

$$V = \bigoplus_{i=1}^n V_i.$$

The result will follow if we can prove $V_i \cong \pi_i$.

To that end, let $f \in V_i$. We claim that the functon $f(a_i\,\cdot) : K \to W$ is then an element of π_i. Indeed, for any $k \in K_i$ and $x \in K$, we have

$$f(a_i k x) = f(a_i k a_i^{-1} a_i x) = \rho(a_i k a_i^{-1}) f(a_i x) = \rho_i(k) f(a_i x).$$

The middle equality holds because $a_i k a_i^{-1}$ is an element of H, and f is an element of $\mathrm{Ind}_H^G \rho$.

The linear map $f \mapsto f(a_i\,\cdot)$ we have just constructed is clearly K-equivariant, because K acts on both representations by right translation. It is also injective: if $f(a_i k) = 0$ for all $k \in K$, then it follows that $f = 0$, because f is supported on the double coset Ha_iK, and $f(ha_ik) = \rho(h) f(a_i k)$.

It remains to prove that this map is surjective. Let φ be a function from π_i. We need $f \in V_i$ such that $f(a_i k) = \varphi(k)$ for every $k \in K$. The candidate is obvious: define $f \in V_i$ by setting

$$f(ha_ik) = \rho(h)\varphi(k) \quad \text{for } h \in H, k \in K,$$

and $f = 0$ outside of Ha_iK. Now we are done as soon as we prove that f is well-defined. To show this, suppose $ha_ik = h'a_ik'$ for $h, h' \in H$ and $k, k' \in K$. Then $k'k^{-1} = a_i^{-1} {h'}^{-1} h a_i$ is an element of K and $a_i^{-1} H a_i$, and hence of K_i. Therefore

$$\begin{aligned} f(h'a_ik') = \rho(h')\phi(k') = \rho(h')\phi(k'k^{-1}k) &= \rho(h')\rho_i(a_i^{-1}{h'}^{-1}ha_i)\phi(k) \\ &= \rho(h')\rho({h'}^{-1}h)\phi(k) = \rho(h)\phi(k) = f(ha_ik). \end{aligned}$$

This proves that f is well-defined and concludes the proof. □

Mackey's theorem gives us a useful irreducibility criterion for induced representations:

Corollary 7.4 (Mackey)**.** *Let H be a subgroup of G and let ρ be a representation of H. Then $\mathrm{Ind}_H^G\rho$ is irreducible if and only if*

- *ρ is irreducible; and*
- *for any $x \in G\backslash H$, the representations ρ_x and $\rho|_{H_x}$ of $H_x = H \cap x^{-1}Hx$ are disjoint:*
$$\mathrm{Hom}_{H_x}(\rho_x, \rho) = 0.$$

Proof. The first bullet is clearly necessary: since Ind_H^G is exact, $\mathrm{Ind}_H^G\rho$ can be irreducible only if ρ is irreducible. As for the second bullet, we know that a representation π of G is irreducible if and only if $\dim \mathrm{Hom}_G(\pi, \pi) = 1$. Therefore, we compute

$$\begin{aligned}\mathrm{Hom}_G(\mathrm{Ind}_H^G\rho, \mathrm{Ind}_H^G\rho) &\cong \mathrm{Hom}_H(\mathrm{Res}_H^G\mathrm{Ind}_H^G\rho, \rho)\\ &\cong \bigoplus_{i=1}^n \mathrm{Hom}_H(\mathrm{Ind}_{H_i}^H\rho_i, \rho)\\ &\cong \bigoplus_{i=1}^n \mathrm{Hom}_{H_i}(\rho_i, \rho|_{H_i}).\end{aligned}$$

Here we use Frobenius, Mackey, and Frobenius again. The sum is taken over a set of coset representatives for $H\backslash G/H$. One of those cosets is just H; in that case, the corresponding term in the sum is $\mathrm{Hom}_H(\rho, \rho)$, which is one-dimensional because ρ is irreducible. Therefore the original Hom-space is one-dimensional if and only if all of the other Hom-spaces $\mathrm{Hom}_{H_i}(\rho_i, \rho|_{H_i})$ vanish. This is precisely the meaning of the second bullet in the statement of the Corollary. □

Additional problems

Problem 23. Describe all irreducible representations of the cyclic group $\mathbb{Z}/n\mathbb{Z}$.

Problem 24. Describe all irreducible representations of the dihedral group. Hint: use induction to build representations, and Mackey theory to check if they are irreducible.

Problem 25. Describe all irreducible representations of the alternating group A_3. For every $\rho \in \mathrm{Irr}\, A_3$, compute the induced representation $\mathrm{Ind}_{A_3}^{S_3}$ and its restriction back to A_3.

Problem 26. Let H be a subgroup of G, V a representation of G, and W a representation of H. Prove
$$V \otimes \mathrm{Ind}_H^G W \cong \mathrm{Ind}_H^G(\mathrm{Res}_H^G V \otimes W).$$
(the tensor products are over $\mathbb{C}$). Note that this specializes to a nice formula for $\mathrm{Ind}_H^G\mathrm{Res}_H^G$:
$$\mathrm{Ind}_H^G\mathrm{Res}_H^G\, V \cong V \otimes \mathrm{Ind}_H^G\, 1.$$

CHAPTER II: REPRESENTATION THEORY OF COMPACT GROUPS

In this chapter, we study the representation theory of compact topological groups. In the previous chapter, we obtained crucial results using the method of averaging. If we are to adapt this method to the setting of topological groups, we need to prove the existence of an invariant integral, or equivalently, a translation-invariant measure on our group. This is our first order of business.

8. Arzelà–Ascoli

Let K be a compact topological space, and let $C(K)$ denote the space of all continuous (complex-valued) functions on K. Note that $C(K)$ is a Banach space with the usual sup-norm $||\cdot||_\infty$. We say that a subset S of $C(K)$ is **equicontinuous** at $x \in K$ if for every $\epsilon > 0$ there exists a neighborhood U of x such that

$$|f(x) - f(y)| < \epsilon \quad \text{for all } y \in U \text{ and } f \in S.$$

The following theorem gives necessary and sufficient conditions for a subset S of $C(K)$ to be relatively compact (i.e. to have compact closure).

Theorem 8.1 (Arzelà–Ascoli)**.** *Let K be a compact topological space and let S be a subset of $C(K)$. Then S is relatively compact if it is bounded and equicontinuous at every point $x \in K$.*

The converse statement is also true, but we do not need it. To prove this theorem, we will need the following lemma.

Lemma 8.2. *Let S be a bounded and equicontinuous subset of $C(K)$ and let (f_n) be a sequence in S. Let $\epsilon > 0$. Then the sequence (f_n) has a subsequence (g_n) such that*

$$||g_k - g_l||_\infty < \epsilon, \quad \textit{for all } k, l.$$

Proof. By equicontinuity, every point $x \in K$ has a neighborhood $U(x)$ such that

$$|f_n(x) - f_n(y)| < \epsilon/3, \quad \text{for all } n \text{ and } y \in U(x).$$

Since K is compact, there is a finite set of points $\{x_1, \ldots, x_d\} \subseteq K$ such that

$$K \subseteq U(x_1) \cup \cdots \cup U(x_d).$$

Since S is bounded, $(f_n(x_1), \ldots, f_n(x_d))_n$ is a bounded sequence in $\mathbb{C}^d$. It therefore has a convergent subsequence. In particular, we can find a subsequence (g_n) of (f_n) such that

$$|g_k(x_i) - g_l(x_i)| < \epsilon/3 \quad \text{for all } k, l \text{ and } i = 1, \ldots, d.$$

Now let $x \in K$ be arbitrary. Then there exists an x_i such that $x \in U(x_i)$. We have

$$|g_k(x) - g_l(x)| \leq |g_k(x) - g_k(x_i)| + |g_k(x_i) - g_l(x_i)| + |g_l(x_i) - g_l| \leq \epsilon/3 + \epsilon/3 + \epsilon/3 = \epsilon.$$

Since x was arbitrary, this shows $||g_k - g_l|| < \epsilon$ for any k, l. $\square$

We are now ready to prove the theorem.

Proof of Theorem 8.1. Let (f_n) be a sequence in S. Taking $\epsilon = \frac{1}{2}$, Lemma 8.2 gives a subsequence $(f_{1,n})_n$ of (f_n) such that

$$||f_{1,k} - f_{1,l}|| < \frac{1}{2}, \quad \text{for all } k, l.$$

Using the Lemma again, we can find a subsequence of this sequence, say $(f_{2,n})_n$, such that

$$||f_{2,k} - f_{2,l}|| < \frac{1}{4}, \quad \text{for all } k, l.$$

Proceeding inductively, we may construct a sequence of subsequences; the m-th subsequence $(f_{m,n})_n$ satisfies

$$||f_{m,k} - f_{m,l}|| < \frac{1}{2^m}, \quad \text{for all } k, l.$$

We set $S_m = \{f_{m,n} : n = 1, 2, \dots\}$ so that

$$S_1 \supseteq S_2 \supseteq S_3 \supseteq \dots$$

and $||f - g|| < \frac{1}{2^m}$ for any $f, g \in S_m$.

Now let $g_n = f_{n,n} \in S_n$ for every n. Let $\epsilon > 0$ be arbitrary. Taking n_0 such that $\epsilon > \frac{1}{2^{n_0}}$, we have

$$||g_k - g_l|| < \epsilon \quad \text{whenever } k, l \geq n_0,$$

showing that (g_k) is a Cauchy sequence. Since $C(K)$ is complete, we conclude that (g_k) has a limit in $C(K)$.

We have thus shown that every sequence in S has a subsequence that converges in $C(K)$. This means that S is relatively compact. □

Problem 27. Prove the other direction of the Arzelà–Ascoli theorem: if $S \subseteq C(K)$ is relatively compact, then it is bounded and equicontinuous.

9. A fixed-point theorem

We prove a fixed-point theorem for compact convex sets in normed spaces. If X is a normed space and $x \in X$, we let $B(x, r)$ (resp. $\overline{B}(x, r)$) denote the open (resp. closed) ball of radius r around x. We will use the following variant of Zorn's Lemma:

Hausdorff Maximal Principle: Every non-empty poset contains a maximal totally ordered subset.

Theorem 9.1 (Kakutani)**.** *Let X be a normed space, $K \subset X$ a compact convex set, and let G be a group of isometries of X. Suppose $AK \subseteq K$ for every $A \in G$. Then there exists an $x \in K$ such that $Ax = x$ for every $A \in G$.*

Proof. Let Ω be the collection of all non-empty compact convex sets $H \subseteq K$ such that $GH \subseteq H$ (we use this as shorthand for $AH \subseteq H$ for every $A \in G$). Then Ω is a non-empty poset (it contains K itself), so it contains a maximal totally ordered subset, say Ω_0. Let

$$H_0 = \bigcap_{H \in \Omega_0} H.$$

A nested family of compact sets necessarily has a non-empty intersection (see Problem 28), so H_0 is non-empty, compact and convex. Since $GH \subseteq H$ for every $H \in \Omega_0$, we also have $GH_0 \subseteq H_0$. The proof will be completed once we show that H_0 is a singleton.

Suppose H_0 contains more than one element. Then H_0 has positive diameter,

$$d = \max\{||x - y|| : x, y \in H_0\}.$$

This is well-defined because the function $f : H_0 \times H_0 \to \mathbb{R}$ defined by $f(x, y) = ||x - y||$ is a continuous function on a compact set. Since H_0 is compact, there exist $x_1, \ldots, x_n \in H_0$ such that

$$H_0 \subseteq \bigcup_{i=1}^{n} B\left(x_i, \frac{d}{2}\right).$$

We set

$$H_1 = H_0 \cap \bigcap_{y \in H_0} \overline{B}\left(y, \left(1 - \frac{1}{4n}\right) d\right).$$

Note that H_1 is an intersection of convex sets, so it is itself convex. Furthermore, it is a closed subset of H_0, so it is compact. Finally, it is clear that H_1 is strictly smaller than H_0, because $\left(1 - \frac{1}{4n}\right) d$ is smaller than the diameter d of H_0. It remains to prove that H_1 is non-empty.

Let $x_0 = \frac{1}{n}(x_1 + \cdots + x_n)$. Then x_0 is in H_0 because H_0 is convex. For any $y \in H_0$, there exists an $i \in \{1, \ldots, n\}$ such that $y \in B\left(x_i, \frac{d}{2}\right)$, that is, $||y - x_i|| < \frac{d}{2}$. For any other $j \neq i$, we have

$$||y - x_i|| < \left(1 + \frac{1}{4n}\right) d$$

by the definition of d. We compute

$$||y - x_0|| = \left|\left|y - \frac{1}{n}\sum_{k=1}^{n} x_k\right|\right| = \frac{1}{n}\left|\left|\sum_{k=1}^{n}(y - x_k)\right|\right| \leq \frac{1}{n}\sum_{k=1}^{n}||(y - x_k)||$$
$$< \frac{1}{n}\left[\frac{1}{2} + (n-1)\left(1 + \frac{1}{4n}\right)\right] \cdot d = \frac{4n^2 - n - 1}{4n^2} \cdot d < \frac{4n^2 - n}{4n^2} \cdot d = \left(1 - \frac{1}{4n}\right) d.$$

This shows $x_0 \in B\left(y, \left(1 - \frac{1}{4n}\right)\right)$ for all $y \in H_0$. Therefore $x_0 \in H_1$, so H_1 is non-empty.

Because G is a group of isometries, we have $g\overline{B}\left(y, \left(1 - \frac{1}{4n}\right)\right) = \overline{B}\left(gy, \left(1 - \frac{1}{4n}\right)\right)$ for every $y \in H_0$ and $g \in G$. This, combined with $GH_0 \subseteq H_0$, shows $GH_1 \subseteq H_1$. To summarize, H_1 is a compact convex subset of K which is mapped into itself by G. However, H_1 is strictly smaller than H_0, so this contradicts the maximality of Ω_0. It follows that H_0 is a singleton. This concludes the proof of the theorem.

□

Problem 28. Prove Cantor's Intersection Theorem: If $\Omega = \{K_i : i \in I\}$ is a family of compact sets in X totally ordered by inclusion, then $\bigcap_{i\in I} K_i$ is non-empty.

10. The Haar measure

Let G be a compact group. We assume that the topology on G is Hausdorff[2]. In this section, we prove the existence (and uniqueness) of a translation-invariant measure on G, called the **Haar measure**.

To construct the Haar measure, we will in fact use the Riesz Representation Theorem, which we now recall. Let X be a compact Hausdorff space. Let $C(X)$ denote the space of continuous functions on X. Recall that $C(X)$ is a Banach space when viewed as a normed space with the usual sup-norm $||\cdot||_\infty$. A linear functional $I : C(X) \to \mathbb{C}$ is said to be positive if $I(f) \geq 0$ whenever $f \geq 0$.

The Riesz Representation Theorem. Let I be a positive functional on $C(X)$. Then there exists a unique regular Borel measure μ on X such that

$$I(f) = \int_X f \; d\mu, \quad \text{for all } f \in C(X).$$

The main result of this section is the following:

Theorem 10.1. *Let G be a compact group. There exists a positive functional $I : C(G) \to \mathbb{C}$ with the following properties:*

(i) left invariance: $I(L(x)f) = I(f)$, for all $f \in C(G)$ and $x \in G$;

(ii) right invariance: $I(R(x)f) = I(f)$, for all $f \in C(G)$ and $x \in G$;

(iii) $I(\check{f}) = I(f)$ for every $f \in C(G)$, where $\check{f}(x) = f(x^{-1})$;

(iv) $I(1) = 1$;

(v) if $f \geq 0$ and $f \neq 0$, then $I(f) > 0$, for every $f \in C(G)$.

Such a functional is determined up to a constant by property (i) (or (ii)), and therefore uniquely determined by (i) (or (ii)) and (iv).

The proof of this theorem will occupy the rest of this section. We start by observing that any positive functional is necessarily continuous:

Lemma 10.2. *Let X be a compact space and I a positive functional on $C(X)$. Then I is continuous, with norm $||I||$ equal to $I(1)$.*

---

[2]Strictly speaking, this assumption is not necessary. See e.g. (5) in the relevant section of the bibliography.

Proof. Since I is positive, we have $I(f) \leq I(g)$ whenever f and g are real-valued functions such that $f \leq g$. We use this observation in what follows:

First, suppose $f \geq 0$ is a non-negative real-valued function. Then $f \leq ||f||_\infty$, so

$$I(f) \leq I(||f||_\infty) = ||f||_\infty \cdot I(1).$$

Now let $f \in C(X)$ be a continuous function and let $I(f) = re^{i\varphi}$. Denote by g and h the real and imaginary part of $fe^{-i\varphi}$, respectively. Then

$$|I(f)| = r = I(fe^{-i\varphi}) = I(g + ih) = I(g) \leq I(|g|) \leq I(|f|) \leq ||f||_\infty \cdot I(1).$$

where the first two inequalities hold because I is positive, and the last equality follows from the above calculation for non-negative functions.

This shows $|I(f)| \leq ||f||_\infty \cdot I(1)$, which is what we needed to prove. □

We now construct the functional I from the theorem. For any $f \in C(G)$, denote by

$$C_l(f) = \text{the convex hull of } \{L(x)f : x \in G\},$$
$$C_r(f) = \text{the convex hull of } \{R(x)f : x \in G\}.$$

Our first goal is to prove

Lemma 10.3. *The sets $C_l(f)$ and $C_r(f)$ are relatively compact in $C(G)$.*

Throughtout this chapter, when working with functions on G, we will denote functions by f, g, h (and occasionally ϕ); by contrast, elements of G will be denoted by x, y, z.

Proof. Let g be an element of $C_l(f)$, that is, $g = \sum_i^n \alpha_i L(x_i)f$ for some $x_1, \dots x_n \in G$ and a choice of positive $\alpha_1, \dots, \alpha_n$ such that $\sum_i^n \alpha_i = 1$. Then

$$||g||_\infty = ||\sum_i^n \alpha_i L(x_i)f||_\infty \leq \sum_i^n \alpha_i ||L(x_i)f||_\infty = \sum_i^n \alpha_i ||f||_\infty = ||f||_\infty.$$

Here we use the fact that $||L(x)f||_\infty = ||f||_\infty$ for any $x \in G$; in other words, $L(x)$ is an isometry on $C(G)$. The above calculation shows that the set $C_l(f)$ is bounded.

Recall that a set $S \subset C(G)$ is said to be equicontinuous at $x \in G$ if, for every $\epsilon > 0$, there exists a neighborhood U of x such that

$$|f(x) - f(y)| < \epsilon \quad \text{for every } y \in U \text{ and } f \in S.$$

But every neighborhood of x in G is of the form xU for some neighborhood U of the identity. This means that S is equicontinuous at x if, for every $\epsilon > 0$, there exists a neighborhood U of e such that

$$|f(x) - f(y)| < \epsilon \quad \text{whenever } x^{-1}y \in U \text{ and } f \in S.$$

Now let $g = \sum_i^n \alpha_i L(x_i)f$ be an element of $C_l(f)$. Since f is a continuous function on a compact space, it is uniformly continuous: for every $\epsilon > 0$ there exists a neighborhood U of e such that

$$|f(x) - f(y)| < \epsilon \quad \text{whenever } x^{-1}y \in U.$$

Taking $x, y \in G$ such that $x^{-1}y \in U$, we see

$$|g(x) - g(y)| \leq \sum_i^n \alpha_i |L(x_i)f(x) - L(x_i)f(y)| = \sum_i^n \alpha_i |f(x_i^{-1}x) - f(x_i^{-1}y)| < \sum_i^n \alpha_i \epsilon = \epsilon.$$

Indeed, the inequality follows because $(x_i^{-1}x)^{-1}(x_i^{-1}y) = x^{-1}y \in U$. This shows that the set $C_l(f)$ is equicontinuous.

By Arzelà–Ascoli, it now follows that $C_l(f)$ is relatively compact. The proof for $C_r(f)$ is analogous. Alternatively, one can deduce it by noting that $f \mapsto \check{f}$ is an isometry of $C(G)$ taking $C_l(f)$ to $C_r(\check{f})$. $\square$

Now G acts on the compact set $\overline{C_l(f)}$ by isometries $L(g)$. Therefore, by Kakutani's fixed point theorem, the set $\overline{C_l(f)}$ contains a fixed point for this action. Notice that a function fixed by all left translations is necessarily constant. Thus, we have proved that $\overline{C_l(f)}$ contains a constant. By the same argument, $\overline{C_r(f)}$ contains a constant. We now show that this constant is unique:

Lemma 10.4. *Let $\alpha \in \overline{C_l(f)}$ and $\beta \in \overline{C_r(f)}$ be constants. Then $\alpha = \beta$. In particular $\overline{C_l(f)}$ (resp. $\overline{C_r(f)}$) contains a unique constant function.*

Proof. Let $\epsilon > 0$. Since α is an element of $\overline{C_l(f)}$, there is a convex combination $\sum \alpha_i L(x_i) f \in C_l(f)$ such that

$$||\alpha - Lf|| < \frac{\epsilon}{2},$$

where we have used L to denote $\sum \alpha_i L(x_i)$. Similarly, we have

$$||\beta - Rf|| < \frac{\epsilon}{2},$$

for some convex combination R of right translations. Since L is a convex combination of isometries, it does not increase the norm:

$$||Lh||_\infty = \sum ||\alpha_i L(x_i)h||_\infty \leq \sum \alpha_i ||L(x_i)h||_\infty = \sum \alpha_i ||h||_\infty = ||h||_\infty,$$

for any $h \in C(G)$. Therefore, applying L to the above inequality (and using $L\beta = \beta$), we get

$$||\beta - LRf|| < \frac{\epsilon}{2}.$$

Similarly, we may apply R to the first inequality to get

$$||\alpha - RLf|| < \frac{\epsilon}{2}.$$

One verifies readily that R and L commute. Therefore,

$$||\alpha - \beta||_\infty = ||\alpha - RLf + LRf - \beta||_\infty < ||\alpha - RLf||_\infty + ||LRf - \beta||_\infty < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon.$$

This shows that $|\alpha - \beta|$ is arbitrarily small; it follows that $\alpha = \beta$. $\square$

We have just proved that $\overline{C_l(f)}$ contains a unique constant function. We denote the corresponding constant by $I(f)$. We have thus constructed a map $f \mapsto I(f)$, and it remains to prove that I is a positive functional satisfying properties (i)–(v).

First, it is clear that $I(f) \geq 0$ whenever $f \geq 0$: in that case, every function in $C_l(f)$ is non-negative, so the constant function $I(f)$ is (by definition) a uniform limit of non-negative functions; therefore, $I(f)$ must itself be non-negative.

Next, for any scalar $\alpha \in \mathbb{C}$, we have $C_l(\alpha f) = \alpha \cdot C_l(f)$ and therefore $\overline{C_l(\alpha f)} = \alpha \cdot \overline{C_l(f)}$. It follows that $I(\alpha f) = \alpha I(f)$.

To prove additivity, we first note that if L is a convex combination of left translations, then $I(Lf) = I(f)$. Indeed, one checks that $C_l(Lf) \subseteq C_l(f)$, and the claim follows: $I(Lf)$ is the unique constant in $\overline{C_l(Lf)}$; since it is also contained in $\overline{C_l(f)}$, it must be equal to $I(f)$.

Now suppose f and g are functions in $C(G)$, and let $\epsilon > 0$. By defintion of I, there exists a convex combination L of left translations such that

$$||Lf - I(f)||_\infty < \frac{\epsilon}{2}.$$

Now we consider the function Lg. Since $I(Lg) = I(g)$, we may find a covex combination L' of left translations such that

$$||L'Lg - I(g)||_\infty < \frac{\epsilon}{2}.$$

Applying L' to the first inequality (and recalling that L' does not increase the norm and does not affect constants), we get

$$||L'Lf - I(f)||_\infty < \frac{\epsilon}{2}.$$

Adding these inequalities, we get

$$||L'L(f + g) - I(f) - I(g)||_\infty < \epsilon.$$

Notice that $L'L$ is just another convex combination of left translations. This shows that the constant $I(f) + I(g)$ can be uniformly approximated (with arbitrary precision) by convex combinations of left translations of the function $f+g$. It follows that $I(f)+I(g) = I(f+g)$, which is what we needed to prove. We now know that I is a positive functional; it remains to prove properties (i)–(v).

Let g in G be arbitrary. One verifies immediately that $C_l(L(g)f) = C_l(f)$. It follows that $I(L(g)f) = I(f)$. This proves (i). Property (ii) follows by the same argument applied to $R(g)$ acting on $C_r(f)$ (indeed, recall that $I(f)$ is also the unique constant in the closure of $C_r(f)$).

For property (iii), observe that $C_l(\check{f}) = C_r(f)^\vee$. The involution $f \to \check{f}$ is a continuous map $C(G) \to C(G)$, so $\overline{C_r(f)^\vee} = \overline{C_r(f)}^\vee$. Furthermore, the unique constant in $\overline{C_r(f)}^\vee$ is the same as the unique constant in $\overline{C_r(f)}$, which is $I(f)$. Therefore $I(\check{f}) = I(f)$.

Property (iv) follows from $C_l(1) = \{1\}$.

Finally, property (v) follows from a compactness argument. Let $f \geq 0$, $f \neq 0$. Let $\epsilon > 0$ and consider the set $U = \{x \in G : f(x) > \epsilon\}$. This is open in G, and non-empty if we take ϵ to be small enough. Since G is compact, there exist finitely many points $x_1, \ldots, x_n$ in G such that the sets $x_i U$ cover G. Now let $g = \sum_{i=1}^n L(x_i) f$. For any $y \in G$, there exists a $k \in \{1, \ldots, n\}$ such that y is contained in $x_k U$. Therefore

$$g(y) = \sum_{i=1}^{n} L(x_i) f(y) \geq L(x_k) f(y) = f(x_k^{-1} y) > \epsilon,$$

because $x_k^{-1} y$ is an element of U. It follows that $g(y) > \epsilon$ for every $y \in G$. Note that $I(g) = n \cdot I(f)$. By positivity, we have

$$n \cdot I(f) = I(g) > I(\epsilon) = \epsilon.$$

Therefore $I(f) > \frac{\epsilon}{n}$. In particular, $I(f) \neq 0$.

Finally, we need to prove uniqueness. Let M be another positive left-invariant functional on $C(G)$. For any function $f \in C(G)$, there is a sequence of functions f_n in $C_l(f)$ uniformly converging to $I(f)$.

Because of left invariance, we have $M(f_n) = M(f)$. By Lemma 10.2, M is continuous, so $f_n \to I(f)$ implies $M(f_n) \to M(I(f)) = I(f) \cdot M(1)$. Therefore $M(f) = I(f) \cdot M(1)$, which proves uniqueness up to a multiplicative constant. This concludes the proof of our theorem.

The measure corresponding to $I : C(G) \to \mathbb{C}$ via the Riesz representation theorem is called the **Haar measure** on G. It is the unique left-invariant regular Borel measure on G. Note that the Haar measure inherits other properties from I, namely right-invariance and invariance under taking the inverse. We write

$$\int_G f(x) dx$$

for the integral of a measurable function f with respect to the Haar measure. Of course, if f is continuous, this is precisely $I(f)$.

Remark 10.5. The construction of the Haar measure presented here only works for compact groups. However, a left-invariant (resp. right-invariant) Haar measure exists on any locally compact Hausdorff topological group. For Lie groups, there is a particularly simple construction:

Let G be an n-dimensional Lie group. Choose a non-zero element $\omega_e \in \Lambda^n T_e^* G$. Recall that G acts on itself by left translations: we denote this action by

$$L_g(h) = gh, \quad \text{for } g, h \in G.$$

In particular, $L_{g^{-1}}$ takes g to e, so we get a pullback $L_{g^{-1}}^* : \Lambda^n T_e^* G \to \Lambda^n T_g^* G$. Letting

$$\omega_g = L_{g^{-1}}^* \omega_e$$

we get a section $\omega : g \mapsto \omega_g$ of $\Lambda^n T^* G$.

We check that this section is left-invariant: for any $g, h \in G$, we have

$$(L_g^* \omega)_h = L_g^* \omega_{gh} = L_g^* L_{h^{-1}g^{-1}}^* \omega_e = (L_{h^{-1}g^{-1}} L_g)^* \omega_e = L_{h^{-1}}^* \omega_e = \omega_h.$$

We thus get a left-invariant volume form (and hence measure) on G. To prove uniqueness, notice that $\dim \Lambda^n T_e^* G = 1$. If ω' is another volume form, then there exist a scalar c such that $\omega'_e = c \cdot \omega_e$. But then

$$\omega'_g = L_{g^{-1}}^* \omega'_e = c \cdot L_{g^{-1}}^* \omega_e = c \cdot \omega_g.$$

This proves that $\omega' = c \cdot \omega$: the left-invariant volume form is unique up to a multiplicative constant. Note that the left Haar measure need not be equal to the right Haar measure if the group is not compact.

11. Representations on Hilbert spaces

In the rest of this chapter, G is a compact Hausdorff group. Let $\mathcal{H}$ be a Hilbert space. We let $B(\mathcal{H})$ denote the space of all bounded operators on $\mathcal{H}$; this is a Banach space. Within $B(\mathcal{H})$ we have $\mathrm{GL}(\mathcal{H})$, the group of invertible bounded operators on $\mathcal{H}$.

A **representation** of G on $\mathcal{H}$ is a homomorphism $\pi : G \to \mathrm{GL}(\mathcal{H})$ such that the map

$$(g, v) \mapsto \pi(g)v$$

from $G \times \mathcal{H}$ to $\mathcal{H}$ is continuous.

Lemma 11.1. *Equivalently, we may require that the map*

$$g \mapsto \pi(g)v$$

from G to $\mathcal{H}$ be continuous for every v.

Proof. Suppose $g \mapsto \pi(g)v$ is continuous for each $v \in \mathcal{H}$. Then $g \mapsto ||\pi(g)v||$ is a continuous function on G, and therefore bounded. Now the uniform boundedness principle shows that there exists an $M > 0$ such that

$$||\pi(g)|| < M \quad \text{for all } G.$$

Now, for any $v, w \in \mathcal{H}$ and $g, h \in G$, we have

$$||\pi(g)v - \pi(h)w|| \leq ||\pi(g)v - \pi(g)w|| + ||\pi(g)w - \pi(h)w|| \leq M \cdot ||v - w|| + ||\pi(g)w - \pi(h)w||,$$

which is enough to show that $(g, v) \mapsto \pi(g)v$ is continuous.

The other direction is immediate: the map $g \mapsto \pi(g)v_0$ is obtained by precomposing $(g, v) \mapsto \pi(g)v$ with the (clearly continuous) map $g \mapsto (g, v_0)$. □

Remark 11.2. In other words, we require the map $\pi : G \to B(\mathcal{H})$ to be continuous, where we consider $B(\mathcal{H})$ with the strong operator topology. Caveat: the SOT is strictly weaker than the norm topology whenever $\mathcal{H}$ is infinte-dimensional.

The following proposition is the first of many applications of the Haar measure:

Proposition 11.3. *Let π be a representation of G on a Hilbert space $\mathcal{H}$. Then there exists a G-invariant inner product on $\mathcal{H}$.*

Proof. Let $(\cdot|\cdot)$ be the inner product on $\mathcal{H}$. For $v, w \in H$ we set

$$\langle v, w\rangle = \int_G (\pi(g)v|\pi(g)w)\ dg.$$

One verifies easily that this formula defines a G-invariant inner product. As an example, we prove that the constructed sesquilinear map is definite.

Let $v \neq 0$. Then $f : g \mapsto (\pi(g)v|\pi(g)v)$ is a non-negative continuous function on G. Since $f(e) = (v|v) > 0$, we see that $f \neq 0$. Now property (v) of Theorem 10.1 shows that the integral of f is positive; this means precisely $\langle v, v\rangle > 0$. $\square$

Problem 29. Show that the norm induced by this inner product is equivalent to the original norm on $\mathcal{H}$: there exist constants $m, M > 0$ such that

$$m(v|v) \leq \langle v, v\rangle \leq M(v|v), \quad \text{for all } v \in \mathcal{H}.$$

Note that every operator $\pi(g)$ is unitary with respect to this new inner product. Because of this, we assume that all representations are unitary from now on.

Remark 11.4 (Finite-dimensional representations)**.** Proposition 11.3 is not the only result that carries over from Chapter I without modification, once we replace the weighted average with an integral over G. In fact, if we restrict our attention to *finite-dimensional* representations of G, we get

- Schur's Lemma;
- Maschke's Theorem: every finite-dimensional representation of G is completely reducible;
- Theorem 2.7;
- The definition of character χ_π and all the results of §4.1.

12. The regular representation

As before, we let $C(G)$ denote the space of all continuous functions on G. We define the **convolution** of f and g in $C(G)$ by

$$(f \star g)(x) = \int_G f(xy^{-1})g(y)\ dy.$$

Problem 30. Prove that $f \star g$ is continuous and that $||f \star g||_\infty \leq ||f||_\infty ||g||_\infty$. Thus, $(C(G), ||\cdot||_\infty, \star)$ is an example of a **Banach algebra**.

The space of continuous functions $C(G)$ comes equipped with the standard inner product:

$$\langle f, g \rangle = \int_G f(x)\overline{g(x)}\ dx.$$

Therefore, we have another norm on $C(G)$, namely $||\cdot||_2$. Note that

$$||f||_2^2 = \int_G |f(x)|^2\ dx \leq ||f||_\infty^2,$$

so $||f||_2 \leq ||f||_\infty$. (Uniform convergence implies L^2 convergence.)

Recall that G acts on $C(G)$ by right translation: $[R(g)f](x) = f(xg)$. Because the Haar measure is translation-invariant, we have

$$||R(g)f||_2 = \int_G |R(g)f|^2 = \int_G |f|^2 = ||f||_2.$$

This shows that $R(G)$ is an isometry on $(C(G), ||\cdot||_2)$.

Now take $f \in C(G)$ and let $\epsilon > 0$. Since f is uniformly continuous, one may find a neighborhood U of the identity such that

$$|f(x) - f(y)| < \epsilon \quad \text{whenever } y^{-1}x \in U.$$

Thus, if we take such x, y, we get

$$|[R(x)f - R(y)f](z)| = |f(zx) - f(zy)| < \epsilon \quad \text{for any } z \in G.$$

This means $||R(x)f - R(y)f||_\infty < \epsilon$, and therefore $||R(x)f - R(y)f||_2 < \epsilon$ as well. In other words, this shows that $x \mapsto R(x)f$ is a continuous map $G \to C(G)$ for any $f \in C(G)$.

The completion of $C(G)$ with respect to the norm $||\cdot||_2$ is the space $L^2(G)$ of square-integrable functions. For any $x \in G$, the operator $R(x)$ uniquely extends to an isometry of $L^2(G)$. Clearly, this extension is still the right translation by x, so we abuse notation and denote it by $R(x)$. We would like to show that $x \mapsto R(x)$ is a representation of G on the Hilbert space $L^2(G)$.

To that end, we need to show that $x \mapsto R(x)\phi$ is a continuous function for any $\phi \in L^2(G)$. So take $\phi \in L^2(G)$. Let $\epsilon > 0$. Because $C(G)$ is dense in $L^2(G)$, we can find a continuous function $f \in C(G)$ such that $||\phi - f||_2 < \frac{\epsilon}{3}$. For any $x, y \in G$, we now have

$$\begin{aligned}
||(R(x) - R(y))\phi||_2 &= ||(R(x) - R(y))(\phi - f)||_2 + ||(R(x) - R(y))f||_2 \\
&\leq ||R(x)(\phi - f)||_2 + ||R(y)(\phi - f)||_2 + ||(R(x) - R(y))f||_2 \\
&\leq ||R(x)||_2 \cdot ||(\phi - f)||_2 + ||R(y)||_2 \cdot ||(\phi - f)||_2 + ||(R(x) - R(y))f||_2 \\
&\leq \frac{\epsilon}{3} + \frac{\epsilon}{3} + ||(R(g) - R(h))f||_2.
\end{aligned}$$

Finally, since we have shown that $x \mapsto R(x)f$ is continuous for $f \in C(G)$, we can find a neighborhood U of the identity such that $y^{-1}x \in U$ implies $||(R(x) - R(y))f||_2 < \frac{\epsilon}{3}$.

To summarize, we have shown that for any $\phi \in L^2(G)$ and any $\epsilon > 0$, there exists a symmetric neighborhood U of the identity such that

$$||(R(x) - R(y))\phi||_2 < \epsilon$$

whenever $y^{-1}x \in U$. Therefore, the map $G \to L^2(G)$ given by $x \mapsto R(x)\phi$ is continuous for every $\phi \in L^2(G)$. This means that R is a representation of G on the Hilbert space $L^2(G)$. We call this the (right) **regular representation** of G. Note that this representation preserves the standard inner product on $L^2(G)$: R is a unitary representation.

Problem 31. Let $G = S^1$ (the circle group). We may identify G with $[0, 1]$; translation by $x \in [0, 1]$ is then given by $y \mapsto y + x \pmod{\mathbb{Z}}$. Consider the right regular representation R of G. Show that the map $g \mapsto R(g)$ from G to $B(L^2(G))$ is *not* continuous if we take the norm topology on $B(L^2(G))$.

This shows why we only require SOT-continuity when defining representations of compact groups: norm-continuity is too restrictive.

Now let $f \in C(G)$ be a continuous function. We define an operator $R(f)$ on $L^2(G)$ by setting

$$[R(f)\phi](x) = \int_G f(y)\phi(xy)\ dg$$

for every $\phi \in L^2(G)$.

Problem 32. Show that $R(f \star g) = R(f)R(g)$.

Problem 33. Let $f \in C(G)$. Define $f^* \in C(G)$ by $f^*(x) = \overline{f(x^{-1})}$. Show that

$$R(f)^* = R(f^*).$$

Remark 12.1. One may define $\pi(f)$ analogously for any representation $(\pi, \mathcal{H})$ of G and $f \in C(G)$. Then $f \mapsto \pi(f)$ is a homomorphism of Banach algebras $C(G) \to B(\mathcal{H})$:

$$\pi(f \star g) = \pi(f)\pi(g), \quad \text{for any } f, g \in C(G).$$

Furthermore, we have $\pi(f)^* = \pi(f^*)$.

One may show that a closed subspace $\mathcal{K}$ of $\mathcal{H}$ is $C(G)$-stable if and only if it is G-stable. This is slightly more complicated than in the finite group case, because the Banach algebra $C(G)$ does not have a unit.

Let $f \in C(G)$. We show that the image of $R(f) : L^2(G) \to L^2(G)$ is contained in $C(G)$.

Let $\epsilon > 0$. Since f is uniformly continuous, there exists a neighborhood U of the identity such that $|f(x) - f(y)| < \epsilon$ whenever $xy^{-1} \in U$. Now, for any $\phi \in L^2(G)$, we compute

$$\begin{aligned} |R(f)\phi(x) - R(f)\phi(y)| &= \left| \int_G f(z)\phi(xz)\ dx - \int_G f(z)\phi(yz)\ dz \right| \\ &= \left| \int_G f(x^{-1}z)\phi(z)\ dz - \int_G f(y^{-1}z)\phi(z)\ dz \right| \\ &= \left| \int_G \left(f(x^{-1}z) - f(y^{-1}z)\right)\phi(z)\ dz \right| \\ &\leq \int_G \left|f(x^{-1}z) - f(y^{-1}z)\right| \cdot |\phi(z)|\ dz \\ &\leq \epsilon \cdot ||\phi||_1 \leq \epsilon \cdot ||\phi||_2. \end{aligned}$$

This proves that $R(f)\phi$ is in $C(G)$.

Remark 12.2. The last inequality used above is a standard application of Cauchy–Schwarz: for any $f \in L^2(G)$, we have

$$||f||_1 = \int_G |f| = \int_G 1 \cdot |f| \leq ||1||_2 \cdot ||f||_2 = ||f||_2.$$

We end this section by studying the image of the unit ball in $L^2(G)$ under $R(f)$. Let $\phi \in L^2(G)$. Then $R(f)\phi$ is a continuous function and we have, for any $x \in G$:

$$\begin{aligned} |[R(f)\phi](x)| = \left| \int_G f(y)\phi(xy)\ dy \right| &\leq \int_G |f(y)\phi(xy)|\ dy \\ \leq \int_G ||f||_\infty \cdot |\phi(xy)|\ dg &\leq ||f||_\infty \cdot ||\phi||_1 \leq ||f||_\infty \cdot ||\phi||_2. \end{aligned}$$

This shows $||R(f)\phi||_\infty = ||f||_\infty \cdot ||\phi||_2$; in particular, $R(f)$ is bounded, if viewed as a map

$$(L^2(G), ||\cdot||_2) \to (C(G), ||\cdot||_\infty).$$

This also shows that the image of the unit ball

$$R(f)B(0,1) = \{R(f)\phi : ||\phi||_2 \leq 1\}$$

is a bounded set in $C(G)$.

Furthermore, the calculation we used above to show that $R(f)\phi$ is continuous shows that for any $\epsilon > 0$, there is a neighborhood U of the identity such that

$$|R(f)\phi(x) - R(f)\phi(y)| < \epsilon \cdot ||\phi||_2$$

whenever xy^{-1} is in U. This implies that

$$R(f)B(0,1) = \{R(f)\phi : ||\phi||_2 \leq 1\}$$

is an equicontinuous set of functions in $C(G)$.

Combining these two observations, we use Arzelà–Ascoli to conclude that the closure of $R(f)B(0,1)$ is compact in $(C(G), ||\cdot||_\infty)$. Finally, recall that $||f||_2 \leq ||f||_\infty$ for any $f \in C(G)$.

That means that the inclusion $(C(G), ||\cdot||_\infty) \to (L^2(G), ||\cdot||_2)$ is continuous. In particular, any set that is compact in $(C(G), ||\cdot||_\infty)$ is also compact in $(L^2(G), ||\cdot||_2)$.

13. Compact operators

Let X be a normed space. We say that an operator $A \in B(X)$ is **compact** if it takes the closed unit ball in X to a relatively compact set. This is equivalent to A taking every bounded set to a relatively compact set.

Example 13.1. Let G be a compact group and let $X = L^2(G)$. Then the discussion from the end of last section shows that $R(f)$ is a compact operator, for any $f \in C(G)$.

Lemma 13.2. *The set of compact operators is a two-sided ideal in $B(X)$.*

Proof. Recall that continuous maps take relatively compact sets to relatively compact sets.

First, we show that compact operators form a subspace of $B(X)$. Indeed, let S, T be compact operators and let $\alpha, \beta \in \mathbb{C}$ be arbitrary scalars. Then $\alpha S + \beta T$ is a composition of two maps:

$$X \to X \times X \text{ given by } x \mapsto (Sx, Tx)$$

and

$$X \times X \to X \text{ given by } (x_1, x_2) \mapsto \alpha x_1 + \beta x_2.$$

The first map takes the unit ball to a relatively compact set; since the second map is continuous, the image of the unit ball under the composition of these two maps is again relatively compact.

Now suppose S is compact and T is bounded. Then the set $SB(0,1)$ is relatively compact, and T is continuous; therefore, $TSB(0,1)$ is relatively compact. On the other hand, the set $TB(0,1)$ is bounded, and S is compact; therefore $STB(0,1)$ is relatively compact. This shows that ST and TS are compact operators. □

On Hilbert spaces, the situation is particularly nice for self-adjoint operators:

Proposition 13.3. *Let T be a* self-adjoint *compact operator on a Hilbert space $\mathcal{H}$. Then either $||T||$ or $-||T||$ is an eigenvalue of T.*

Proof. Without loss of generality, we may assume $||T|| = 1$ (if not, we simply rescale T). By definition of $||T||$, we have

$$1 = \sup\{||Tv|| : v \in B(0,1)\}.$$

Therefore, we can find a sequence v_n in $B(0,1)$ such that $(||Tv_n||)$ converges to 1. Since $TB(0,1)$ is relatively compact, we may as assume that the sequence (Tv_n) converges to a limit, say w. Of course, $||w|| = 1$.

Now $Tv_n \to w$ implies $T^2v_n \to Tw$. Therefore

$$\begin{aligned}1 = ||T|| \cdot ||w|| \geq ||Tw|| = \lim_n ||T^2v_n|| \geq \limsup_n ||T^2v_n|| \cdot ||v_n|| \\ \geq \limsup_n |\langle T^2v_n, v_n\rangle| = \lim_n |\langle Tv_n, Tv_n\rangle| = \lim_n ||Tv_n||^2 = 1.\end{aligned}$$

Here we use Cauchy–Schwarz to obtain the first inequality in the second row. This shows that $||Tw|| = 1$.

Thus, we get

$$1 = ||Tw||^2 = \langle Tw, Tw\rangle = \langle T^2w, w\rangle \leq ||T^2w|| \cdot ||w|| \leq ||T||^2||w||^2 = 1.$$

The inequality in the middle is Cauchy–Schwarz again; however, the computation shows that we in fact have equality. This implies that the two vectors are collinear: $T^2w = \lambda w$ for some $\lambda \in \mathbb{C}$. Furthermore, $\lambda = 1$, because

$$\lambda = \lambda\langle w, w\rangle = \langle T^2w, w\rangle = 1.$$

Finally, either $Tw = w$ (so w is an eigenvector for eigenvalue 1), or $Tw - w$ is non-zero, and therefore an eigenvector for -1:

$$T(Tw - w) = T^2w - Tw = w - Tw = -(Tw - w).$$ □

We conclude the section with another useful fact:

Lemma 13.4. *Let T be a compact operator on a normed space X. Assume $\lambda \neq 0$ is an eigenvalue for T. Then the λ-egienspace of T is finite-dimensional.*

Proof. Let X_λ be the λ-eigenspace for T. The restriction of T to X_λ is still compact. But it is also equal to $\lambda \cdot I$, and this operator is compact if and only if the unit ball in X_λ is compact, which is equivalent to X_λ being finite-dimensional. □

Remark 13.5. On a Hilbert space, every compact operator is a limit (in the norm topology) of a sequence of finite-rank operators. This property fails in Banach spaces: The first counterexample famously earned Per Enflo a live goose.

14. Matrix coefficients

In this section, we work in $(C(G), ||\cdot||_\infty)$. We say that a function $f \in C(G)$ is **right-finite** if the set

$$\{R(x)f : x \in G\}$$

spans a finite-dimensional subspace of $C(G)$. We define left-finite functions analogously.

Lemma 14.1. *Let $f \in C(G)$. The following are equivalent:*

(i) f is right-finite;

(ii) f is left-finite;

(iii) there exists an integer n and functions $a_1, \ldots, a_n$ and $b_1, \ldots, b_n$ in $C(G)$ such that

$$f(xy) = \sum_{i=1}^{n} a_i(x) b_i(y), \quad \textit{for all } x, y \in G$$

Proof. Let f be right-finite. Then $V = \operatorname{span}\{R(y)f : y \in G\}$ is finite-dimensional. Let $\{a_1, \ldots, a_n\}$ be a basis for V. Then $R(y)f$ can be written as a linear combination of a_i's, where the coefficients depend on g:

$$R(y)f = \sum_{i=1}^{n} b_i(y) a_i.$$

Recall that $g \mapsto R(y)f$ is continuous, and a function with values in a finite-dimensional vector space is continuous if and only if the coefficients with respect to any given basis are continuous functions. Thus $b_1, \ldots, b_n$ are continuous functions. Evaluating the above equality at $x \in G$, we get

$$f(xy) = \sum_{i=1}^{n} b_i(y) a_i(x).$$

This shows that (i) implies (iii). Conversely, if the above equation holds for all $x, y \in G$, then $R(y)f = \sum_{i=1}^{n} b_i(y) a_i$, which means that the space V is spanned by $a_1, \ldots, a_n$. In other words, f is right-finite. Thus (iii) implies (i).

The equivalence of (ii) and (iii) is analogous. □

We let $M(G)$ denote the space of all finite functions.

Proposition 14.2. *$M(G)$ is a subalgebra of $C(G)$ (under pointwise operations).*

Proof. First, it is clear that $M(G)$ is a subspace: a linear combination of right-finite functions is still right-finite. Now let f and g be functions in $M(G)$. Then, by part (iii) of the Lemma, we can find functions a_i, b_i, c_j, d_j such that

$$f(xy) = \sum_{i=1}^{n} a_i(x) b_i(y) \quad \text{and} \quad g(xy) = \sum_{j=1}^{m} c_j(x) d_j(y)$$

holds for all $x, y \in G$. But now

$$fg(xy) = \sum_{i=1}^{n} \sum_{j=1}^{m} (a_i c_j)(x) \cdot (b_i d_j)(y),$$

which shows that fg is finite as well. □

Remark 14.3. For f finite, we may write $f(xy) = \sum_{i=1}^{n} a_i(x) b_i(y)$. Recall that the a_i's were obtained in the proof of Lemma 14.1 by taking a basis $a_1, \ldots, a_n$ for the space $V = \operatorname{span}\{R(y)f : y \in G\}$. Now Proposition 14.2 shows that V is contained in $M(G)$. Therefore we may assume $a_i \in M(G)$.

We record two more useful consequences of Lemma 14.1:

Lemma 14.4. *Let $f \in M(G)$. Then $\overline{f}$ and $\check{f}$ are in $M(G)$ as well.*

Proof. This follows immediately from part (iii) of Lemma 14.1. $\square$

To state the next result, recall the operator $R(f)$ defined in §12.

Lemma 14.5. *Suppose $f \in C(G)$. The space $M(G)$ is $R(f)$-invariant.*

Proof. Let ϕ be a function in $M(G)$. Then $\phi(xy) = \sum_{i=1}^n a_i(x)b_i(y)$ for some continuous functions a_i, b_i. By Remark 14.3, we may assume that the a_i's are elements of $M(G)$. We have

$$[R(f)\phi](x) = \int_G f(y)\phi(xy)\ dy = \int_G f(y)\left(\sum_{i=1}^n a_i(x)b_i(y)\right) dy = \sum_{i=1}^n a_i(x)\int_G f(y)b_i(y)\ dy.$$

This shows that $R(f)\phi$ is a linear combination of $a_1, \ldots, a_n$ for any $f \in C(G)$. Since a_i's are contained in $M(G)$, the claim follows from Proposition 14.2. $\square$

We now come to the main result of this section. Suppose π is a representation of G on a finite-dimensional space V. Choosing a basis for V, we may form matrix coefficients $\pi(x)_{ij}$ for $i, j = 1, \ldots, \dim V$. Equivalently, one may think of matrix coefficients as functions of the form $v^*(\pi(x)v)$ for some $v \in V$ and v^* in V^*. Again, these are continuous functions on G.

Proposition 14.6. *Let $f \in C(G)$. The following are equivalent:*

(i) f is in $M(G)$;

(ii) f is a matrix coefficient of a finite-dimensional representation.

Proof. Suppose f is a matrix coefficient of (π, V), say $f(x) = \pi(x)_{kl}$. If we write $\pi(xy) = \pi(x)\pi(y)$, using matrices, we get

$$\pi(xy)_{kl} = \sum_{i=1}^n \pi(x)_{ki}\pi(y)_{il}.$$

This shows that f satisfies property (iii) of Lemma 14.1; therefore, $f \in M(G)$.

Conversely, let f be finite. Then, as before, $R(y)f = \sum_{i=1}^n b_i(y) \cdot a_i$. We may assume that $a_1, \ldots, a_n$ are independent. Then $\{a_1, \ldots, a_n\}$ is a basis of the R-invariant space V it spans. Now let $v = f$ and choose $v^* \in V^*$ so that $a_i(1) = v^*(a_i)$, for $i = 1, \ldots, n$. Then

$$v^*(R(y)v) = \sum_{i=1}^n b_i(y) \cdot v^*(a_i) = \sum_{i=1}^n b_i(y) \cdot a_i(1) = f(y).$$

Therefore, f is a matrix coefficient. $\square$

One can proceed to analyze the algebra $M(G)$ of matrix coefficients just like in the finite group case. The Haar measure enables us to use the method of averaging to show that Schur Orthogonality holds in this setting. This immediately gives us the decomposition

$$M(G) = \bigoplus_{\pi \in \hat{G}} M(\pi),$$

where $\hat{G}$ denotes the set of equivalence classes of finite-dimensional irreducible representations of G, and $M(\pi)$ is the space of matrix coefficients of π. As before, we have $\dim M(\pi) = (\dim \pi)^2$.

A guided exercise. In this exercise, we imitate the proof of Proposition 3.7 to show that

$$M(\pi) = \pi \otimes \pi^\vee,$$

as representations of $G \times G$. The idea is the same: one shows that $f \mapsto \pi(f)$ is the required isomorphism from $M(\pi)$ to $\mathrm{End}(\pi)$. However, there is a technical point that we must address here. A crucial fact that we used in proving Proposition 3.7 was the existence of a unit (namely, δ_e) for the algebra $(\mathbb{C}[G], \star)$. In the compact group setting, the algebra $(C(G), \star)$ does not have a unit. The point of this guided exercise is to show how one can circumvent this problem.

Problem 34. Let U be an open neighborhood of $e \in G$. Show that there exists a continuous function δ_U which vanishes outside of U, such that

$$\int_G \delta_U = 1.$$

Hint: Urysohn's Lemma.

Recall that the collection of neighborhoods $\{U\}$ of e forms a filter. In particular, we may study convergence properties of nets indexed by $\{U\}$.

Problem 35. Let $f \in C(G)$. Show that the net $f \star \delta_U$ converges to f uniformly. Conclude that any closed set in $C(G)$ that contains all of $f \star \delta_U$'s must contain f as well.

With this problem, we may carry out the proof of Proposition 3.7:

Problem 36. For $\pi \in \hat{G}$, let

$$N(\pi) = \bigcap_{F \in \mathrm{Hom}_G(R, \pi)} \ker F.$$

Here, we require every F to be continuous. In particular, $\ker F$ is closed for every F, which implies $N(\pi)$ is closed, too. Show that

a) $N(\pi)$ does not contain π as a subrepresentation;

b) the quotient $C(G)/N(\pi)$ is a $G \times G$-module isomorphic to $\pi \otimes \pi^\vee$.

Hint for a). Suppose $V \cong \pi$ is a subrepresentation of $C(G)$. You will need to construct a *bounded* G-intertwining from $C(G)$ to V. You can do so using the method of averaging (Theorem 2.7), but you need to ensure that the linear projector you are starting with is bounded. Since V is finite-dimensional, this is automatic, courtesy of Hahn–Banach.

15. Peter–Weyl

The following is the main result of representation theory of compact groups:

Theorem 15.1 (Peter–Weyl)**.** *The algebra $M(G)$ is dense in $L^2(G)$.*

Remark 15.2. In light of the guided exercise from the previous section, we may write

$$L^2(G) = \widehat{\bigoplus_{\pi \in \hat{G}}} \pi \otimes \pi^{\vee}.$$

The proof uses the operator $R(f)$ defined in Section 12:

$$R(f)\phi(x) = \int_G f(y)\phi(xy)\ dy.$$

In Section 12 we showed that the image of $R(f)$ is contained in $C(G)$. We recall that the operator $R(f)$ is compact (Example 13.1); it follows that $R(f)^*R(f) = R(f^*)R(f)$ is (positive) self-adjoint and compact. One verifies immediately that $R(f)$ commutes with left translation $L(x)$, for any $x \in G$. We shall need the following:

Lemma 15.3. *Suppose $\lambda > 0$ is an eigenvalue of $R(f^*)R(f)$; let V denote the corresponding eigenspace. Then V is contained in $M(G)$.*

Since we are viewing $R(f^*)R(f)$ as an operator on $L^2(G)$ (whose elements are equivalence classes), we should be careful: an element of $L^2(G)$ is said to be contained in $M(G)$ if the corresponding equivalence class has a representative in $M(G)$.

Proof. Let ϕ be a function such that $R(f^*)R(f)\phi = \lambda\phi$. Then ϕ is in the image of $R(f^*)R(f)$, and is therefore continuous. Moreover, since $R(f^*)R(f)$ commutes with L, we see that V is L-invariant. But recall that V is finite-dimensional, by Lemma 13.4. Therefore ϕ is finite. $\square$

Before proving the Theorem, we recall a few facts:

- If T is a self-adjoint compact operator on a Hilbert space, then either $||T||$ or $-||T||$ is an eigenvalue of T (Proposition 13.3);
- $M(G)$ is $R(f)$-invariant (Lemma 14.5).

Proof of Theorem 15.1. Let W be the orthogonal complement of $M(G)$ in $L^2(G)$. Since $M(G)$ is $R(f)^*R(f)$-invariant, so is W (this holds because $R(f)^*R(f)$ is self-adjoint). Thus we get a positive self-adjoint compact operator on the Hilbert space W. (Recall that the orthogonal complement is always a closed subspace.)

Assume that this operator is non-zero. Then its norm is non-zero, and therefore $R(f^*)R(f)$ has a non-zero eigenvalue, say λ. Let ϕ be an eigenvector for λ. Then ϕ is in W, but also in $M(G)$, by Lemma 15.3. Therefore $\phi = 0$, a contradiction. We conclude that the restriction of $R(f^*)R(f)$ to W is 0.

Now let $\phi \in W$. Since $R(f^*)R(f)\phi = 0$, we have

$$0 = \langle R(f^*)R(f)\phi, \phi\rangle = \langle R(f)\phi, R(f)\phi\rangle = ||R(f)\phi||_2^2.$$

Therefore $R(f)\phi = 0$, for any $\phi \in W$. Now

$$0 = [R(f)\phi](1) = \int_G f(x)\phi(x)\ dx.$$

This shows that ϕ is orthogonal to $\overline{f}$.

Since $f \in C(G)$ was arbitrary, and since $C(G)$ is dense in $L^2(G)$, we conclude that $\phi = 0$. Therefore $W = 0$. $\square$

15.1. **The continuous version.** Under mild assumptions, the Peter–Weyl theorem can be proven in a much simpler way, as we now explain. Assume that G has a faithful finite-dimensional representation (π, V). (Faithful means $g \mapsto \pi(g)$ is injective.) Then π is an embedding of G into $\mathrm{GL}(V)$. Therefore G is a **linear group** (we will also sometimes say G is a **matrix group**).

As before, let $M(G)$ denote the space of matrix coefficients. Without even using the results of §14, we make the following observations:

- $M(G)$ is an algebra. Indeed, if π and ρ are finite-dimensional representations of G, then the matrix coefficients of $\pi \otimes \rho$ are products of matrix coefficients of π and ρ. Thus $M(G)$ is closed under multiplication.
- $M(G)$ is closed under complex conjugation: for each representation π, we may consider its complex conjugate $\overline{\pi}$.
- $M(G)$ separates points. Recall that we have an injective representation π. If $h \neq g \in G$, then there is at least one matrix coefficient which in which $\pi(h)$ and $\pi(g)$ differ. Therefore the matrix coefficients of π separate points.
- $M(G)$ contains the constant functions (because of the trivial representation).

Now the fact that $M(G)$ is dense in $(C(G), ||\cdot||_\infty)$ is an immediate consequence of Stone–Weierstrass. Of course, this also means that $M(G)$ is dense in $C(G), ||\cdot||_2)$. Since $C(G)$ is dense in $L^2(G)$, we conclude that $M(G)$ is dense in $L^2(G)$!

The fact that $M(G)$ is dense in $(C(G), ||\cdot||_\infty)$ is known as the continuous version of Peter–Weyl. We now prove that this holds without assuming that G is a matrix group. We will be able to deduce this using the Stone–Weierstrass Theorem as soon as we show that $M(G)$ separates points.

To that end, let $x \in G$, $x \neq e$. Then, one may find a closed neighborhood U of e such that Ux and U are disjoint. One can show that there exists a positive function f which restricts to 0 on U and 1 on Ux. We have

$$||R(x)f - f||_2^2 = \int_G |f(yx) - f(y)|^2 \, dy \geq \int_U |f(yx) - f(y)|^2 \, dy = \mu(U) > 0,$$

where μ denotes the Haar measure. In particular, $R(x) \neq I$. Since $M(G)$ is dense in $L^2(G)$ by the L^2-version of the Peter–Weyl theorem, the restriction of $R(x)$ to $M(G)$ is also not equal to the identity operator.

Corollary 15.4. *The algebra $M(G)$ separates points: for any $x_1 \neq x_2 \in G$, there exists a function $f \in M(G)$ such that $f(x_1) \neq f(x_2)$.*

Proof. Let $x_1 \neq x_2 \in G$; set $y = x_1^{-1}x_2$. Since $R(y) \neq I$, we may find a function $\phi \in M(G)$ such that $R(y)\phi \neq \phi$. In other words, $R(x_1)\phi \neq R(x_2)\phi$. Thus we may find an element $z \in G$ such that $\phi(zx_1) \neq \phi(zx_2)$. Now $f = L(z^{-1})\phi$ is the function we need. □

This proves the continuous version of Peter–Weyl:

Theorem 15.5. *The algebra $M(G)$ is dense in $(C(G), ||\cdot||_\infty)$.*

15.2. **An example.** Let $G = \mathrm{SO}(2)$, the group of rotations of the two-dimensional plane. Recall that $\mathrm{SO}(2)$ can be identified with the group of all matrices of the form

$$g(\varphi) = \begin{bmatrix} \cos\varphi & -\sin\varphi \\ \sin\varphi & \cos\varphi \end{bmatrix}$$

Note that $\varphi \mapsto g(\varphi)$ is a surjective homomorphism $\mathbb{R} \mapsto \mathrm{SO}(2)$. Its kernel is $2\pi\mathbb{Z}$, so

$$\mathrm{SO}(2) \cong \mathbb{R}/2\pi\mathbb{Z} \cong S^1$$

This shows that G is a **Lie group**, i.e. a group which is also a manifold. In this case, the underlying manifold is the unit circle S^1; therefore, the group G is compact.

We also see that G is Abelian. Consequently, by Schur's Lemma, all finite-dimensional representations of G are one-dimensional. Let us determine them.

A one-dimensional representation (character) χ of G is a continuous homomorphism $G \mapsto \mathbb{C}^\times$. Composing with the homomorphism $\varphi \mapsto g(\varphi)$, we get a homomorphism $F : \mathbb{R} \mapsto \mathbb{C}^\times$. In other words, F satisfies

$$F(s+t) = F(s)F(t) \quad \text{for all } s, t \in \mathbb{R}.$$

This is precisely the functional equation defining the exponential function:

Problem 37. Let $F : \mathbb{R} \mapsto \mathbb{C}^\times$ be a continuous function such that $F(s+t) = F(s)F(t)$ for all $s, t \in \mathbb{R}$. Prove that there exists a $\lambda \in \mathbb{C}$ such that

$$F(t) = e^{\lambda \cdot t} \quad \text{for all } t \in \mathbb{R}.$$

For characters of G, the function $F(t) = e^{\lambda \cdot t}$ needs to satisfy $F(2\pi) = 1$. This is true precisely when λ is an integer multiple of the imaginary unit, i. We thus get a list of all characters of G:

$$\hat{G} = \{\chi_n : n \in \mathbb{Z}\},$$

where $\chi_n(g(\varphi)) = e^{in\varphi}$.

Schur orthogonality guarantees that these characters are mutually orthogonal (and that $||\chi_n||_2 = 1$ for each n), if we view them as functions on $S^1 \cong \mathbb{R}/2\pi\mathbb{Z}$. Finally, by Peter–Weyl, the characters of G span a dense subspace of $L^2(S^1)$. Thus we recover a classical result of harmonic analysis: The set

$$\{\chi_n : n \in \mathbb{Z}\}$$

is an orthonormal basis for $L^2(S^1)$!

Having analyzed $\mathrm{SO}(2)$, we may now describe the representation theory of the full orthogonal group, $\mathrm{O}(2)$. Recall that this is the group of all linear isometries of $\mathbb{R}^2$, and we have $\mathrm{O}(2) \cong \mathrm{SO}(2) \rtimes \mathbb{Z}/2\mathbb{Z}$. The non-trivial element of $\mathbb{Z}/2\mathbb{Z}$ can be represented by the matrix

$$\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

which acts on $\mathrm{SO}(2)$ by conjugation. (Geometrically speaking, any isometry is composed of a rotation, and a flip over a fixed axis). Thus $\mathrm{SO}(2)$ is a normal subgroup of index 2 in $\mathrm{O}(2)$.

For each n, we may define

$$\rho_n = \mathrm{Ind}_{\mathrm{SO}(2)}^{\mathrm{O}(2)} \chi_n.$$

Note that ρ_n is two-dimensional. Using Mackey theory, we compute $\rho_n|_{\mathrm{SO}(2)} = \chi_n \oplus \chi_{-n}$. This shows that ρ_n is irreducible for any $n \neq 0$; moreover, $\rho_n \cong \rho_{-n}$. Finally, for $n = 0$, ρ_0 reduces, so it must break up into two one-dimensional representations of $\mathrm{O}(2)$. The two one-dimensional representations are 1 (the trivial representation) and det. To summarize, we have

$$\widehat{\mathrm{O}(2)} = \{\rho_n : n = 1, 2, \dots\} \cup \{1, \det\}.$$

16. Complete reducibility

In this section, we prove the following result:

Theorem 16.1. *Let π be a representation of G on a Hilbert space $\mathcal{H}$. Then π is a (Hilbert space) direct sum of finite-dimensional irreducible representations.*

So far, we have only worked with finite-dimensional irreducible representations. In general, we say that a representation π of G is **irreducible** if it has no *closed* G-invariant subspaces aside from 0 and itself. A direct consequence of the above theorem is

Corollary 16.2. *Every irreducible representation of G is finite-dimensional.*

To prove Theorem 16.1, we expound Remark 12.1: for any representation π of G on a Hilbert space $\mathcal{H}$, and any function $f \in L^2(G)$, one can define a bounded operator $\pi(f)$. We would like to define this map by setting

$$\pi(f)v = \int_G f(x)\pi(x)v \; dx.$$

For this definition to make sense, we would need to develop a machinery of $\mathcal{H}$-valued integrals. To bypass this technical issue, we proceed as follows: for any $w \in \mathcal{H}$, the map

$$A_w : v \mapsto \int_G f(x)\langle \pi(x)v, w\rangle \; dx$$

is a bounded linear functional on $\mathcal{H}$. Moreover, the map $w \mapsto A_w$ from $\mathcal{H}$ to $\mathcal{H}^*$ is *antilinear*. Therefore, there exists a bounded operator on $\mathcal{H}$, denoted $\pi(f)$, such that

$$\langle \pi(f)v, w\rangle = \int_G f(x)\langle \pi(x)v, w\rangle \; dx.$$

Note that

$$\begin{aligned} |\langle \pi(f)v, w\rangle| \le \int_G |f(x)| \cdot |\langle \pi(x)v, w\rangle| \; dx &\le \int_G |f(x)| \cdot ||\pi(x)v|| \cdot ||w|| \\ &= \int_G |f(x)| \cdot ||v|| \cdot ||w|| = ||f||_1 \cdot ||v|| \cdot ||w||. \end{aligned}$$

This implies $||\pi(f)v|| \le ||f||_1||v||$. We will need the following observation:

Problem 38. Let π be a representation of G on a Hilbert space $\mathcal{H}$. Let U be a neighborhood of $e \in G$, and let $\mathbb{1}_U$ denote the characteristic function of U. Prove that the net

$$\frac{1}{\mu(U)} \cdot \pi(\mathbb{1}_U)$$

converges to the identity operator

a) in the weak operator topology;

b) in the strong operator topology.

Proof of Theorem 16.1. Let U be a maximal orthogonal sum of all finite-dimensional irreducible subrepresentations of π; the existence of such a U is guaranteed by Zorn's Lemma. Let $V = U^\perp$; note that V is G-invariant. For the sake of contradiction, assume $V \neq 0$ and fix an element $v \neq 0$ of V.

Now suppose S is a finite-dimensional subspace of $L^2(G)$ that is stable with respect to left translation. Let $f_1, \ldots, f_n$ be a basis for S. We claim that $W = \operatorname{span}\{\pi(f_1)v, \ldots, \pi(f_n)v\}$ is a G-invariant subspace of V. To prove this, we need to show that $\pi(x)\pi(f)v$ is an element of W for any $f \in S$ and $x \in G$.

Equivalently, one may show that $\langle \pi(x)\pi(f)v, w\rangle = 0$ for any $w \in W^\perp$. We compute

$$\begin{aligned}\langle \pi(x)\pi(f)v, w\rangle = \langle \pi(f)v, \pi(x^{-1})w\rangle &= \int f(y)\langle \pi(y)v, \pi(x^{-1})w\rangle \; dy \\ &= \int f(y)\langle \pi(x)\pi(y)v, w\rangle \; dy = \int f(x^{-1}y)\langle \pi(y)v, w\rangle \; dy \\ &= \int [L(x)f](y)\langle \pi(y)v, w\rangle \; dy.\end{aligned}$$

But $L(x)f$ is a linear combination of f_i's: $L(x)f = \sum \alpha_i \cdot f_i$. Therefore the above integral becomes

$$\sum \alpha_i \int_G f_i(y)\langle \pi(y)v, w\rangle \; dy = \sum \alpha_i \langle \pi(f_i)v, w\rangle.$$

This is equal to 0 because w is an element of $W^\perp$.

To arrive at a contradiction, it suffices to find a matrix coefficient f such that $\pi(f)v \neq 0$. Then the space S of left f-translates is finite-dimensional, and the above construction produces a subspace $W \subset V$ that is finite-dimensional and G-invariant, contradicting the maximality of U. We find f as follows.

By Problem 38, there exists an open neighborhood U of e such that $\pi(\mathbb{1}_U)v \neq 0$. By Peter–Weyl, $M(G)$ is dense in $L^2(G)$, so we can find a matrix coefficient f such that

$$||\mathbb{1}_U - f||_2 \leq \frac{1}{2} \cdot \frac{||\pi(\mathbb{1}_U)v||}{||v||}.$$

Then

$$||\pi(\mathbb{1}_U)v - \pi(f)v|| = ||\pi(\mathbb{1}_U - f)v|| \leq ||\mathbb{1}_U - f||_1||v|| \leq ||\mathbb{1}_U - f||_2||v|| \leq \frac{1}{2}||\pi(\mathbb{1}_U)v||.$$

This shows

$$||\pi(f)v|| \geq ||\pi(\mathbb{1}_U)v|| - ||\pi(f)v + \pi(\mathbb{1}_U)v|| \geq ||\pi(\mathbb{1}_U)v|| - \frac{1}{2}||\pi(\mathbb{1}_U)v|| = \frac{1}{2}||\pi(\mathbb{1}_U)v|| > 0.$$

Thus $\pi(f)v \neq 0$, which brings us to a contradiction and concludes our proof. $\square$

CHAPTER III: COMPACT LIE GROUPS

17. Lie groups and Lie algebras

17.1. **Lie groups.** A **Lie group** is a smooth manifold with a group structure; the group operation and the inverse map are required to be smooth.

Example 17.1. Examples of Lie groups include:

1. $\mathrm{GL}_n(\mathbb{R})$ and $\mathrm{GL}_n(\mathbb{C})$.
2. $\mathrm{SL}_n(\mathbb{R})$ and $\mathrm{SL}_n(\mathbb{C})$.
3. $\mathrm{O}(n) = \{g \in \mathrm{GL}_n(\mathbb{R}) : g^t g = I\}$ and $\mathrm{SO}(n) = \{g \in \mathrm{O}(n) : \det(g) = 1\}$.
4. $\mathrm{Sp}(2n) = \{g \in \mathrm{GL}_{2n}(\mathbb{R}) : g^t J g = J\}$. Here J is a non-singular skew-symmetric matrix, e.g.
$$J = \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix}.$$
5. $\mathrm{U}(n) = \{g \in \mathrm{GL}_n(\mathbb{C}) : g^* g = I\}$, $\mathrm{SU}(n) = \{g \in \mathrm{U}(n) : \det g = 1\}$.

In this chapter, we will be interested in *compact* Lie groups, that is, Lie groups whose underlying manifold is compact. One can show (as a consequence of the Peter–Weyl theorem) that any compact Lie group is necessarily a linear group: it is isomorphic to a closed subgroup of $\mathrm{GL}_n(\mathbb{C})$, for some positive integer n (see Appendix B).

17.2. **Lie algebras.** The additional manifold structure unlocks new ways of studying the representation theory of our groups. Crucial here is the concept of a **Lie algebra**. A (real or complex) Lie algebra is a vector space $\mathfrak{g}$ together with a binary operation
$$[\cdot,\cdot] : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$$
which is

- bilinear;
- anticommutative: $[x, y] = -[y, x]$ for all $x, y \in \mathfrak{g}$; and
- satisfies the **Jacobi identity**:
$$[x, [y, z]] + [y, [z, x]] + [z, [x, y]] = 0$$
for all $x, y, z \in \mathfrak{g}$.

All of our Lie algebras will be finite-dimensional.

Example 17.2. Let A be an associative algebra. Setting $[x, y] = xy - yx$ endows A with the structure of a Lie algebra.

A **homomorphism** of Lie algebras is a linear map compatible with the Lie brackets. A subspace $\mathfrak{h}$ of $\mathfrak{g}$ is called a **Lie subalgebra** if it is closed under the Lie bracket. An **ideal** is a subspace $\mathfrak{i}$ of $\mathfrak{g}$ that satisfies a stronger requirement: $[\mathfrak{g}, \mathfrak{i}] \subseteq \mathfrak{i}$.

A **representation** of a Lie algebra $\mathfrak{g}$ on a vector space V is a linear map $\phi : \mathfrak{g} \to \mathrm{End}(V)$ which satisfies

$$\phi([x, y]) = \phi(x)\phi(y) - \phi(y)\phi(x), \quad \text{for all } x, y \in \mathfrak{g}.$$

If we let $\mathfrak{gl}(V)$ denote the Lie algebra of all linear maps on V, then a representation is simply a homomorphism of Lie algebras $\mathfrak{g} \to \mathfrak{gl}(V)$. Every Lie algebra comes equipped with the **adjoint representation**. This is a representation ad of $\mathfrak{g}$ on the space $\mathfrak{g}$ itself given by

$$\mathrm{ad}(x)y = [x, y].$$

For any $x, y \in \mathfrak{g}$ we may define

$$B_{\mathfrak{g}}(x, y) = \mathrm{tr}(\mathrm{ad}(x)\mathrm{ad}(y)).$$

Then $B_{\mathfrak{g}}$ is a symmetric bilinear form on $\mathfrak{g}$, called the **Killing form**.

A Lie algebra is said to be **abelian** if its Lie bracket is identically equal to zero. For any Lie algebra $\mathfrak{g}$, we define its **lower central series** as

$$\mathfrak{g}_0 = \mathfrak{g}, \quad \mathfrak{g}_n = [\mathfrak{g}, \mathfrak{g}_{n-1}] \text{ for } n > 0.$$

We say that $\mathfrak{g}$ is **nilpotent** if its lower central series terminates in 0. Similarly, one may define the derived series of $\mathfrak{g}$ by setting

$$\mathfrak{g}^{(0)} = \mathfrak{g}, \quad \mathfrak{g}^{(n)} = [\mathfrak{g}^{(n-1)}, \mathfrak{g}^{(n-1)}] \text{ for } n > 0.$$

If the derived series terminates in 0, we say that $\mathfrak{g}$ is **solvable**. Clearly, every nilpotent Lie algebra is solvable.

Example 17.3. Let $\mathfrak{b}_n$ denote the Lie algebra of all upper-triangular $n \times n$ matrices. Let $\mathfrak{u}_n$ denote the Lie algebra of all *strictly* upper-triangular $n \times n$ matrices.

(i) $\mathfrak{u}_n$ is nilpotent, but not abelian for $n > 2$.

(ii) $\mathfrak{b}_n$ is solvable, but not nilpotent for $n > 1$.

We say that a Lie algebra $\mathfrak{g}$ is **simple** if it is not abelian and its only ideals are 0 and $\mathfrak{g}$ itself. More generally, $\mathfrak{g}$ is called **semisimple** if it is a direct sum of simple subalgebras. There are many other characterizations of semisimplicity; see Appendix B. Finally, we say that $\mathfrak{g}$ is **reductive** if the adjoint representation is completely reducible. It follows from the above theorem that every semisimple Lie algebra is reductive. In fact, one may show that $\mathfrak{g}$ is reductive if and only if it is a direct sum of an abelian Lie algebra and a semisimple Lie algebra.

17.3. **The Lie algebra of a Lie group.** Let M be a manifold. A **vector field** on M is a derivation of $C^\infty(M)$, that is, a linear map $X : C^\infty(M) \to C^\infty(M)$ which satisfies

$$X(fg) = X(f)g + fX(g) \quad \text{for } f, g \in C^\infty(M).$$

One checks that the commutator of two derivations is itself a derivation. Thus, the vector fields on M form a Lie algebra.

Now let G be a Lie group. We say that a vector field X on G is left-invariant if $X(L(g)f) = X(f)$ for every $g \in G$. The **Lie algebra of** G is the Lie algebra $\mathfrak{g}$ of all left-invariant vector fields on G. Because of left invariance, we may identify $\mathfrak{g}$ with the tangent space of G at the identity, T_eG.

One can go in the opposite direction ($\mathfrak{g} \to G$) by means of the **exponential map**. If $X \in \mathfrak{g}$ is a tangent vector of G at the identity, then there exists a unique **one-parameter subgroup**, that is, a Lie group homomorphism

$$\gamma : \mathbb{R} \to G$$

such that $d\gamma(0) = X$. We set $\exp(X) = \gamma(1)$.

These concepts become much more concrete once we restrict our attention to a matrix group $G \subset \mathrm{GL}_n(\mathbb{C})$. In that case, the Lie algebra of G may be defined as the set of all matrices $X \in M_n(\mathbb{C})$ such that e^{tX} is an element of G for all $t \in \mathbb{R}$. The exponential map $\mathfrak{g} \to G$ is simply the matrix exponential.

Example 17.4. Let us determine the Lie algebras of Lie groups listed above.

1. Let $G = \mathrm{GL}_n(\mathbb{C})$. Then e^{tX} is invertible for every $t \in \mathbb{R}$ and $X \in M_n(\mathbb{C})$. Consequently, $\mathfrak{g} = M_n(\mathbb{C})$. This Lie algebra is denoted $\mathfrak{gl}_n(\mathbb{C})$.

   Now let $G = \mathrm{GL}_n(\mathbb{R})$. Note e^{tX} is an invertible matrix with real entries for every $t \in \mathbb{R}$ and $X \in M_n(\mathbb{R})$. Conversely, if $X \in M_n(\mathbb{C})$ and e^{tX} has real entries for every t, it follows that
   $$\frac{d}{dt} e^{tX}|_{t=0} = X$$
   has real entries as well. We conclude $\mathfrak{g} = M_n(\mathbb{R})$. This Lie algebra is denoted by $\mathfrak{gl}_n(\mathbb{R})$.

2. Now let $G = \mathrm{SL}_n(\mathbb{C})$. Recall that $\det(e^X) = e^{\mathrm{tr}(X)}$. This shows that the corresponding Lie algebra is
   $$\mathfrak{sl}_n(\mathbb{C}) = \{X \in M_n(\mathbb{C}) : \mathrm{tr}(X) = 0\}.$$
   Similarly, $\mathfrak{sl}_n(\mathbb{R})$ is the Lie algebra of traceless matrices with real entries.

3. Let $G = \mathrm{O}(n)$. Recall that the exponential is compatible with the transpose: $(e^X)^t = e^{X^t}$. Suppose that $X \in M_n(\mathbb{R})$ satisfies
   $$(e^{sX^t})e^{sX} = I$$
   for every $s \in \mathbb{R}$. Equivalently, $e^{s(X^t+X)} = I$. Differentiating at $s = 0$, we get $X^t + X = 0$. One checks that this is also a sufficient condition:
   $$\mathfrak{so}(n) = \{X \in M_n(\mathbb{R}) : X^t + X = 0\}.$$

Note that $\mathrm{O}(n)$ and $\mathrm{SO}(n)$ have the same Lie algebra.

4. Similarly, one obtains:
$$\mathfrak{sp}(2n) = \{X \in M_n(\mathbb{R}) : X^t J + JX = 0\}.$$

5. We have
$$\mathfrak{u}(n) = \{X \in M_n(\mathbb{C}) : X^* + X = 0\}$$
and $\mathfrak{su}(n) = \mathfrak{u}(n) \cap \mathfrak{sl}_n(\mathbb{C})$.

The main idea of this chapter is to exploit the connection between a Lie group and its Lie algebra. We state the main results which establish this connection:

Lie's third theorem. Every finite-dimensional real Lie algebra is the Lie algebra of some simply connected Lie group.

The homomorphism theorem. Let G and H be Lie groups and let $\mathfrak{g}$ and $\mathfrak{h}$ denote their respective Lie algebras. Suppose G is simply connected. For every homomorphism $\phi : \mathfrak{g} \to \mathfrak{h}$ there exists a Lie group homomorphism $f : G \to H$ such that $\phi = df$.

The homomorphsim theorem is particularly important for us: if we set $H = \mathrm{GL}_n(\mathbb{C})$, it follows that every representation of $\mathfrak{g}$ comes from a smooth representation of G, provided that G is simply connected. Furthermore, one can show that smoothness is automatic for finite-dimensional representations of G. Thus, if one is interested in finite-dimensional representations of G, one may instead study the finite-dimensional representations of $\mathfrak{g}$.

The Lie algebra $\mathfrak{g}$ of a Lie group G is, by definition, a real vector space. Because we prefer working with complex Lie algebras, we will often consider the **complexification** of $\mathfrak{g}$:
$$\mathfrak{g}_{\mathbb{C}} = \mathbb{C} \otimes \mathfrak{g}.$$
By the appropriate universal property, there is a bijective correspondence between (complex) representations of $\mathfrak{g}$ and $\mathfrak{g}_{\mathbb{C}}$. One can show that certain important properties of $\mathfrak{g}$ — such as nilpotence, solvability, and semisimplicity (but not simplicity!) — are all preserved under complexification.

We end this section with the following result:

Proposition 17.5. *The Lie algebra $\mathfrak{g}$ of a compact Lie group G is reductive. If G has finite center, then $\mathfrak{g}$ is moreover semisimple.*

Because we will be able to handle the center in a reasonable way, from now on we restrict our attention to *complex semismple Lie algebras.*

18. The Jordan–Chevalley decomposition

All results in this section assume we are working in a field of characteristic 0.

A crucial result about the structure theory of semsimple Lie algebra is the **Jordan–Chevalley** decomposition, which generalizes the following linear algebra fact:

Jordan decomposition. Let V be a finite-dimensional vector space. For every linear operator A on V, there exist unique operators A_s and A_n such that

- A_s is semisimple;
- A_n is nilpotent;
- $A_sA_n = A_nA_s$;
- $A = A_s + A_n$.

Moreover, A_s (and hence also A_n) can be written as a polynomial in A.

Now let $\mathfrak{g}$ be a semisimple Lie algebra. We say that $x \in \mathfrak{g}$ is nilpotent (resp. semisimple) if the operator $\operatorname{ad}(x)$ is nilpotent (resp. semisimple).

Jordan–Chevalley decomposition. Let $\mathfrak{g}$ be a finite-dimensional semisimple Lie algebra. For every $x \in \mathfrak{g}$, there exist unique elements $x_{(s)}$ and $x_{(n)}$ of $\mathfrak{g}$ such that

- $x_{(s)}$ is semisimple;
- $x_{(n)}$ is nilpotent;
- $[x_{(s)}, x_{(n)}] = 0$;
- $x = x_{(s)} + x_{(n)}$.

One of the most important facts about semisimple Lie algebras is

Weyl's theorem on complete reducibility. Every finite-dimensional representation of a semisimple Lie algebra is completely reducible.

Using Weyl's theorem on complete reducibility, one may prove the following fact:

Proposition. Let $\mathfrak{g}$ be a semisimple Lie algebra and let V be a finite-dimensional vector space. If $\pi : \mathfrak{g} \to \mathfrak{gl}(V)$ is a representation of V, then

$$\pi(x_{(s)}) = \pi(x)_s \quad \text{and} \quad \pi(x_{(n)}) = \pi(x)_n.$$

Recall that we will mostly be interested in semisimple Lie algebras attached to matrix groups. In that case, $\mathfrak{g}$ is a subalgebra of $\mathfrak{gl}(V)$ for some V. In particular, taking π to be the inclusion, we see that the Jordan–Chevalley decomposition of $x \in \mathfrak{g}$ coincides with the Jordan decomposition of the operator x on V.

19. The three-dimensional simple Lie algebra

In this section, we study the Lie algebra $\mathfrak{sl}_2(\mathbb{C})$. Recall that

$$\mathfrak{sl}_2(\mathbb{C}) = \left\{ \begin{bmatrix} a & b \\ c & -a \end{bmatrix} : a, b, c \in \mathbb{C} \right\}.$$

This is a three-dimensional Lie algebra. We consider the following basis:

$$e = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \quad h = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \quad f = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}.$$

It is easy to check the following:

$$[h, e] = 2e; \quad [h, f] = -2f; \quad [e, f] = h.$$

Problem 39. Using the above commutation relations, show that $\mathfrak{sl}_2(\mathbb{C})$ is simple.

Problem 40. Show that every complex three-dimensional simple Lie algebra has a basis $\{e, h, f\}$ which satisfies the above relations. In particular, $\mathfrak{sl}_2(\mathbb{C})$ is the unique three-dimensional simple Lie algebra.

Now let π be a representation of $\mathfrak{g} = \mathfrak{sl}_2(\mathbb{C})$ on a finite-dimensional vector space V. Set

$$E = \pi(e); \quad H = \pi(h); \quad F = \pi(f).$$

Then H is semisimple, whereas E and F are nilpotent. The eigenvalues of H are called the **weights** of π. Given an eigenvalue λ of H, the corresponding eigenspace is called the **weight space** of λ. Since H is semisimple, V decomposes into a direct sum of eigenspaces

$$V_\lambda = \{v \in V : Hv = \lambda v\}.$$

Using the above commutation relations, one checks that

$$EV_\lambda = V_{\lambda+2} \quad \text{and} \quad FV_\lambda = V_{\lambda-2}.$$

Now let $v \neq 0$ be a vector such that $Ev = 0$. Such a vector exists because E is nilpotent. Moreover, we may assume that v is an eigenvector of H; let λ be the corresponding eigenvalue.

Lemma 19.1. *Let $v \neq 0$ be a vector as above: $Ev = 0$ and $Hv = \lambda v$. Set*

$$v_{-1} = 0, \quad v_0 = v, \quad v_n = \frac{F^n}{n!} v \quad \textit{for } n > 0.$$

Then

$$Hv_n = (\lambda - 2n)v_n, \quad Ev_n = (\lambda - n + 1)v_{n-1}, \quad Fv_n = (n+1)v_{n+1} \quad \textit{for } n > 0.$$

Proof. The third equality is immediate from the definition of v_n; the first one follows from $Y^n v_\lambda \subseteq V_{\lambda-2n}$. We prove the second equality by induction. The base case is easy:

$$Ev_1 = Ev = EFv = FEv + Hv = Hv = \lambda v.$$

For the induction step, assume the equality holds for $n > 0$. Then

$$\begin{aligned} Ev_{n+1} &= \frac{1}{n+1} EFv_n = \frac{1}{n+1}(FEv_n + Hv_n) = \frac{1}{n+1}\left(F(\lambda - n + 1)v_{n-1} + (\lambda - 2n)v_n\right) \\ &= \frac{1}{n+1}\left((\lambda - n + 1)nv_n + (\lambda - 2n)v_n\right) = (\lambda - n)v_n, \end{aligned}$$

which we needed to prove. $\square$

Note that v_n are eigenvectors corresponding to different eigenvalues of H. As such, they are independent (if non-zero). Since V is finite-dimensional, we conclude that there must exist a non-negative m such that $v_m \neq 0$ and $v_{m+1} = 0$. The above lemma then shows that the span of

$$\{v_0, \dots, v_m\}$$

is π-invariant. If π is irreducible, this means that we have found a basis for V. Thus, V is $(m+1)$-dimensional. Finally, we have $0 = Ev_{m+1} = (\lambda - m)v_m$, so we get $\lambda = m$.

Conversely, suppose that m is a non-negative integer and let V be a complex $(m+1)$-dimensional space with basis $\{v_0, \dots, v_m\}$. Let us define operators E, H, F by their action on the basis:

$$\begin{gathered} Hv_n = (m-2n)v_n, \quad n = 0, \dots, m; \\ Ev_0 = 0; \quad Ev_n = (m-n+1)v_{n-1}, \quad n = 1, \dots, m; \\ Fv_m = 0; \quad Fv_n = (n+1)v_{n+1}, \quad n = 0, \dots, m-1. \end{gathered}$$

Then one checks (by a direct computation) that

$$[H, E]v_n = 2Ev_n, \quad [H, F]v_n = -2Fv_n, \quad [R, F]v_n = Hv_n$$

for all $n = 0, \dots, m$. This shows that

$$\pi(\alpha e + \beta h + \gamma f) = \alpha E + \beta H + \gamma F$$

is a representation of $\mathfrak{g}$ on V. One checks that π is irreducible. Indeed, suppose that W is a non-zero $\{E, H, F\}$-invariant subspace of V. Then it contains an eigenvector for H. Thus W must contain a vector v_n for some $n \in \{0, \dots, m\}$. Then, by E- and F-invariance (and the definitions of E, F), W must contain all other v_n's as well. Therefore $W = V$.

Let us summarize our discussion:

Theorem 19.2. *Let $\mathfrak{g} = \mathfrak{sl}_2(\mathbb{C})$.*

(i) For every non-negative integer m there exists a unique (up to equivalence) representation of $\mathfrak{g}$ of dimension m. We denote this representation by π_m.

(ii) The weights of π_m are $m, m-2, \dots, 2-m, -m$; each weight space is one-dimensional.

(iii) Up to scaling, there exists a unique vector v in π_m annihilated by e. This is the weight vector for m, and is therefore called the ***highest weight vector****.*

20. Roots

The decomposition of a representation into weight spaces and the commutation relations between e, h, and f were crucial in the last section. We would now like to generalize these observations to other semisimple Lie algebras.

Let $\mathfrak{g}$ be a Lie algebra. A **Cartan subalgebra** (**CSA**) of $\mathfrak{g}$ is a nilpotent subalgebra $\mathfrak{h}$ of $\mathfrak{g}$ which is equal to its own normalizer:

$$\mathfrak{h} = N_{\mathfrak{g}}(\mathfrak{h}) := \{x \in \mathfrak{g} : [x, h] \in \mathfrak{h} \text{ for all } h \in \mathfrak{h}\}.$$

In a finite-dimensional complex Lie algebra $\mathfrak{g}$, a subalgebra $\mathfrak{h}$ is Cartan if and only if the following conditions are fulfilled:

(i) $\mathfrak{h}$ is a maximal abelian subalgebra;

(ii) every element of $\mathfrak{h}$ is semisimple.

Cartan subalgebras exist in any complex finite-dimensional Lie algebra. In fact, all Cartan subalgebras of a complex finite-dimensional Lie algebra $\mathfrak{g}$ are conjugate (and therefore isomorphic) under automorphisms of $\mathfrak{g}$. The dimension of a CSA is called the **rank** of $\mathfrak{g}$.

CSA's will play the role of h from the previous section. Because all elements of a CSA are semisimple, their action on a finite-dimensional complex vector space can be diagonalized. Moreover, since a CSA is abelian, we can simultaneously diagonalize these actions. To be precise, let (π, V) be a finite-dimensional representation of a complex finite-dimensional semisimple Lie algebra $\mathfrak{g}$. Let $\mathfrak{h}$ be a CSA of $\mathfrak{g}$. For each $\alpha \in \mathfrak{h}^*$, set

$$V_\alpha = \{v \in V : \pi(h)v = \alpha(h)v \text{ for all } h \in \mathfrak{h}\}.$$

Then we have a direct sum decomposition

$$V = \bigoplus V_\alpha.$$

An element $\alpha \in V^*$ is called a **weight** of the representation (π, V) if the **weight space** V_α is non-zero. In particular, one can apply this to the adjoint representation of $\mathfrak{h}$ on $\mathfrak{g}$. One gets

$$\mathfrak{g} = \mathfrak{h} \oplus \bigoplus_{\alpha \neq 0} \mathfrak{g}_\alpha$$

where $\mathfrak{g}_\alpha = \{x \in \mathfrak{g} : [h, x] = \alpha(h)x \text{ for all } h \in \mathfrak{h}\}$. In this case, an element $\alpha \neq 0 \in \mathfrak{h}^*$ is called a **root** if the **root space** $\mathfrak{g}_\alpha$ is non-zero. The set of all roots is called the **root system** of $\mathfrak{g}$ with respect to $\mathfrak{h}$ and is denoted by $R(\mathfrak{g}, \mathfrak{h})$. The following result summarizes the main properties of root spaces of a complex finite-dimensional semisimple Lie algebra $\mathfrak{g}$.

Theorem. (Properties of root spaces) Let $\mathfrak{h}$ be a CSA of $\mathfrak{g}$; set $R = R(\mathfrak{g}, \mathfrak{h})$. Let (π, V) be a finite-dimensional representation of $\mathfrak{g}$.

(i) $[\mathfrak{g}_\alpha, \mathfrak{g}_\beta] \subseteq \mathfrak{g}_{\alpha+\beta}$. Moreover, if $\alpha, \beta \in R$ and $\alpha + \beta \in R$, then $[\mathfrak{g}_\alpha, \mathfrak{g}_\beta] = \mathfrak{g}_{\alpha+\beta}$.

(ii) $\dim \mathfrak{g}_\alpha = 1$ for every $\alpha \in R$.

(iii) For every $\alpha \in R$, $\dim[\mathfrak{g}_\alpha, \mathfrak{g}_{-\alpha}] = 1$. In particular, there exists a unique element $h_\alpha \in [\mathfrak{g}_\alpha, \mathfrak{g}_{-\alpha}]$ such that $\alpha(h_\alpha) = 2$.

(iv) For every $\alpha \in R$ and $e_\alpha \in \mathfrak{g}_\alpha$, there exists an element $f_\alpha \in \mathfrak{g}_{-\alpha}$ such that $[e_\alpha, f_\alpha] = h_\alpha$. Then $\{e_\alpha, h_\alpha, f_\alpha\}$ is an **$\mathfrak{sl}_2$-triple**:

$$[h_\alpha, e_\alpha] = 2e_\alpha; \quad [h_\alpha, f_\alpha] = -2f_\alpha; \quad [e_\alpha, f_\alpha] = h_\alpha.$$

We will denote this copy of $\mathfrak{sl}_2$ by $\mathfrak{s}_\alpha$.

The Killing form is non-degenerate when restricted to $\mathfrak{h}$. We may thus use it to identify $\mathfrak{h}$ with its dual, $\mathfrak{h}^*$. If we use this identification to transfer the Killing form to a bilinear form on $\mathfrak{h}^*$, we get an inner product $(\cdot \mid \cdot)$ on the $\mathbb{R}$-span of R. The following result summarizes the main properties of roots:

Theorem.

(i) $\mathfrak{h}^* = \text{span } R$.

(ii) $R = -R$.

(iii) For every root α, $\mathbb{C}\alpha \cap R = \{\pm\alpha\}$.

(iv) If $(\alpha|\beta) < 0$, then $\alpha + \beta \in R \cup \{0\}$.

(v) For every $\alpha, \beta \in R$, the number $2\dfrac{(\alpha|\beta)}{(\alpha|\alpha)}$ is an integer.

20.1. **The Weyl group.** Let E be a real vector space with an inner product $(\cdot|\cdot)$. For any non-zero vector $x \in E$, we may consider the reflection of E with respect to x. This is a linear map σ_x defined as follows: every vector $v \in E$ decomposes uniquely as $v = cx + y$, where c is a scalar and $(x|y) = 0$. We define

$$\sigma_x(v) = -cx + y.$$

Note that $(v|x) = c(x|x)$, so $c = \dfrac{(v|x)}{(x|x)}$ and hence $y = v - \dfrac{(v|x)}{(x|x)}x$. We thus get a formula for the reflection with respect to x:

$$\sigma_x(v) = v - 2\frac{(v|x)}{(x|x)}x.$$

Now let E denote the $\mathbb{R}$-span of the root system R. For each root $\alpha \in R$, we may form a reflection s_α of E. Let W denote the group of isometries of E generated by $\{s_\alpha : \alpha \in R\}$. This is the **Weyl group** of R.

Proposition. For every $\alpha \in R$, $s_\alpha R = R$.

Corollary. The Weyl group is finite.

20.2. **Weyl chambers, positivity, simple roots.** We will call an element x of E regular if $(\alpha|x) \neq 0$ for every $\alpha \in R$. The set of E^{reg} of all regular elements is the (set theoretic complement) of the union of hyperplanes $\alpha^\perp$, $\alpha \in R$. The connected components of E^{reg} are called **Weyl chambers**. One shows:

Proposition. The Weyl group acts freely transitively on the set of Weyl chambers. Given a Weyl chamber C and an element $x \in E$, exactly one element of the orbit Wx is contained in the closure of C.

Next, fix $x \in E^{\text{reg}}$. Define

$$R^+ = R^+(x) = \{\alpha \in R : (\alpha|x) > 0\} \quad \text{and} \quad R^- = R^-(x) = \{\alpha \in R : (\alpha|x) < 0\}.$$

(These sets depend only on the Weyl chamber, not on the particular choice of x.) Then R is the disjoint union of R^+ and R^-; moreover, $R^- = -R^+$.

We say that a root α in R^+ is **simple** (with respect to the Weyl chamber that defines R^+) if it cannot be represented as a sum $\alpha = \beta + \gamma$ for some $\beta, \gamma \in R^+$. We let Δ denote the set of all simple roots in R^+; we say that Δ is a **base** of R. Of course, Δ depends on our initial choice of Weyl chamber.

Proposition.

(i) Δ is a basis for E.

(ii) For any $\alpha \neq \beta \in \Delta$, we have $(\alpha|\beta) \leq 0$.

(iii) Every $\beta \in R^+$ can be written as a linear combination

$$\beta = \sum_{\alpha \in \Delta} n_\alpha \alpha$$

where n_α are non-negative integers.

(iv) For every $\alpha \in R$ there exists a Weyl chamber with respect to which α is a simple root.

We end this section by stating a crucial result about the Weyl group:

Proposition. Let Δ be the set of simple roots with respect to a Weyl chamber. The set $\{s_\alpha : \alpha \in \Delta\}$ generates the Weyl group.

20.3. **Integral and dominant elements; ordering.** We continue using the notation from the previous section. We fix a base Δ of R. We say that an element $\lambda \in E$ is **dominant** (with respect to Δ) if

$$(\lambda|\alpha) \geq 0 \quad \text{for all } \alpha \in \Delta.$$

We say that λ is **strictly dominant** if the above inequality is strict for every $\alpha \in \Delta$. We note that every orbit of the Weyl group contains a unique dominant element.

We say that $\lambda \in E$ is **integral** if $2\dfrac{(\lambda|\alpha)}{(\alpha|\alpha)}$ is an integer for every $\alpha \in R$. Clearly, it suffices to require this condition for every simpler root $\alpha \in \Delta$. If $\Delta = \{\alpha_1, \ldots, \alpha_r\}$, then we can find a dual basis $\{\lambda_1, \ldots, \lambda_r\}$ with respect to the Killing form:

$$2\frac{(\lambda_i|\alpha_j)}{(\alpha_j|\alpha_j)} = \delta_{ij}, \quad \text{for all } i, j = 1, \ldots, r.$$

The λ_i's are called the **fundamental weights** of $\mathfrak{g}$. Note that $\lambda \in E$ is integral if and only if it is a $\mathbb{Z}$-linear combination of the fundamental weights. The set of all integral elements thus forms a lattice in E, called the **weight lattice**.

We now introduce a partial order on E. This order depends on the choice of Δ, which we fix. For any $\lambda, \mu \in E$, we say that $\lambda \geq \mu$ if $\lambda - \mu$ is a linear combination of elements of Δ with non-negative real coefficients. We state some basic results about this partial order:

Proposition. If λ is dominant, then $\lambda \geq 0$.

Proposition. If λ is dominant, then $w\lambda \leq \lambda$ for every $w \in W$.

Proof. Since $W\lambda$ is finite, it contains a maximal element μ with respect to our partial ordering. Then, for every $\alpha \in \Delta$, we have $(\mu|\alpha) \geq 0$. Indeed, if $(\mu|\alpha)$ were negative, then

$$s_\alpha \mu = \mu - 2\frac{(\mu|\alpha)}{(\alpha|\alpha)}\alpha$$

would be higher than μ, contradicting maximality. This proves μ is dominant. But we know that every Weyl orbit contains a unique dominant element, so $\lambda = \mu$. Thus λ is the unique maximal element of the Weyl orbit $W\lambda$; therefore λ is also the highest element. $\square$

Proposition.

(i) If λ and μ are dominant, then $\lambda \leq \mu$ if and only if λ is contained in the convex hull of $W\mu$.

(ii) If λ is dominant, then μ is contained in the convex hull of $W\lambda$ if and only if $w\mu \leq \lambda$ for every $w \in W$.

21. Examples: the classical Lie algebras

We retain the notation introduced in the previous sections: given a semisimple Lie algebra $\mathfrak{g}$, $\mathfrak{h}$ will denote its Cartan subalgebra; we also use R, R^+, Δ, E to denote the corresponding root system and related objects.

21.1. A_n**.** Let $\mathfrak{g} = \mathfrak{sl}_{n+1} = \{X \in M_{n+1}(\mathbb{C}) : \operatorname{tr}(X) = 0\}$. The CSA $\mathfrak{h}$ consists of diagonal matrices in $\mathfrak{g}$:

$$\mathfrak{h} = \{\operatorname{diag}(x_1, \ldots, x_{n+1}) : \sum x_i = 0\}.$$

Taking typical elements $h \in \mathfrak{h}$ and $X \in \mathfrak{g}$, and computing $[h, X]$ we see that the roots are given by functionals

$$\operatorname{diag}(x_1, \ldots, x_{n+1}) \mapsto x_i - x_j \quad \text{for } i \neq j.$$

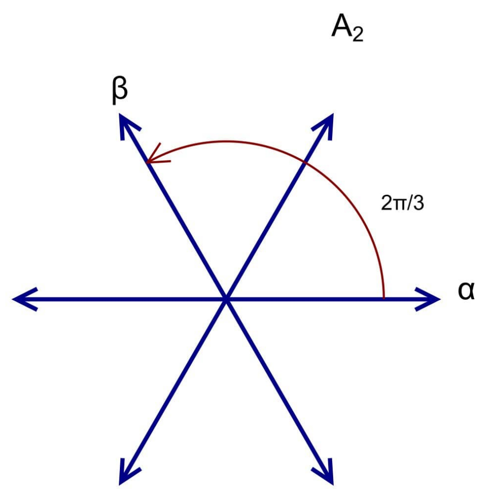
Thus E is the subspace of $\mathbb{R}^{n+1}$ consisting of vectors whose coordinates sum to 0. Using the standard basis $\{e_i : i = 1, \ldots, n+1\}$ for E, the roots can be written as $e_i - e_j$, where $i \neq j$.

A choice of positive roots is

$$R^+ = \{e_i - e_j : i < j\}.$$

The simple roots are then

$$\Delta = \{e_i - e_{i+1} : i = 1, \ldots, n\}.$$



The A_2 root system, drawn in the plane $\{(x_1, x_2, x_3 \in \mathbb{R}^3 : x_1 + x_2 + x_3 = 0\}$.
$\alpha = (1, -1, 0), \beta = (0, 1, -1)$.

21.2. B_n. Let $\mathfrak{g} = \mathfrak{so}_{2n+1}(\mathbb{C})$. Recall that we have defined $\mathfrak{so}_{2n+1} = \{X \in M_{2n+1}(\mathbb{C}) : X + X^t = 0\}$. However, since all non-degenerate quadratic forms on $\mathbb{C}^n$ are equivalent, we may choose a different way to describe $\mathfrak{so}_{2n+1}$. It will be convenient to set

$$\mathfrak{so}_{2n+1} = \{X \in M_{2n+1}(\mathbb{C}) : X^t J + JX = 0\},$$

where J is the $(2n+1) \times (2n+1)$ matrix with 1's on the anti-diagonal and zeroes elsewhere:

$$J = \begin{bmatrix} & & 1 \\ & \iddots & \\ 1 & & \end{bmatrix}.$$

With this convention, $\mathfrak{so}_{2n+1}$ consists of all matrices that are anti-symmetric with respect to the anti-diagonal. The CSA $\mathfrak{h}$ again consists of diagonal matrices in $\mathfrak{g}$:

$$\mathfrak{h} = \{\mathrm{diag}(x_1, \ldots, x_n, 0, -x_n, \ldots, -x_1)\}.$$

Taking typical elements $h \in \mathfrak{h}$ and $X \in \mathfrak{g}$, and computing $[h, X]$ we see that the roots are given by functionals

$$\pm(x_i - x_j) \text{ and } \pm(x_i + x_j) \quad \text{for } i < j, \quad \text{and } x_i \text{ for } i = 1, \ldots, n.$$

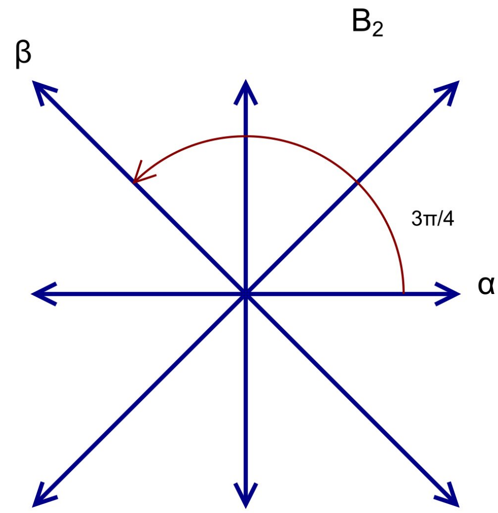
Thus E is isomorphic to $\mathbb{R}^n$; using the standard basis again the roots can be written as $\pm(e_i - e_j), \pm(e_i + e_j)$ (where $i \neq j$) and e_i.

A choice of positive roots is

$$R^+ = \{e_i - e_j, e_i + e_j : i < j\} \cup e_i : i = 1, \ldots, n.$$

The simple roots are then

$$\Delta = \{e_i - e_{i+1} : i = 1, \ldots, n-1\} \cup \{e_n\}.$$



The B_2 root system. $\alpha = (1, 0), \beta = (-1, 1)$.

21.3. C_n**.** Let $\mathfrak{g} = \mathfrak{sp}_{2n}(\mathbb{C}) = \{X \in M_{2n}(\mathbb{C}) : X^t J + JX = 0\}$ where J is the $(2n) \times (2n)$ anti-diagonal matrix with n -1's followed by n 1's:

$$J = \begin{bmatrix} & & -1 \\ & \iddots & \\ 1 & & \end{bmatrix}.$$

With this convention, $\mathfrak{sp}_{2n}$ consists of block 2×2 matrices that look like

$$\begin{bmatrix} A & B \\ C & -A^\tau \end{bmatrix}.$$

Here τ denotes the trasnpose with respect to the anti-diagonal, and B, C are symmetric. The CSA $\mathfrak{h}$ again consists of diagonal matrices in $\mathfrak{g}$:

$$\mathfrak{h} = \{\mathrm{diag}(x_1, \ldots, x_n, -x_n, \ldots, -x_1)\}.$$

Computing $[h, X]$ for typical elements $h \in \mathfrak{h}$ and $X \in \mathfrak{g}$, we see that the roots are given by functionals

$$\pm(x_i - x_j) \text{ and } \pm(x_i + x_j) \quad \text{for } i < j, \quad \text{and } 2x_i \text{ for } i = 1, \ldots, n.$$

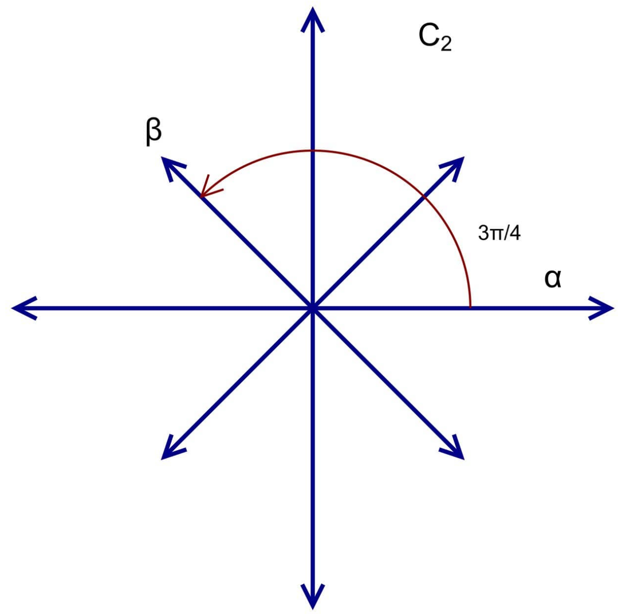
Thus E is isomorphic to $\mathbb{R}^n$; using the standard basis again, the roots can be written as $\pm(e_i - e_j), \pm(e_i + e_j)$ (where $i \neq j$) and $2e_i$.

A choice of positive roots is

$$R^+ = \{e_i - e_j, e_i + e_j : i < j\} \cup \{2e_i : i = 1, \ldots, n\}.$$

The simple roots are then

$$\Delta = \{e_i - e_{i+1} : i = 1, \ldots, n-1\} \cup \{2e_n\}.$$



The C_2 root system. $\alpha = (2,0), \beta = (-1,1)$.

Remark 21.1. Note that the root systems B_2 and C_2 are isomorphic. This suggests a connection between $\mathfrak{sp}_4$ and $\mathfrak{so}_5$. Indeed, one may construct a 2-to-1 Lie homomorphism $\mathrm{Sp}_4 \to \mathrm{SO}_5$ which differentiates to an isomorphism of the corresponding Lie algebras.

21.4. D_n**.** Let $\mathfrak{g} = \mathfrak{so}_{2n}(\mathbb{C}) = \{X \in M_{2n}(\mathbb{C}) : X^t J + JX = 0\}$ where J is the $(2n) \times (2n)$ anti-diagonal matrix

$$J = \begin{bmatrix} & & 1 \\ & \iddots & \\ 1 & & \end{bmatrix}.$$

With this convention, $\mathfrak{so}_{2n}$ consists of matrices that are anti-symmetric with respect to the anti-diagonal. Again, the CSA is

$$\mathfrak{h} = \{\mathrm{diag}(x_1, \dots, x_n, -x_n, \dots, -x_1)\}.$$

Computing $[h, X]$ for typical elements $h \in \mathfrak{h}$ and $X \in \mathfrak{g}$, we see that the roots are given by functionals

$$\pm(x_i - x_j) \text{ and } \pm(x_i + x_j) \quad \text{for } i < j.$$

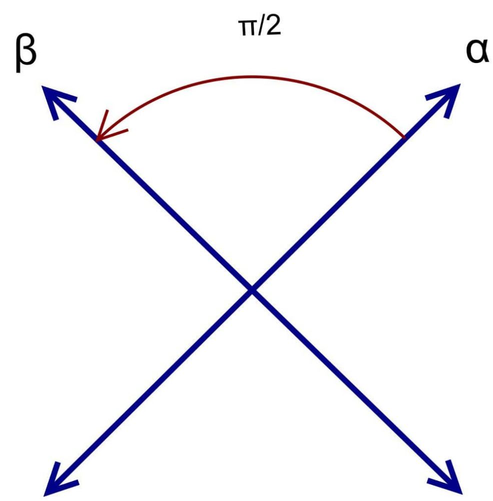
Thus E is isomorphic to $\mathbb{R}^n$; using the standard basis again the roots can be written as $\pm(e_i - e_j), \pm(e_i + e_j)$ (where $i \neq j$).

A choice of positive roots is

$$R^+ = \{e_i - e_j, e_i + e_j : i < j\}.$$

The simple roots are then

$$\Delta = \{e_i - e_{i+1} : i = 1, \dots, n-1\} \cup \{e_{n-1} + e_n\}.$$



The D_2 root system. $\alpha = (1,1), \beta = (-1,1)$.

Remark 21.2. Note that the root system D_2 is isomorphic to $A_1 \times A_1$ are isomorphic. Once again, this suggests a connection between $\mathfrak{so}_4$ and $\mathfrak{sl}_2 \times \mathfrak{sl}_2$. As before, one may construct a 2-to-1 Lie homomorphism $\mathrm{SL}_2 \times \mathrm{SL}_2 \to \mathrm{SO}_4$ which differentiates to an isomorphism of the corresponding Lie algebras. Another exceptional isomorphism is $D_3 = A_3$, which shows that $\mathfrak{sl}_4 \cong \mathfrak{so}_6$.

We say that $R' \subseteq R$ is a sub-root system of R if $R' = E' \cap R$ where E' denotes the span of R'. We say that R is **irreducible** if it cannot be decomposed into a disjoint union of sub-root systems $R = R' \cup R''$ with $R' \perp R''$.

Example 21.3. The root system A_n is irreducible for every n. On the other hand $D_2 = A_1 \times A_1$ is not irreducible.

22. Theorem of the highest weight

In this section, we state — and prove one direction of — the Theorem of the highest weight. Throughout this section $\mathfrak{g}$ is a complex semisimple Lie algebra and $\mathfrak{h}$ is a CSA. This gives us a root system of $\mathfrak{g}$ with respect to $\mathfrak{h}$; we fix (once and for all) a choice of positive roots. We will freely identify $\mathfrak{h}$ with its dual by means of the Killing form.

Let (π, V) be a representation of $\mathfrak{g}$. Recall that $\lambda \in \mathfrak{h}$ is called a weight of V if there exists a non-zero vector $v \in V$ such that

$$\pi(h)v = (\lambda|h)v \quad \text{for all } h \in \mathfrak{h}.$$

The weight space V_λ is the space of all vectors in V which satisfy the above equation. We call $\dim V_\lambda$ the multiplicity of λ in V.

Before we begin our discussion of weights, let us recall the results of Section 20: for any positive root α, the space $[\mathfrak{g}_\alpha, \mathfrak{g}_{-\alpha}] \subset \mathfrak{h}$ is one-dimensional. In particular, it contains a unique element h_α such that $(\alpha|h_\alpha) = 2$. Clearly, this element is given by

$$h_\alpha = 2\frac{\alpha}{(\alpha|\alpha)}$$

(we call it the coroot attached to α). Furthermore, for any $e_\alpha \in \mathfrak{g}_\alpha$ there exists an $f_\alpha \in \mathfrak{g}_{-\alpha}$ such that $\{e_\alpha, h_\alpha, f_\alpha\}$ is an $\mathfrak{sl}_2$-triple. We shall denote the corresponding copy of $\mathfrak{sl}_2$ by $\mathfrak{sl}_{2,\alpha}$. Given a representation π, we will often write E_α, H_α, and F_α for $\pi(e_\alpha), \pi(h_\alpha)$, and $\pi(f_\alpha)$, respectively.

We begin with a few simple observations:

Lemma 22.1. *Let λ be a weight of V and let α be a root of $\mathfrak{g}$. Then $\pi(\mathfrak{g}_\alpha)V_\lambda \subseteq V_{\lambda+\alpha}$.*

Proof. Let $x \in \mathfrak{g}_\alpha$. By definition, this means $[h, x] = (\alpha|h)x$. For $v \in V_\lambda$, we have

$$\pi(h)\pi(x)v = \pi(x)\pi(h)v + \pi([h, x])v = (\lambda|h)\pi(x)v + (\alpha|h)\pi(x)v = (\lambda + \alpha|h)\pi(x)v.$$

This proves the Lemma. ☐

Lemma 22.2. *Let λ be a weight of V. Then λ is an integral element.*

Proof. Let α be a root and consider the action of $\mathfrak{sl}_{2,\alpha}$ on V. Then λ is also a weight for this representation. Since all weights of $\mathfrak{sl}_2$ representations are integers, we conclude that $(\lambda|h_\alpha)$ is an integer. But this means that

$$2\frac{(\lambda|\alpha)}{(\alpha|\alpha)}$$

is an integer. Since this holds for every root α, it follows that λ is integral. ☐

Proposition 22.3. *The (multi)set of weights of a finite-dimensional representation V is invariant under the action of the Weyl group.*

Proof. Let α be a root. Consider the operator

$$S_\alpha = e^{E_\alpha} e^{-F_\alpha} e^{E_\alpha}$$

on $\mathfrak{g}$. Let $h \in \mathfrak{h}$. Suppose $(h|\alpha) = 0$. This means that h commutes with e_α and f_α; it follows that $\pi(h)$ commutes with S_α. On the other hand, one shows that

$$S_\alpha \pi(h_\alpha) S_\alpha^{-1} = -\pi(h_\alpha).$$

Therefore,

$$S_\alpha \pi(h) S_\alpha^{-1} = \pi(s_\alpha h)$$

for all $h \in H$.

Now let λ be a weight of V and let v be a weight vector. Then

$$\pi(h) S_\alpha^{-1} v = S_\alpha^{-1} \pi(s_\alpha h) v = (\lambda|s_\alpha h) S_\alpha^{-1} v = (s_\alpha^{-1}\lambda|h) S_\alpha^{-1} v$$

Therefore $S_\alpha^{-1} v$ is a weight vector for $s_\alpha^{-1}\lambda$. We have thus constructed an isomorphism between the weight spaces for λ and $s_\alpha^{-1}\lambda$. ☐

We are now ready to state the main result.

Theorem 22.4 (Theorem of the highest weight)**.** *(i) Every irreducible finite-dimensional representation of $\mathfrak{g}$ has a highest weight.*

(ii) Two irreducible finite-dimensional representations of $\mathfrak{g}$ are isomorphic if they have the same highest weight.

(iii) The highest weight is a dominant integral element.

(iv) Let μ be a dominant integral element. Then there exists an irreducible finite-dimensional representation of $\mathfrak{g}$ with highest weight μ.

We say that a representation (π, V) of $\mathfrak{g}$ is **highest weight cyclic** with weight μ if there exists a non-zero vector $v \in V$ such that

- v is a weight vector for μ;
- $E_\alpha v = 0$ for every positive root α;
- v generates all of V.

To prove the first three points of the above theorem, we will need the following result:

Proposition 22.5. *Let (π, V) be highest weight cyclic with weight μ. Then*

- *μ is indeed the highest weight of V;*
- *the corresponding weight space is one-dimensional;*
- *if V is finite-dimensional, then it is irreducible.*

This proposition, in turn, relies on the following observation. Let $\alpha_1, \ldots, \alpha_k$ be the positive roots of $\mathfrak{g}$. For each α_i, consider the corresponding $\mathfrak{sl}_2$-triple $\{e_{\alpha_i}, h_{\alpha_i}, f_{\alpha_i}\}$. Then the root space decomposition guarantees that $\mathfrak{g}$ has a basis of the form

$$B = \{e_{\alpha_1}, \ldots, e_{\alpha_k}, h_1, \ldots, h_r, f_{\alpha_1}, \ldots, f_{\alpha_k}\}$$

where $h_1, \ldots, h_r \in \mathfrak{h}$ (here r is the rank of $\mathfrak{g}$). We then have

Lemma 22.6. *Let (π, V) be a representation of $\mathfrak{g}$. Any expression $\pi(x_1) \cdot \cdots \cdot \pi(x_N)$, with $x_i \in B$ can be rearranged — using the commutation relations — to a linear combination of similar expressions, where e's act first, h's act second, and f's act last.*

Proof. Let α and β be positive roots. Since $[\mathfrak{g}_\alpha, \mathfrak{g}_\beta] \subset \mathfrak{g}_{\alpha+\beta}$, we have $\{[e_\alpha, f_\alpha] = cx\}$, for some scalar c and $x = e_{\alpha+\beta}$ or $f_{\alpha+\beta}$, depending on whether $\alpha + \beta$ is positive or negative.

Thus $\pi(e_\alpha)\pi(f_\beta) = \pi(f_\beta)\pi(e_\alpha) + c\pi(x)$. This means that we can replace $\pi(e_\alpha)\pi(f_\beta)$ by $\pi(f_\beta)\pi(e_\alpha)$ in our expression, at the price of introducing a shorter expression (involving $c\pi(x)$ in place of $\pi(e_\alpha)\pi(f_\beta)$). For expressions of the form $\pi(h)\pi(f_\beta)$ and $\pi(e_\alpha)\pi(h)$ (with $h \in \mathfrak{h}$), one can proceed similarly. This allows us to prove the lemma by inducing on the total length of the expression; the details are left to the reader. $\square$

Proof of Proposition 22.5. Let v be a vector as in the definition of a highest weight cyclic representation. Consider the subspace W of V spanned by all elements of the form

$$F_{\alpha_{i_1}} \cdot \dots \cdot F_{\alpha_{i_k}} v.$$

We claim that this is a subrepresentation of V.

The above lemma shows the following: applying $\pi(x)$ for any $x \in B$ to an expression of the form $F_{\alpha_{i_1}} \cdot \dots \cdot F_{\alpha_{i_k}} v$, we get a linear combination of similar expressions: indeed, any $\pi(e)$'s will annihilate v, and $\pi(h)$'s act by scalars. This shows that the subspace spanned W is $\mathfrak{g}$-invariant. Since it contains v, we must have $W = V$.

We therefore have a basis for V that consists of elements of the form

$$F_{\alpha_{i_1}} \cdot \dots \cdot F_{\alpha_{i_k}} v.$$

By Lemma 22.1, each of these is a weight vector; moreover, the corresponding weight is lower than μ. Therefore μ is indeed the highest weight appearing in V, and the corresponding weight space is spanned by v.

Finally, by Weyl's theorem, a finite-dimensional representation V is completely reducible:

$$V = \bigoplus V_i$$

where each V_i is irreducible. Now each of the V_i's decomposes into a sum of weight spaces; μ has to appear in at least one of them. In fact, μ appears in exactly one of them, say V_j, because V_μ is one-dimensional. Since the highest weight vector generates all of V, we must have $V = V_j$. This proves the last claim of the lemma. $\square$

Now let (π, V) be a finite-dimensional irreducible representation of $\mathfrak{g}$. The set of weights is finite, so it contains a maximal element, say μ. Because μ is maximal, we have $V_{\mu+\alpha} = 0$ for any positive root α. In particular, $E_\alpha V_\mu = 0$. Thus, any non-zero vector $v \in V_\mu$ is annihilated by E_α's and generates all of V (because V is irreducible). It follows that V is highest weight cyclic. Thus μ is the highest weight of V, by the above proposition. This proves part (i) of the Theorem.

To prove part (ii), suppose V and W are irreducible representations with the same highest weight μ. Take non-zero vectors $v \in V_\mu$ and $w \in W_\mu$. Consider the representation $V \oplus W$, and let U be the smallest subrepresentation generated by (v, w). Then U is highest weight cyclic with weight μ. In particular, U is irreducible. The projections $p_1 : U \to V$ and $p_2 : U \to W$ are $\mathfrak{g}$-morphisms. Note that $p_1(v, w) = v \neq 0$ and $p_1(v, w) = w \neq 0$, which shows that p_1 and p_2 are non-zero. Now, because V, W, and U are all irreducible, p_1 and p_2 must be isomorphisms. Therefore $V \cong W$.

For part (iii), let μ be the highest weight of V. For every simple root $\alpha \in \Delta$, we consider V as a representation of $\mathfrak{sl}_{2,\alpha}$. Let v be a non-zero vector in V_μ. Now H_α acts on this vector by a scalar, and we have $E_\alpha v = 0$, which means that v is the highest weight of an irreducible subrepresentation of V (viewed as an $\mathfrak{sl}_{2,\alpha}$-representation). From the representation theory of $\mathfrak{sl}_2$ we know that this implies that h_α acts on v by a non-negative integer. But the action of h_α on v is given by

$$(\mu|h_\alpha) = 2\frac{(\mu|\alpha)}{(\alpha|\alpha)}.$$

Therefore $2\dfrac{(\mu|\alpha)}{(\alpha|\alpha)}$ is a non-negative integer for every $\alpha \in \Delta$, which means that μ is dominant integral.

We have thus proven the first three points of the Theorem. To prove part (iv), we need to construct an irreducible representation with highest weight λ for every dominant integral element λ. This will be carried out in the next few sections.

23. Verma modules

In this section, we introduce the universal enveloping algebra and use it to define Verma modules, which play a crucial role in our proof of the Theorem of the highest weight.

Let $\mathfrak{g}$ be a Lie algebra over a field F. There exists an associative unital algebra $U(\mathfrak{g})$ equipped with a map $i : \mathfrak{g} \to U(\mathfrak{g})$ satisfying the following properties:

(i) i is a Lie algebra homomorphism;

(ii) the algebra $U(\mathfrak{g})$ is generated by $i(\mathfrak{g})$;

(iii) for any associative unital algebra $\mathcal{A}$ and any Lie algebra homomorphism $j : \mathfrak{g} \to \mathcal{A}$ there exists a homomorphism of unital associative algebras $\phi : U(\mathfrak{g}) \to \mathcal{A}$ such that $j = \phi \circ i$.

By the universal property (iii), such an algebra is unique up to isomorphism. Note that the universal property also implies that we may lift any representation V of $\mathfrak{g}$ to a homomorphism of associative algebras $U(\mathfrak{g}) \to \mathfrak{gl}(V)$.

We construct $U(\mathfrak{g})$ as follows. Let $T(\mathfrak{g})$ denote the tensor algebra of $\mathfrak{g}$:

$$T(\mathfrak{g}) = F \;\oplus\; \mathfrak{g} \;\oplus\; \mathfrak{g}\otimes\mathfrak{g} \;\oplus\; \mathfrak{g}\otimes\mathfrak{g}\otimes\mathfrak{g} \;\oplus\; \dots$$

Now let J be the two-sided ideal generated by all elements of the form

$$xy - yx - [x,y], \quad x, y \in \mathfrak{g}.$$

One checks that $U(\mathfrak{g}) = T(\mathfrak{g})/J$ has all the required properties.

The following is a key result about the structure of the universal enveloping algebra:

Theorem (Poincaré–Birkhoff–Witt). Let $\{x_1, \dots, x_k\}$ be a basis for $\mathfrak{g}$. Then a basis for $U(\mathfrak{g})$ is given by

$$\{i(x_1)^{n_1} \cdot \dots \cdot i(x_k)^{n_k} : n_1, \dots, n_k \in \mathbb{Z}_{\geq}\}.$$

In particular, $i(x_1), \dots, i(x_k)$ are independent, which implies that $i : \mathfrak{g} \to U(\mathfrak{g})$ is injective. Because of this, we often omit i from the notation and simply view $\mathfrak{g}$ as a subspace of $U(\mathfrak{g})$.

We are now ready to define Verma modules. Let $\mathfrak{n}^+$ (resp. $\mathfrak{n}^-$) denote the subspace of $\mathfrak{g}$ spanned by all $\mathfrak{g}_\alpha$ for $\alpha \in R^+$ (resp. R^-). Then

$$\mathfrak{g} = \mathfrak{n}^- \oplus \mathfrak{h} \oplus \mathfrak{n}^+.$$

We refer to elements of $\mathfrak{n}^+$ (resp. $\mathfrak{n}^-$) as raising (resp. lowering) operators. Fix $\mu \in \mathfrak{h}$ (the CSA of $\mathfrak{g}$) and let I_μ denote the left ideal of $U(\mathfrak{g})$ generated by all elements of the form

$$h - (\mu|h)1, \text{for } h \in \mathfrak{h} \quad \text{and} \quad e \in \mathfrak{n}^+.$$

We let W_μ denote the quotient $U(\mathfrak{g})/I(\mu)$, on which $U(\mathfrak{g})$ acts by left multiplication. This is the **Verma module** with highest weight μ.

Proposition 23.1. *W_μ is a non-zero highest weight cyclic representation with highest weight μ; its highest weight vector is* 1.

Remark 23.2. In contrast with the finite-dimensional case discussed in the previous section, an infinite-dimensional highest weight cyclic representation need not be irreducible.

The fact that W_μ is highest weight cyclic will follow from the construction as soon as we can show that $1 \neq 0$ in W_μ. Indeed, it is then immediate that 1 is a weight vector for μ which is annihilated by all the raising operators $e \in \mathfrak{n}^+$. Furthermore, 1 obviously generates all of W_μ. Thus, it remains to prove that $1 \notin I_\mu$. We begin with an intermediate step.

Set $\mathfrak{b} = \mathfrak{h} \oplus \mathfrak{n}^+$ so that $\mathfrak{g} = \mathfrak{n}^- \oplus \mathfrak{b}$. Note that $\mathfrak{b}$ is a subalgebra of $\mathfrak{g}$; consequently, we may view $U(\mathfrak{b})$ as a subalgebra of $U(\mathfrak{g})$.

Lemma 23.3. *Let J_μ be the left ideal in $U(\mathfrak{b})$ generated by the elements which generate I_μ. Then $1 \notin J_\mu$.*

Proof. Define a one-dimensional representation σ_μ of $\mathfrak{b}$ by setting

$$\sigma_\mu(h+e) = (\mu|h) \quad \text{for } e \in \mathfrak{n}^+ \text{ and } h \in \mathfrak{h}.$$

This is indeed representation: the commutator in $\mathbb{C}$ is identically 0, whereas the commutator of two elements of $\mathfrak{b}$ is an element of $\mathfrak{n}^+$, and is therefore in the kernel of σ_μ.

On the one hand, the kernel of σ_μ is easily seen to contain the elements which generate I_μ. It therefore contains J_μ as well. On the other hand, the universal property implies that σ_μ lifts to a homomorphism of unital algebras $\sigma_\mu : U(\mathfrak{b}) \to \mathbb{C}$ (we abuse the notation here). In particular, we have $\sigma_\mu(1) = 1$. This shows that 1 is not in the kernel, so $1 \notin J_\mu$. □

Now let $f_1, \ldots, f_k$ be a basis for $\mathfrak{n}^-$. By the PBW theorem, every element u of $U(\mathfrak{g})$ can be written as

$$u = f_1^{n_1} \cdot \ldots \cdot f_k^{n_k} \cdot b_{n_1,\ldots,n_k}$$

with $b_{n_1,\ldots,n_k} \in \mathfrak{b}$. Moreover, this expression is unique.

Thus any element $u \in I_\mu$ can be written as a sum of elements of the form

$$f_1^{n_1} \cdot \ldots \cdot f_k^{n_k} \cdot b_{n_1,\ldots,n_k} \cdot (h - (\mu|h)1) \quad \text{and} \quad f_1^{n_1} \cdot \ldots \cdot f_k^{n_k} \cdot b_{n_1,\ldots,n_k} \cdot e$$

But then $b_{n_1,\ldots,n_k} \cdot (h - (\mu|h)1)$ and $f_1^{n_1} \cdot \ldots \cdot f_k^{n_k} \cdot b_{n_1,\ldots,n_k} \cdot e$ are elements of J_μ. Therefore, if we write $u \in I_\mu$ as

$$u = f_1^{n_1} \cdot \ldots \cdot f_k^{n_k} \cdot b'_{n_1,\ldots,n_k},$$

it follows that every b' is contained in J_μ. Now suppose $u = 1$ is an element of I_μ. Then 1 has a unique representation as a sum of elements of the form $f_1^{n_1} \cdot \ldots \cdot f_k^{n_k} \cdot b'_{n_1,\ldots,n_k}$; moreover, the above discussion shows that the b' are in J_μ. But clearly this unique representation of 1 is obtained by taking only one summand, namely $n_1 = \cdots = n_k = 0$ and $b'_{0,\ldots,0} = 1$. Thus $b'_{0,\ldots,0} = 1$ must then be an element of J_μ. But this is impossible by Lemma 23.3. We thus arrive at a contradiction. This proves Proposition 23.1.

Corollary 23.4. *The set*

$$\{f_1^{n_1} \cdot \ldots \cdot f_k^{n_k}, \quad n_1, \ldots, n_k \in \mathbb{Z}_\geq\}$$

is a basis for the Verma module W_μ.

Proof. We have just proved that W_μ is a highest weight cyclic representation with highest weight vector 1. This implies that the elements $f_1^{n_1} \cdot \ldots \cdot f_k^{n_k}$ span all of W_μ. To prove that they are independent, assume that there exist scalars $c_{n_1,\ldots,n_k} \in \mathbb{C}$ such that

$$0 = \sum f_1^{n_1} \cdot \ldots \cdot f_k^{n_k} \cdot c_{n_1,\ldots,n_k}.$$

Note that $0 \in I_\mu$, so the above discussion shows that each $c_{n_1,\ldots,n_k}$ is an element of J_μ. But Lemma 23.3 shows that 0 is the only scalar in J_μ. Therefore all the $c_{n_1,\ldots,n_k}$'s are equal to 0. This proves independence. □

24. Irreducible quotients of Verma modules

As remarked in the previous section, W_μ is not necessarily irreducible. In this section we prove that W_μ has a (unique) largest proper invariant subspace U_μ, and that the irreducible quotient $V_\mu = W_\mu/U_\mu$ has highest weight μ.

From Corollary 23.4 it follows that each Verma module W_μ is a direct sum of its weight spaces, all of which are lower than μ. We may therefore uniquely decompose each vector $w \in W_\mu$ as $w = w_1 \oplus w'$, where w_1 belongs to the weight space spanned by 1 (corresponding to the highest weight μ), and w' lies in the span of weight spaces corresponding to weights other than μ. We call w_1 the μ-**component** of w.

Let $e_1, \ldots, e_k$ be a basis for $\mathfrak{n}^+$. We define U_μ as the subspace of W_μ consisting of all the vectors w for which the μ-component of

$$e_1^{n_1} \cdot \cdots \cdot e_k^{n_k} \cdot w$$

is 0, for every $n_1, \ldots, n_k \in \mathbb{Z}_\geq$.

Once again, we use reordering as in Lemma 22.6 to show that this is a $\mathfrak{g}$-invariant subspace of W_μ. Indeed, let B be the basis of $\mathfrak{g}$ as in that lemma, and let $x \in B$. Suppose $w \in U_\mu$; we want to show that $x \cdot w$ is still in U_μ. In other words, we need to show that the μ-component of any expression of the form

$$e_1^{n_1} \cdot \cdots \cdot e_k^{n_k} \cdot x \cdot w.$$

is zero. Rearranging the above terms, we obtain a linear combination of expressions in which e's act first, h's act second, and f's act last. By definition of U_μ, e's acting on w will still produce a vector whose μ-component is zero. Since h simply scales each weight space, acting by h will still give us a vector with that property. Finally, acting by f's will only shift the existing weight spaces down, which shows that the weight space attached to μ cannot appear.

This leads to the main result of this section:

Proposition 24.1. *The quotient $V_\mu = W_\mu / U_\mu$ is an irreducible representation of $\mathfrak{g}$.*

Proof. We need to show that any subspace of W_μ strictly containing U_μ is equal to W_μ. Suppose X is such a subspace, and let v be a vector from X not contained in U_μ. We may assume (by applying a certain number of raising operators to the vector v) that the μ-component of v is non-zero.

We may thus write v as a linear combination

$$v = c_\mu + \sum_\lambda c_\lambda v_\lambda$$

where the sum is taken over a finite set of weights $\lambda \neq \mu$; v_λ is a vector in the weight space of λ, and each c_λ is a scalar. (Recall that the weight vector of μ is simply 1, which is why we omit v_μ.) Now fix $\lambda \neq \mu$, and let $h \in \mathfrak{h}$ be an element such that $(\mu - \lambda | h) \neq 0$. Applying $h - (\lambda|h)1$ to v, we get a vector whose μ-component is $c_\mu \cdot (\mu - \lambda | h)$; in particular, it is non-zero. Meanwhile, the λ-component of this vector is now zero. Therefore, after several such steps, we can annihilate all the λ-components (for $\lambda \neq \mu$) in the sum, obtaining a non-zero constant. This shows that $1 \in X$, and therefore $X = W_\mu$. □

Since 1 is not an element of U_μ (its μ-component is non-zero), it follows that V_μ is still a highest weight cyclic representation, with highest weight μ and the corresponding weight space spanned by (the coset of) 1.

25. Finite-dimensional quotients

In the last section, we attached an irreducible (possibly infinite-dimensional representation) V_μ to each element $\mu \in \mathfrak{h}$. In this section, we show that V_μ is finite-dimensional if μ is dominant integral.

We will prove this by showing that the (multi)set of weights of V_μ is invariant under the action of the Weyl group. To do this, we would like to mimic the proof of Proposition 22.3 which asserts Weyl-invariance for weights of a finite-dimensional representation. Recall that the main ingredient of the proof was the operator $S_\alpha = e^{E_\alpha} e^{-F_\alpha} e^{E_\alpha}$ attached to a simple root $\alpha \in \Delta$. For a finite-dimensional-representation, the operators E_α and F_α are nilpotent, which implies that their exponentials (and hence S_α) are well-defined.

For an infinite-dimensional representation, we need the following definition. We say that a linear operator A on a (possibly infinite-dimensional) space V is **locally nilpotent** if for every $v \in V$ there exists a non-negative integer n such that $A^n v = 0$. Of course, on a finite-dimensional space, locally nilpotent is the same as nilpotent. Assuming A is locally nilpotent, we may define its exponential e^A by setting

$$e^A v = \sum_{n \geq 0} A^n v \quad \text{for every } v \in V.$$

Our first goal, therefore, will be to show the following:

Proposition 25.1. *Let $\mu \in \mathfrak{h}$ be dominant integral. For any simple root $\alpha \in \Delta$, the operators e_α and f_α are locally nilpotent.*

We will need the following lemma:

Lemma 25.2. *Let $\mu \in \mathfrak{h}$ and consider the quotient $V_\mu = W_\mu / U_\mu$. Let $\{e_\alpha, h_\alpha, f_\alpha\}$ be the $\mathfrak{sl}_2$-triple attached to a simple root $\alpha \in \Delta$. Suppose $m := (\mu | h_\alpha)$ is a non-negative integer. Then*

$$f_\alpha^{m+1} = 0$$

as an element of V_μ.

Proof. For simplicity, set $e = e_\alpha, f = f_\alpha, h = h_\alpha$. In W_μ, we have $h \cdot 1 = (\mu | h) \cdot 1$. This also means $(ef - fe) \cdot 1 = [e, f] \cdot 1 = m \cdot 1$. Since $e \cdot 1 = 0$, this translates to $ef = m$. Similarly,

$$ef^2 = ef \cdot f = (fe + [e, f]) \cdot f = (fe + h) \cdot f.$$

We already know $fef = f \cdot m = mf$. Moreover, $hf = fh + [h, f] = mf - 2f$. Thus $ef^2 = 2(m-1)f$. Continuing inductively, we see that

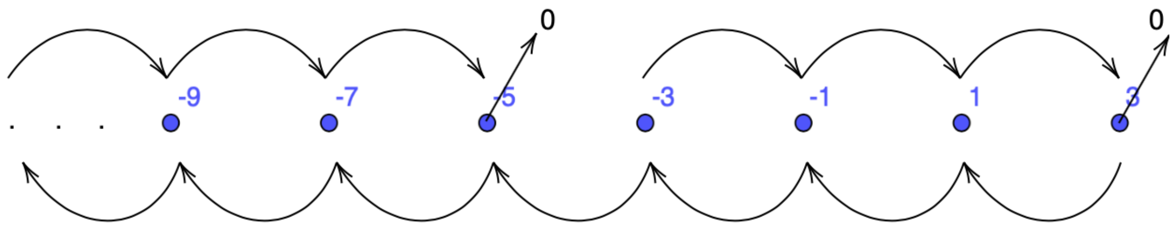
$$ef^k = k(m - (k-1))f^{k-1}$$

in W_μ. In particular, $ef^{m+1} = (m+1) \cdot (m - m) \cdot f^m = 0$.

Furthermore, we have $e_\beta f^{m+1} = 0$ for any positive root $\beta \neq \alpha$. Indeed, $e_\beta f^{m+1}$ is a vector in the weight space for $\lambda = \mu - (m+1)\alpha + \beta$, which is not lower than μ. Since W_μ has highest weight μ, this implies $e_\beta f^{m+1} = 0$.

To summarize, the vector $f^{m+1} \in W_\mu$ has μ-component 0, and is annihilated by every raising operator. This implies $f^{m+1} \in U_\mu$; in other words, $f^{m+1} = 0$ in V_μ. $\square$

Example 25.3. We look at the result of this Lemma in the setting of $\mathfrak{sl}_2$-representations. In the image below, we consider the Verma module of $\mathfrak{sl}_2$ with highest weight $\mu = m = 3$. By Corollary 23.4, we know that a basis for W_m is given by powers of f, where $\{e, h, f\}$ is the standard $\mathfrak{sl}_2$-triple. For any non-negative integer n, the vector f^n spans the weight space for weight $m - 2n$. Thus, in the image below, the dot labeled k represents $f^{\frac{3-k}{2}}$.



Right-pointing arrows represent the action of e; left-pointing arrows are the action of f. The picture shows what we proved in the Lemma: $ef^{m+1} = 0$ (this is the arrow pointing from -5 to 0). Furthermore, it is clear from the picture that the dots labeled $\{-5, -7, \dots\}$ form a subrepresentation of W_μ; this is U_μ.

Proof of Proposition 25.1. Fix a simple root $\alpha \in \Delta$. We say that a vector $v \in V_\mu$ is $\mathfrak{sl}_{2,\alpha}$-**finite** if it is contained in a finite-dimensional $\mathfrak{sl}_{2,\alpha}$-invariant subspace.

Let $m = (\mu|h_\alpha)$. Since μ is dominant integral, this is a non-negative integer. For every non-negative integer n, set $v_n = f_\alpha^n \in V_\mu$. Then the span of the v_n's is $\mathfrak{sl}_{2,\alpha}$-invariant; this follows essentially from the calculation in Lemma 25.2. On the other hand, this space is finite-dimensional, because $v_n = 0$ for $n > m$, again by the same lemma. This shows that the span of $\{v_0, \dots, v_m\}$ is a finite-dimensional $\mathfrak{sl}_{2,\alpha}$-invariant subspace. In particular, there exists a non-zero $\mathfrak{sl}_{2,\alpha}$-finite vector, namely $1 \in V_\mu$.

Now let T_α denote the set of $\mathfrak{sl}_{2,\alpha}$-finite vectors in V_μ. We claim that T_α is a $\mathfrak{g}$-invariant subspace of V_μ. To show this, let $v \in T_\alpha$. By definition, there exists a finite-dimensional $\mathfrak{sl}_{2,\alpha}$-invariant subspace S of T_α. Let S' be the span of all vectors of the form xs for $x \in \mathfrak{g}$, $s \in S$. Clearly S' is finite-dimensional because $\mathfrak{g}$ and S are. Furthermore, S' is $\mathfrak{sl}_{2,\alpha}$-invariant. Indeed, let xs be a typical element of S', and let $y \in \mathfrak{sl}_{2,\alpha}$. Then

$$yxs = x(ys) + [y, x]s$$

which is an element of S' (note that $ys \in S$ because S is $\mathfrak{sl}_{2,\alpha}$-invariant).

This shows that, for any $x \in \mathfrak{g}$, xv is contained in S', which is a finite-dimensional $\mathfrak{sl}_{2,\alpha}$-invariant subspace. This means xv is in T_α.

We have thus shown that T_α is a non-zero subrepresentation of V_μ. Since V_μ is irreducible, it follows that $V_\mu = T_\alpha$. Thus every vector in V_μ is $\mathfrak{sl}_{2,\alpha}$-finite, that is, every vector is contained in a finite-dimensional representation of $\mathfrak{sl}_{2,\alpha}$. The claim of the Proposition now follows from the results on representation theory of $\mathfrak{sl}_2$ (Section 19). $\square$

Proposition 25.4. *If μ is dominant integral, the (multi)set of weights of V_μ is invariant under the action of the Weyl group.*

Proof. We imitate the proof of Proposition 22.3. The Weyl group is generated by simple reflections; it therefore suffices to show that the weights of V_μ are invariant under each $s_\alpha, \alpha \in \Delta$.

We just proved that e_α and f_α are locally nilpotent. We may therefore define the operator

$$S_\alpha = e^{e_\alpha} e^{-f_\alpha} e^{e_\alpha}$$

Now let $h \in \mathfrak{h}$. If $(\alpha|h) = 0$, then $[h, e_\alpha] = [h, f_\alpha] = 0$. Thus h commutes with with e_α and f_α, and therefore also with S_α.

On the other hand, any vector $v \in V_\mu$ is $\mathfrak{sl}_{2,\alpha}$-finite. We can thus find a finite-dimensional, $\mathfrak{sl}_{2,\alpha}$-invariant subspace containing v. We may now apply (the proof of) Proposition 22.3 to show

$$S_\alpha h_\alpha S_\alpha^{-1} v = -h_\alpha v.$$

It follows that

$$S_\alpha h S_\alpha^{-1} v = s_\alpha(h) v.$$

holds for every $h \in \mathfrak{h}$ and $v \in V_\mu$. The rest of the proof is the same as in Proposition 22.3. $\square$

Finally, we are ready to prove part (iv) of Theorem 22.4: for every dominant integral element μ, there is a finite-dimensional irreducible representation of $\mathfrak{g}$ with highest weight μ.

Proof. Let μ be dominant integral. Then any weight λ of V_μ is integral, and is lower than μ: $\lambda \leq \mu$. Since the weights of V_μ are Weyl-invariant, we also have $w\lambda \leq \mu$ for all $w \in W$. This implies, by §20.3, that λ is in the convex hull of the Weyl orbit $W\mu$. In turns, this implies that $||\lambda|| = \sqrt{(\lambda|\lambda)}$ is smaller that $||\mu||$: indeed, the convex hull of $W\mu$ is contained in the circle of radius $||\mu||$ around the origin. Clearly there are only finitely many weights λ which satisfy $||\lambda|| \leq ||\mu||$: we are looking at lattice points inside a ball of radius $||\mu||$. Since we know (by Corollary 23.4) that V_μ is a direct sum of its weight spaces, each of finite dimension, it follows that V_μ is finite-dimensional. $\square$

26. Back to groups

The theorem of the highest weight classifies the representations of a complex semisimple Lie algebra. In this section we investigate how this classification translates back to representations of compact Lie groups.

Let us consider the three-dimensional rotation group:

$$\mathrm{SO}(3) = \{A \in M_3(\mathbb{R}) : A^t A = I\}.$$

This is a compact Lie group, so we should be able to say something about the representation theory of SO(3) by looking at its Lie algebra. However, SO(3) is not simply connected, so we cannot expect all of the representations of the Lie algebra $\mathfrak{so}(3)$ to exponentiate back to representations of the group SO(3).

To remedy this, we pass to the universal cover. We consider the special unitary group

$$\mathrm{SU}(2) = \{A \in M_2(\mathbb{C}) : A^* A = I\} = \{\begin{bmatrix} \alpha & -\overline{\beta} \\ \beta & \overline{\alpha} \end{bmatrix} : |\alpha|^2 + |\beta|^2 = 1\}.$$

It is immediate that SU(2) is homeomorphic to the 3-sphere, and is therefore simply connected.

The Lie algebra $\mathfrak{su}(2)$ consists of 2×2 skew-hermitian complex matrices with trace 0. This is a (real) 3-dimensional space which comes equipped with a symmetric real-valued (real) bilinear form $\langle x, y \rangle = \mathrm{tr}(xy)$. Note that SU(2) acts on $\mathfrak{su}(2)$ by

$$(g, x) \mapsto gxg^*.$$

This action preserves the trace form, so it induces a homomorphism $\mathrm{SU}(2) \to \mathrm{SO}(3)$. One may check that this homomorphism is surjective; its kernel is easily seen to be $\{\pm I\}$. This shows that SU(2) is the universal cover of SO(3).

The complexification of $\mathfrak{su}(2)$ is $\mathfrak{su}(2) \oplus i \cdot \mathfrak{su}(2) = \mathfrak{sl}_2(\mathbb{C})$. The discussion of §17.3 now shows that there is a bijective correspondence between representations of SU(2) and $\mathfrak{sl}_2(\mathbb{C})$. In particular, for each non-negative integer d there exists a unique irreducible representation of SU(2) of dimension d, which we denote by π_d.

A nice way to realize the π_d is via the natural action of SU(2) on the space of homogeneous polynomials of degree $d-1$ in two variables. Equivalently, one may view π_d as the natural representation of SU(2) on the $(d-1)$-th symmetric power of $\mathbb{C}^2$.

To get to representations of SO(3), we simply need to take those representations of the cover SU(2) which factor through the covering map $\mathrm{SU}(2) \to \mathrm{SO}(3)$. We see that $\pi_d(-I) = I$ if and only if d is odd. Therefore, we obtain the list of irreducible representations for SO(3): for each *odd* positive integer d, there exists a unique d-dimensional representation of SO(3).

These results generalize to the case when G is an arbitrary connected compact Lie group. If G is not simply connected, we cannot expect every (finite-dimensional) representation of the complexified Lie algebra to integrate to a representation of G. However, we can still obtain a bijective correspondence between irreducible representations of G and highest weights if we modify our notion of integral elements. We sketch the theory below.

Let $\mathfrak{g}$ be the Lie algebra of G, and let $\mathfrak{h}$ denote a fixed CSA in $\mathfrak{g}$. The image of $\mathfrak{h}$ under the exponential map $\exp : \mathfrak{g} \to G$ is a maximal torus T in G. Denote by Γ the kernel of the exponential map from $\mathfrak{h}$ to T

$$\Gamma = \{h \in \mathfrak{h} : \exp(h) = 1\}.$$

Let $\mathfrak{h}_{\mathbb{C}}$ denote the complexification of $\mathfrak{h}$. As before, we identify $(\mathfrak{h}_{\mathbb{C}})^*$ with $\mathfrak{h}_{\mathbb{C}}$ itself using the Killing form. Recall that an element $\lambda \in \mathfrak{h}_{\mathbb{C}}$ is said to be integral if

$$2\frac{(\lambda|\alpha)}{(\alpha|\alpha)}$$

is an integer for every simple root $\alpha \in \Delta$. We say that $\lambda \in \mathfrak{h}_{\mathbb{C}}$ is **analytically integral** if

$$(\lambda|h) \in 2\pi i\mathbb{Z} \quad \text{whenever} \quad h \in \Gamma.$$

The highest weight theorem for (connected) compact groups asserts that there is a bijective correspondence

$$\{\text{irreducible representations}\} \leftrightarrow \{\text{analytically integral dominant elements}\}.$$

Example 26.1. Let $G = \mathrm{SU}(2)$. Then $\mathfrak{g}$ consists of anti-hermitian matrices of trace 0, and we may take

$$\mathfrak{h} = \{\begin{pmatrix} ix & 0 \\ 0 & -ix \end{pmatrix} : x \in \mathbb{R}\}.$$

We get

$$\Gamma = \{\begin{pmatrix} ix & 0 \\ 0 & -ix \end{pmatrix} : x \in 2\pi\mathbb{Z}\}.$$

Under the identification of $\mathfrak{h}_{\mathbb{C}}$ with its dual, the roots are given by

$$\pm\alpha = \pm\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

Every element of $\mathfrak{h}_{\mathbb{C}}$ can be written as $c\alpha$ for a unique complex number c. By definition, $\lambda = c\alpha$ is integral if

$$2\frac{(\lambda|\alpha)}{(\alpha|\alpha)} = 2c$$

is an integer. It follows that integral elements are $\dfrac{n}{2}\alpha$ for $n \in \mathbb{Z}$.

On the other hand, $c\alpha$ will is analytically integral if

$$(c\alpha|h) \in 2\pi i\mathbb{Z}$$

for every $h \in \Gamma$. Given our description of Γ, this means $2icx \in 2\pi i\mathbb{Z}$ whenever $x \in 2\pi\mathbb{Z}$. This implies $c = \dfrac{n}{2}$ where n is an integer. Thus, in the case of $\mathrm{SU}(2)$, there is no difference between integral and analytically integral elements. Consequently, every finite-dimensional representation of $\mathfrak{g}_{\mathbb{C}}$ lifts to a representation of G. Of course, we already know that this happens whenever G is simply connected.

Example 26.2. Now let $G = \mathrm{SO}(3)$. Then $\mathfrak{g}$ consists of anti-symmetric matrices, and we may take

$$\mathfrak{h} = \{h_x = \begin{pmatrix} 0 & 0 & x \\ 0 & 0 & 0 \\ -x & 0 & 0 \end{pmatrix} : x \in \mathbb{R}\}.$$

One checks that

$$\Gamma = \{h_x : x \in 2\pi\mathbb{Z}\}.$$

Under the identification of $\mathfrak{h}_{\mathbb{C}}$ with its dual, the roots are given by

$$\pm\alpha = \pm\begin{pmatrix} 0 & 0 & \frac{i}{2} \\ 0 & 0 & 0 \\ -\frac{i}{2} & 0 & 0 \end{pmatrix};$$

furthermore, we have $(\alpha|h_x) = ix$.

Again, every element of $\mathfrak{h}_{\mathbb{C}}$ can be written as $c\alpha$ for a unique complex number c. The condition

$$2\frac{(\lambda|\alpha)}{(\alpha|\alpha)} = 2c \in \mathbb{Z}$$

once again implies $c = \dfrac{n}{2}$ for $n \in \mathbb{Z}$.

However, $c\alpha$ will be analytically integral if

$$(c\alpha | h_x) \in 2\pi i\mathbb{Z}$$

for every $h_x \in \Gamma$. Thus $cix \in 2\pi i\mathbb{Z}$ whenever $x \in 2\pi\mathbb{Z}$. This implies c is an integer. To summarize: analytically integral elements are integer multiples of α, whereas integral elements are half-integer multiples of α. We conclude that only half of the representations of $\mathfrak{g}_{\mathbb{C}}$ lift back to a representation of SO(3), verifying what we have already observed earlier in this section.

APPENDIX A: SOME ANALYSIS

We list some standard results from (functional) analysis. This is meant to serve as a quick handbook, rather than an exhaustive reference. As a result, we do not strive for utmost generality; the results are stated in the form in which we use them in class. We do not include proofs, but we do provide relevant bibliography.

Theorem (Stone–Weierstrass)**.** *Let X be a compact Hausdorff space. Denote by $C(X)$ the space of continuous complex-valued functions on X. We view $C(G)$ as an algebra, with pointwise operations. Let $\mathcal{A}$ be a subalgebra of $C(G)$ such that*

- *$\mathcal{A}$ is closed under complex conjugation: $f \in \mathcal{A} \Rightarrow \overline{f} \in \mathcal{A}$;*
- *$\mathcal{A}$ separates points: for any $x, y \in X$, there exists a function $f \in \mathcal{A}$ such that $f(x) \neq f(y)$;*
- *$\mathcal{A}$ does not vanish at any point: $\forall x \in X \ \exists f \in \mathcal{A} : f(x) \neq 0$.*

Then the closure of $\mathcal{A}$ is equal to $C(X)$.

Proof. See [Folland, §4.7]. □

Theorem (Banach–Steinhaus; Uniform Boundendness Principle)**.** *Let X be a Banach space, and let Y be a normed space. Denote by $B(X, Y)$ the space of all bounded operators from X to Y. Let $\mathcal{T}$ be a subset of $B(X, Y)$. If the set*

$$\{||Tx|| : T \in \mathcal{T}\}$$

is bounded for every $x \in X$, then the set

$$\{||T|| : T \in \mathcal{T}\}$$

is also bounded.

Proof. See [Folland, §5.3]. □

Theorem (Riesz Representation Theorem)**.** *Let X be a locally compact Hausdorff space. Denote by $M(X)$ the space of complex Radon measures on X, and by $C_0(X)$ the space of functions which vanish at infinity.*

For any $\mu \in M(X)$ define $I_\mu : C_0(X) \to \mathbb{C}$ by

$$I_\mu(f) = \int_X f \ d\mu.$$

Then $\mu \mapsto I_\mu$ is an isometric isomorphism from $M(X)$ to $C_0(X)^*$.

Proof. See [Folland, §7.3]. □

Theorem. *Let X be a locally compact Hausdorff space equipped with a regular Borel measure which is finite on all compact sets. Then the space $C_c(X)$ of compactly supported functions on X is dense in $L^p(X)$, for all $1 \leq p \leq \infty$.*

Proof. See link. $\square$

APPENDIX B: REFERENCES FOR LIE THEORY

Theorem (Closed subgroup theorem)**.** *Let G be a Lie group and let H be a closed subgroup of G. Then H is a Lie group with manifold structure inherited from G.*

Proof. See [Lee, Theorem 20.10]. ☐

Theorem. *Every compact Lie group is a matrix group.*

Proof. This is a consequence of the Peter–Weyl theorem; see this math.SE post. ☐

Theorem. *The category of compact Lie groups is equivalent to the category of $\mathbb{R}$-anisotropic reductive $\mathbb{R}$-groups all of whose components have $\mathbb{R}$ points.*

Proof. This result is essentially due to Chevalley; see this MO post. ☐

Theorem (Engel)**.** *A finite-dimensional Lie algebra $\mathfrak{g}$ is nilpotent if and only if $ad(X)$ is a nilpotent operator for every $X \in \mathfrak{g}$.*

Proof. See [Fulton–Harris, Theorem 9.9] or the relevant Wikipedia article. ☐

Theorem (Weyl; complete reducibility)**.** *Let $\mathfrak{g}$ be a semisimple Lie algebra over a field of characteristic zero. Every finite-dimensional representation of $\mathfrak{g}$ is completely reducible.*

Proof. See [Hall, Theorem 10.9]. ☐

Theorem (Cartan's criterion for semisimplicity)**.** *Let $\mathfrak{g}$ be a Lie algebra over a field of characteristic zero. Then $\mathfrak{g}$ is semisimple if and only if the Killing form is non-degenerate.*

Proof. See [Jacobson, III.4]. ☐

Theorem (Poincaré–Birkhoff–Witt). Let $\{X_1, \ldots, X_k\}$ be a basis for $\mathfrak{g}$. Then a basis for $U(\mathfrak{g})$ is given by

$$\{i(X_1)^{n_1} \cdot \ldots \cdot i(X_k)^{n_k} : n_1, \ldots, n_k \in \mathbb{Z}_{\geq}\}.$$

Proof. See [Hall, §9.4]. ☐

Appendix C of [Fulton–Harris] provides a nice reference for the results of §18. Chapters 7 and 8 of [Hall] are a good reference for §20.

FURTHER READING

General

In designing the course and these notes, I have often relied on the following texts:

(1) W. Fulton and J. Harris: *Representation Theory. A first course.*
Graduate Texts in Mathematics, 129. Springer-Verlag. ISBN:0-387-97527-6.

(2) H. Kraljević: *Topics in Representation Theory*, unpublished notes (in Croatian); available online (link)

(3) D. Miličić: *Lectures on Representation Theory*, unpublished notes; available online (link)

(4) Brian C. Hall: *Lie Groups, Lie Algebras, and Representations.*
Graduate Texts in Mathematics, 222. Springer-Verlag. ISBN: 978-3-319-13466-6

Variants of Frobenius reciprocity

(1) J. Bernstein: *Representations of p-adic groups.* Lecture notes by K. Rumelhart. Unpublished, available online (link).

- Contains a proof of the second (Bernstein) adjointness theorem between parabolic induction and the Jacquet functor, for smooth (complex) representations of p-adic groups.

(2) J.-F. Dat, D. Helm, R. Kurinczuk, G. Moss: *Finiteness for Hecke algebras of p-adic groups.* arXiv preprint arXiv:2203.04929

- Contains a proof of the second adjointness theorem for modular representations of p-adic groups.

Haar measure, topological groups, analysis

(1) A. Haar: *Der Massbegriff in der Theorie der kontinuierlichen Gruppen.*
Annals of Mathematics, 2, vol. 34, no. 1, pp. 147–169.

(2) G. B. Folland: *Real Analysis: Modern Techniques and Their Applications.* Wiley-Interscience. ISBN: 978-0-471-31716-6.

- Contains proofs of many classical theorems: Riesz, Arzelà–Ascoli, Stone–Weierstrass, Banach–Steinhaus, Hahn–Banach
- A good overview of L^p spaces

(3) P. Halmos: *A Hilbert space problem book.* Springer-Verlag. ISBN: 978-0-387-90685-0

(4) J. von Neumann: *Zum Haarschen Maß in topologischen Gruppen*
Compositio Math. 1 (1935), 106–114.

- The construction of Haar measure presented in class is originally due to von Neumann.

(5) W. Rudin: *Fourier Analysis on Groups.* Wiley-Interscience. ISBN:9780470744819

- See Appendix B for a quick overview of topological groups.
- See Appendix E for a discussion on the Riesz Representation Theorem.

(6) W. Rudin: *Functional Analysis.* Mcgraw-Hill. ISBN: 978-0070542365

- Our exposition closely follows the approach of this book.

(7) L. Valentini: *Haar measure for non-Hausdorff locally compact groups.* arXiv preprint arXiv:2309.07644

- Discusses the ways to extend the notion of Haar measure to non-Hausdorff groups.

(8) A. Weil: *L'intégration dans les groupes topologiques et ses applications.*
Actualités Scientifiques et Industrielles, vol. 869, Paris: Hermann

Lie groups and Lie algebras

(1) Bourbaki: *Lie groups and Lie algebras.*

(2) John M. Lee: *Introduction to Smooth Manifolds.*
Graduate Texts in Mathematics, 218. Springer-Verlag. ISBN: 978-1-4419-9981-8.

(3) Nathan Jacobson: *Lie Algebras.*
Dover publications. ISBN: 9780486638324.