THE REPRESENTATION THEORY OF GL_n

We would like to say something about the representation theory of the following groups:

$$\mathrm{SL}_n(\mathbb{R}), \mathrm{GL}_n(\mathbb{R}), \mathrm{SL}_n(\mathbb{C}), \mathrm{GL}_n(\mathbb{C}).$$

Throughout this note, we will be interested in *finite-dimensional* representations. Henceforth, by a representation we mean a finite-dimensional one.

What we talk about when we talk about representations

For a finite group G, a representation on a complex vector space V is simply a group homomorphism

$$G \to \mathrm{GL}(V)$$

In this note, however, we are not dealing with finite groups. The following examples show that we need to impose additional restrictions if we want to avoid pathological examples.

Example 1. $G = \mathrm{SL}_n(\mathbb{R})$. Let σ be an embedding of $\mathbb{R}$ into $\mathbb{C}$. Then

$$g \mapsto \sigma(g)$$

is an n-dimensional representation of G. Note that most σ's are discontinuous.

Example 2. $G = \mathrm{GL}_n(\mathbb{R})$. Then

$$g \mapsto |\det g|^{\sqrt{2}}$$

is a 1-dimensional representation (i.e. a character) of G.

Example 3. $G = \mathrm{GL}_n(\mathbb{R})$. Then

$$g \mapsto \begin{pmatrix} 1 & \log|\det g| \\ 0 & 1 \end{pmatrix}$$

is a 2-dimensional representation of G.

Example 4. $G = \mathrm{SL}_n(\mathbb{C})$. Let σ denote complex conjugation. Then

$$g \mapsto \sigma(g)$$

is an n-dimensional representation of G.

Example 1 suggests that we should restrict our attention to *continuous* representations of G, i.e. homomorphisms $\pi : G \to \mathrm{GL}(V)$ for which the map $G \times V \to V$ given

$$(g, v) \mapsto \pi(g)v$$

is continuous.[1] It turns out that all continuous finite-dimensional representations of $G = \mathrm{SL}_n(\mathbb{R})$ are automatically algebraic [2, 5], that is, the matrix entries of $\pi(g)$ are rational functions in g.

[1] On a finite-dimensional space, this condition is equivalent to the continuity of the map $g \mapsto \pi(g)$ from G to the space of linear operators on V.

For $G = \mathrm{GL}_n(\mathbb{R})$, this is not true, as demonstrated by examples 2 and 3 (in which the representations are continuous — and moreover, smooth — but not algebraic). This suggests that we should focus on algebraic representations of G.

Finally, $\mathrm{GL}_n(\mathbb{C})$ and $\mathrm{SL}_n(\mathbb{C})$ can be viewed as real Lie groups, but they also possess a complex structure. When viewing them as complex Lie groups, we require their representations to be holomorphic. It turns out that this is the same as requiring them to be algebraic [1]. Example 4 shows a rational representation of $\mathrm{SL}_n(\mathbb{C})$ (viewed as a real Lie group) that is not holomorphic.

Weyl's unitary trick

Weyl's key observation is that one can use the compact form of a group (or its Lie algebra) to answers questions about representation theory. The unitary trick tells us that there is a bijective correspondence between:

(1) algebraic representations of $\mathrm{SL}_n(\mathbb{R})$;

(2) holomorphic representations of $\mathrm{SL}_n(\mathbb{C})$;

(3) representations of $\mathfrak{sl}_n(\mathbb{C})$;

(4) smooth (or equivalently, algebraic) representations of $\mathrm{SU}(n)$

(see [4, Chapter II, §1]). Of course, the name of the unitary trick refers to part (4): since $\mathrm{SU}(n)$ is irreducible, all its representations are unitary. An immediate consequence of this result is that the representations in (1), (2), and (3) are also completely reducible. Weyl's unitary trick also reduces the problem of classifying irreducible algebraic representations of $\mathrm{SL}_n(\mathbb{R})$ and $\mathrm{SL}_n(\mathbb{C})$ to the classification of $\mathfrak{sl}_n(\mathbb{C})$-representations. The irreducible representations of $\mathfrak{sl}_n(\mathbb{C})$ are in turn classified by the theorem of the highest weight.

Representations of $\mathrm{SL}_n(\mathbb{C})$ and $\mathrm{SL}_n(\mathbb{R})$

We can write the roots of $\mathfrak{sl}_n(\mathbb{C})$ as vectors in $\mathbb{R}^n$: they are given by $e_i - e_j$, $i \neq j \in \{1, \ldots, n\}$. (Here e_i denotes the i-th standard basis vector for $\mathbb{R}^n$.) With this identification, the weight lattice is generated by elements

$$L_i = (-\frac{1}{n}, \ldots, -\frac{1}{n}, \frac{n-1}{n}, -\frac{1}{n}, \ldots, -\frac{1}{n}) = e_i - \frac{1}{n}\sum_{j=1}^{n} e_j, \quad i = 1, \ldots, n.$$

The integral elements are thus given by $\mathbb{Z}$-linear combinations

$$\lambda_1 L_1 + \cdots + \lambda_n L_n.$$

Such an element is dominant precisely when $\lambda_1 \geq \ldots \geq \lambda_n$. Since $\sum L_i = 0$, we have $\lambda_1 L_1 + \cdots + \lambda_n L_n = (\lambda_1 + k)L_1 + \cdots + (\lambda_n + k)L_n$ for any integer k. Thus (taking $k = -\lambda_n$) we may assume that $\lambda_n = 0$. To summarize, the irreducible representations of $\mathfrak{sl}_n(\mathbb{C})$ correspond bijectively to tuples of integers $\Lambda = (\lambda_1, \ldots, \lambda_{n-1})$ with

$$\lambda_1 \geq \cdots \geq \lambda_{n-1} \geq 0.$$

As a consequence of the above discussion about the unitary trick, this classifies the irreducible representations of $\mathrm{SL}_n(\mathbb{C})$ as well.

A concrete realization of the $\mathrm{SL}_n(\mathbb{C})$-representation corresponding to Λ can be obtained using the Schur–Weyl duality, which we recall here. To each tuple $\Lambda = (\lambda_1, \ldots, \lambda_{n-1})$ as above we may associate the corresponding irreducible representation of the symmetric group S_d, where $d = \sum \lambda_i$. Recall the construction of this representation: from the Young tableau corresponding to Λ, we build the corresponding Young symmetrizer c_Λ. The corresponding representation of S_n is then realized by the natural action of S_d on the space $\mathbb{C}[S_d]c_\Lambda$ (the image of right multiplication by c_Λ on the group algebra $\mathbb{C}[S_d]$).

Now let $V = \mathbb{C}^n$ be the standard representation of $\mathrm{SL}_n(\mathbb{C})$. Given Λ as above, we consider the d-th tensor power of V:

$$V^{\otimes d} = V \otimes \cdots \otimes V.$$

This space comes equipped with commuting actions of S_d (say, on the right) and $\mathrm{SL}_n(\mathbb{C})$ (on the left). Since the actions commute, the space $V^{\otimes d}c_\Lambda$ is $\mathrm{SL}_n(\mathbb{C})$-invariant, and one shows that this is an irreducible representation of highest weight Λ, i.e.

$$\lambda_1 L_1 + \cdots + \lambda_{n-1} L_{n-1}.$$

We denote this representation by π_Λ. The corresponding irreducible representation of $\mathrm{SL}_n(\mathbb{R})$ is realized on the same space, by restricting the $\mathrm{SL}_n(\mathbb{C})$-action.

See [3, §4,§6,§15] for a more detailed exposition of this construction.

Representations of $\mathrm{GL}_n(\mathbb{C})$ and $\mathrm{GL}_n(\mathbb{R})$

The above construction essentially allows us to recover all irreducible representations of $\mathrm{GL}_n(\mathbb{C})$ and $\mathrm{GL}_n(\mathbb{R})$; the only difference is that here we need to take into account the determinant.

Observe that $\mathrm{GL}_n(\mathbb{C})$ can be obtained by gluing $\mathrm{SL}_n(\mathbb{C})$ and $\mathbb{C}^\times$ along the center:

$$\mathrm{GL}_n(\mathbb{C}) = \mathrm{SL}_n(\mathbb{C}) \times_{\Gamma_n} \mathbb{C}^\times,$$

where Γ_n denotes the group of n-th roots of unity. To see this, note that the map $\mathbb{C}^\times \times \mathrm{SL}_n(\mathbb{C}) \to \mathrm{GL}_n(\mathbb{C})$ given by

$$(z, g) \mapsto zg$$

is surjective, and its kernel is $\{(z, \mathrm{diag}(z^{-1}, \ldots, z^{-1})\}$. Of course, $\mathrm{diag}(z^{-1}, \ldots, z^{-1})$ is an element of $\mathrm{SL}_n(\mathbb{C})$ precisely when $z^n = 1$, so the kernel is indeed isomorphic to Γ_n.

Specifying an irreducible representation of $\mathrm{GL}_n(\mathbb{C})$ thus amounts to choosing an irreducible representation π_Λ of $\mathrm{SL}_n(\mathbb{C})$ and a (holomorphic) character of $\mathbb{C}^\times$ which agrees with the representation π_Λ on the group Γ_n. Note that every holomorphic character of $\mathbb{C}^\times$ is of the form $z \mapsto z^k$ for some integer k; if z^k is a character which agrees with the representation

π_Λ, then all other characters are given by z^{k+nl}, where $l \in \mathbb{Z}$. To summarize, the irreducible representations of $\mathrm{GL}_n(\mathbb{C})$ are parametrized by pairs (Λ, l) where Λ is a tuple as above and l is an integer.

There is another way of seeing this, which follows the construction from the previous section more closely. We view $V = \mathbb{C}^n$ as a representation of $\mathrm{GL}_n(\mathbb{C})$. Let $\Lambda = (\lambda_1, \dots, \lambda_{n-1})$ be an $(n-1)$-tuple as before. Then $V^{\otimes d} c_\Lambda$ is an irreducible representation of $\mathrm{GL}_n(\mathbb{C})$. Moreover, every representation of $\mathrm{GL}_n(\mathbb{C})$ is obtained by twisting such a representation by a power of the determinant. If we set

$$\Lambda + k = (\lambda_1 + k, \dots, \lambda_{n-1} + k, k),$$

for a non-negative integer k, then $V^{\otimes d} c_{\Lambda+k} = \left(V^{\otimes d} c_\Lambda\right) \otimes (\det)^k$.

Thus the irreducible representations of $\mathrm{GL}_n(\mathbb{C})$ correspond to n-tuples (note the extra term!) of integers $\Lambda = (\lambda_1, \dots, \lambda_{n-1}, \lambda_n)$ with

$$\lambda_1 \geq \dots \geq \lambda_{n-1} \geq \lambda_n,$$

where the representation corresponding to Λ is $V^{\otimes d} c_\Lambda$ if λ_n is non-negative, and $\left(V^{\otimes d} c_{\Lambda - \lambda_n}\right) \otimes (\det)^{\lambda_n}$ when λ_n is negative.

As before, the irreducible representations $\mathrm{GL}_n(\mathbb{R})$ are obtained by restricting those of $\mathrm{GL}_n(\mathbb{C})$. It is worth pointing out that, in contrast to the beginning of this section, we cannot view $\mathrm{GL}_n(\mathbb{R})$ as a quotient of $\mathbb{R}^\times \times \mathrm{SL}_n(\mathbb{R})$ if n is even. Indeed, the map

$$(t, g) \mapsto tg$$

is no longer surjective (multiplication by t scales the determinant by t^2, so the image is the set of all matrices of positive determinant). Consequently, we cannot obtain the irreducibles by gluing together irreducible representation of $\mathrm{SL}_n(\mathbb{R})$ with characters of $\mathbb{R}^\times$. However, we can salvage this approach by first inducing the irreducible representations of $\mathrm{SL}_n(\mathbb{R})$ to the group

$$\mathrm{SL}_n^{\pm}(\mathbb{R}) = \{g \in \mathrm{GL}_n(\mathbb{R}) : \det g \in \{\pm 1\}\},$$

and then tensoring the resulting representations with characters of $\mathbb{R}^+$.

Representations of $\mathrm{SL}_n(\mathbb{C})$ and $\mathrm{GL}_n(\mathbb{C})$ as real Lie groups

Finally, we may also view $\mathrm{SL}_n(\mathbb{C})$ and $\mathrm{GL}_n(\mathbb{C})$ as real Lie groups. We thus allow representations such as the one appearing in Example 4.

Since $\mathrm{SL}_n(\mathbb{C})$ is simply connected, its smooth (but not necessarily holomorphic) representations correspond to real-linear representations of $\mathfrak{sl}_n(\mathbb{C})$, or equivalentnly, complex-linear representations of the complexification, $\mathbb{C} \otimes_\mathbb{R} \mathfrak{sl}_n(\mathbb{C})$. Every element of the complexification can be uniquely written as $1 \otimes x + i \otimes y$ for $x, y \in \mathbb{C}$; one checks that

$$1 \otimes x + i \otimes y \mapsto (x + iy, \overline{x} + i\overline{y})$$

is a $\mathbb{C}$-isomorphism of Lie algebras $\mathbb{C} \otimes_\mathbb{R} \mathfrak{sl}_n(\mathbb{C}) \to \mathfrak{sl}_n(\mathbb{C}) \oplus \mathfrak{sl}_n(\mathbb{C})$. An irreducible representation of $\mathfrak{sl}_n(\mathbb{C}) \oplus \mathfrak{sl}_n(\mathbb{C})$ is a tensor product of irreducible representations of the factors.

One verifies this by using the unitary trick to translate this statement to the setting of compact groups; the claim now follows from the fact that every irreducible representation of $\mathrm{SU}(n) \times \mathrm{SU}(n)$ is a tensor product of irreducibles.

The irreducible representations of $\mathrm{SL}_n(\mathbb{C})$ are thus parametrized by *pairs* of dominant integral elements (Λ_1, Λ_2). If $\pi_{\Lambda_1} \otimes \pi_{\Lambda_2}$ is the corresponding representation of $\mathfrak{sl}_n(\mathbb{C}) \oplus \mathfrak{sl}_n(\mathbb{C})$, it restricts back to the original Lie algebra $\mathfrak{sl}_n(\mathbb{C})$ as

$$x \mapsto \pi_{\Lambda_1}(x) \otimes \pi_{\Lambda_2}(\overline{x})$$

(note the composition with complex conjugation in the second factor). To summarize, irreducible representations of $\mathrm{SL}_n(\mathbb{C})$ look like

$$\pi_{\Lambda_1} \otimes \overline{\pi_{\Lambda_2}},$$

where we use $\overline{\pi}$ to denote the complex conjugate of π. See [4, Chapter II, §3].

The situation for $\mathrm{GL}_n(\mathbb{C})$ is analogous; every irreducible representation is given by tensor product of two irreducible representations, one of which is holomorphic, and the other anti-holomorphic (i.e. the complex conjugate of a holomorphic representation). As before, one can deduce this from the corresponding classification of $\mathrm{SL}_n(\mathbb{C})$-representations: irreducible representations of $\mathrm{SL}_n(\mathbb{C})$ are obtained by tensoring $\mathrm{SL}_n(\mathbb{C})$-representations with characters of $\mathbb{C}^\times$; only this time, we are viewing $\mathbb{C}^\times$ as a real Lie group, so its characters are of the form

$$z \mapsto z^k \overline{z}^l, \quad k, l \in \mathbb{Z}.$$

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