Newcomb, Benford, Pareto, Heaps, and Zipf
Are arbitrary numbers random?

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Numbers and distributions

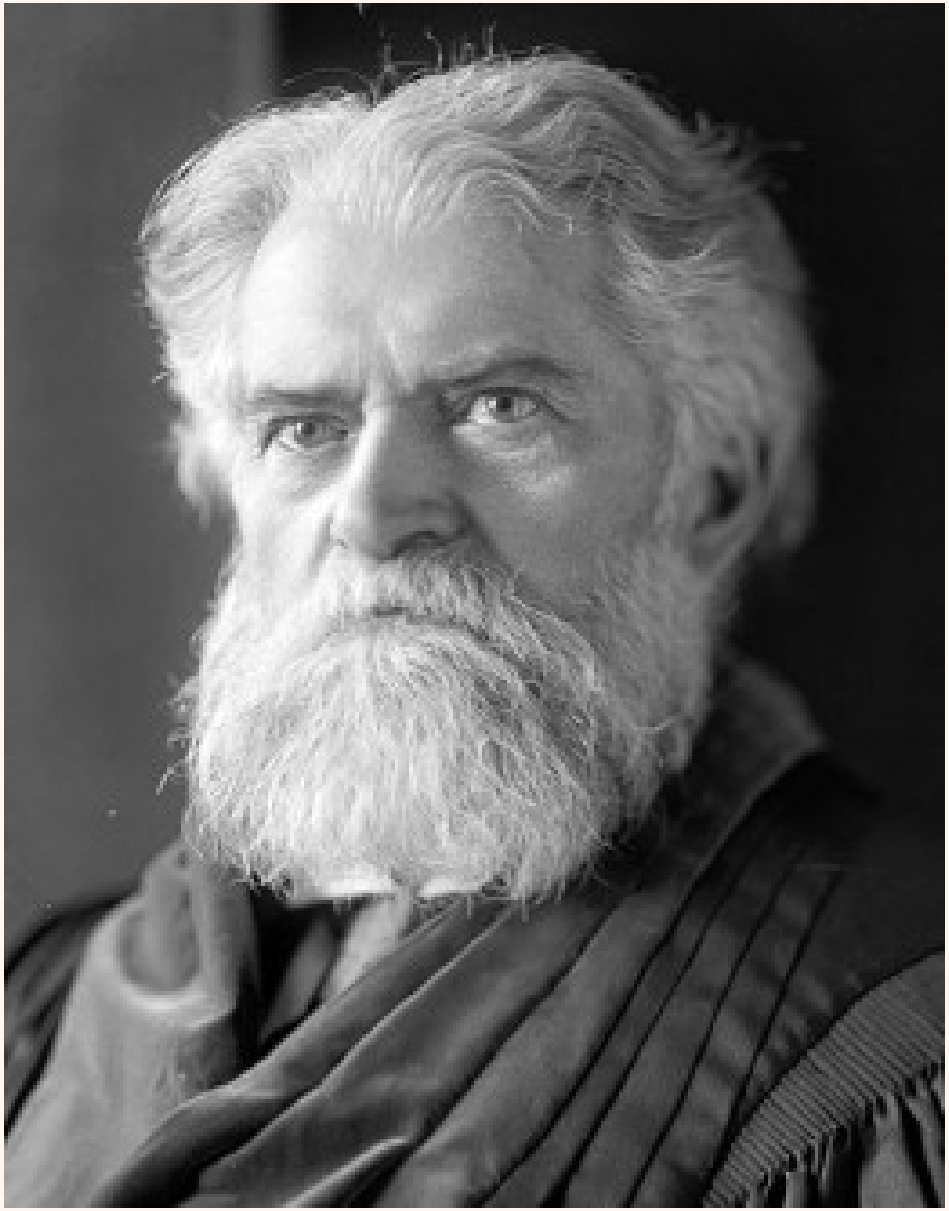
Simulations usually need a source of numeric data, and random values are sometimes a suitable source.

However, random numbers may conform to different distributions: uniform, normal, exponential, logarithmic, Poisson, ...

The key question is:

Do numbers in real data match a uniform distribution?

A negative answer



Simon Newcomb (1835–1909) Canadian / American astronomer, mathematician, economist, linguist, mountaineer

Note on the frequency of use of the different digits in natural numbers, *American Journal of Mathematics*, **4**(1–4) 39–40 (1881). The short note begins:

That the ten digits do not occur with equal frequency must be evident to any one making much use of logarithmic tables, and noticing how much faster the first pages wear out than the last ones.

But wait...

[Newcomb was a co-founder of the **American Astronomical Society**, and its first President (1899–1904).]

Consider the integers from, say, 100 to 999. There are 100 in $[100, 199]$, 100 more in $[200, 299]$, and so on up to the last 100 in $[900, 999]$.

We conclude that for random numbers from a uniform distribution:

leading digits have equal likelihood.

There are nine such digits, $1, 2, \ldots, 9$, so their probabilities are **$1/9 \approx 0.111$**.

Newcomb's prediction

The law of probability of the occurrence of numbers is such that all mantissæ of their logarithms are equally probable.

digit	first	second
0		0.1197
1	0.3010	0.1139
2	0.1761	0.1088
3	0.1249	0.1043
4	0.0969	0.1003
5	0.0792	0.0967
6	0.0669	0.0934
7	0.0580	0.0904
8	0.0512	0.0876
9	0.0458	0.0850

In the case of the third figure the probability will be nearly the same for each digit, and for the fourth and following ones the difference will be inappreciable.

Newcomb's conclusion

It is curious to remark that this law would enable us to decide whether a large collection of independent numerical results were composed of natural numbers or logarithms.

Then Newcomb's work was forgotten for 57 years. . .

Benford's rediscovery

American physicist **Frank Benford** (1883–1948), in ***The Law of Anomalous Numbers***, *Proceedings of the American Philosophical Society*, **78**(4) 551–572, March (1938), perhaps unaware of Newcomb's work (but he mentions the *dirty pages* phenomenon), rediscovered the same curiosity.

Benford's paper was noticed, and the law is named after him.

[photograph ca. 1912, age 29]

Benford's rediscovery [continued]

Benford illustrated the phenomenon with a great variety of data:

river (drainage?) areas	$1/n$, $\sqrt{n}$
land area	design data generators
US population	*Reader's Digest*
physical constants	cost data for concrete
newspaper items	X-ray volts
specific heats	American League baseball (1936)
pressure lost in air flow	black-body radiation
H.P. lost in air flow	AMS street addresses
drainage	$n^1, n^2, n^3, \ldots, n!$
atomic & molecular weights	death rates
house numbers	river drainage rates

He gave frequency data for each, and a cumulative report with first-digit frequencies: **0.306, 0.185, 0.124, 0.094, 0.080, 0.064, 0.051, 0.049**, and **0.047**.

Why Benford got a Law, and Newcomb did not

Benford gave much more data, and provided more mathematical arguments, in support of his **Law of Anomalous Numbers**, than Newcomb did in 1881.

> *Benford's paper was published in 1938 in a journal of rather limited circulation and not usually read by mathematicians. It so happened that it was immediately followed in the same issue by a physics paper which became of some importance for secret nuclear work during World War II. That is why Benford's paper caught the attention of physicists in the early 1940's and was much discussed.*
>
> ***Jonothan L. Logan and Samuel A. Goudsmit**, **The First Digit Phenomenon**, Proceedings of the American Philosophical Society, **122**(4) 193–197, 18 August (1978).*

Boring and Raimi uncover Newcomb's work

Newcomb is briefly cited by **Edwin G. Boring**, ***The Logic of the Normal Law of Error in Mental Measurement***, *The American Journal of Psychology*, **31**(1) 1–33 (1920), but only about randomness of digits in transcendental numbers.

Newcomb's work seems to have been uncovered next by **Ralph A. Raimi**, ***The first digit problem***, *American Mathematical Monthly*, **83**(7) 521–538, August 1976, 95 years later. Raimi wrote:

> *This assertion, whatever it may mean, will be called* ***Benford's Law*** *because it has been thought by many writers to have originated with the General Electric Company physicist Frank Benford. . . . There is ample precedent for naming laws and theorems for persons other than their discoverers, else half of analysis would be named after Euler. Besides, even Newcomb implied that the observation giving rise to the Benford law was an old one in his day. One would hate to change the name of the law now only to find later that another change was called for.*

Benford's Law for first digits

The frequency of the first digit [in measured data] follows closely the logarithmic relation:

$$F_a = \log_{10}\left(\frac{a+1}{a}\right), \qquad \textit{Benford's original,}$$
$$= \log_{10}(1 + 1/a), \qquad \textit{modern form}.$$

Here, a is a *nonzero* leading decimal digit $1, 2, \ldots, 9$.
Benford's leading-digit frequencies are *identical* to those in Newcomb's table: **0.301, 0.176, 0.125, 0.097, 0.079, 0.067, 0.058, 0.051**, and **0.046**.
The partial sums produce cumulative frequencies given by

$$C_a = \log_{10}(1 + a)$$

with these approximate values: **0.301, 0.477, 0.602, 0.699, 0.778, 0.845, 0.903, 0.954**, and **1**. Thus, 60% start with 1, 2, or 3.

Benford's Law for second digits

For a number beginning with decimal digits $ab\cdots$

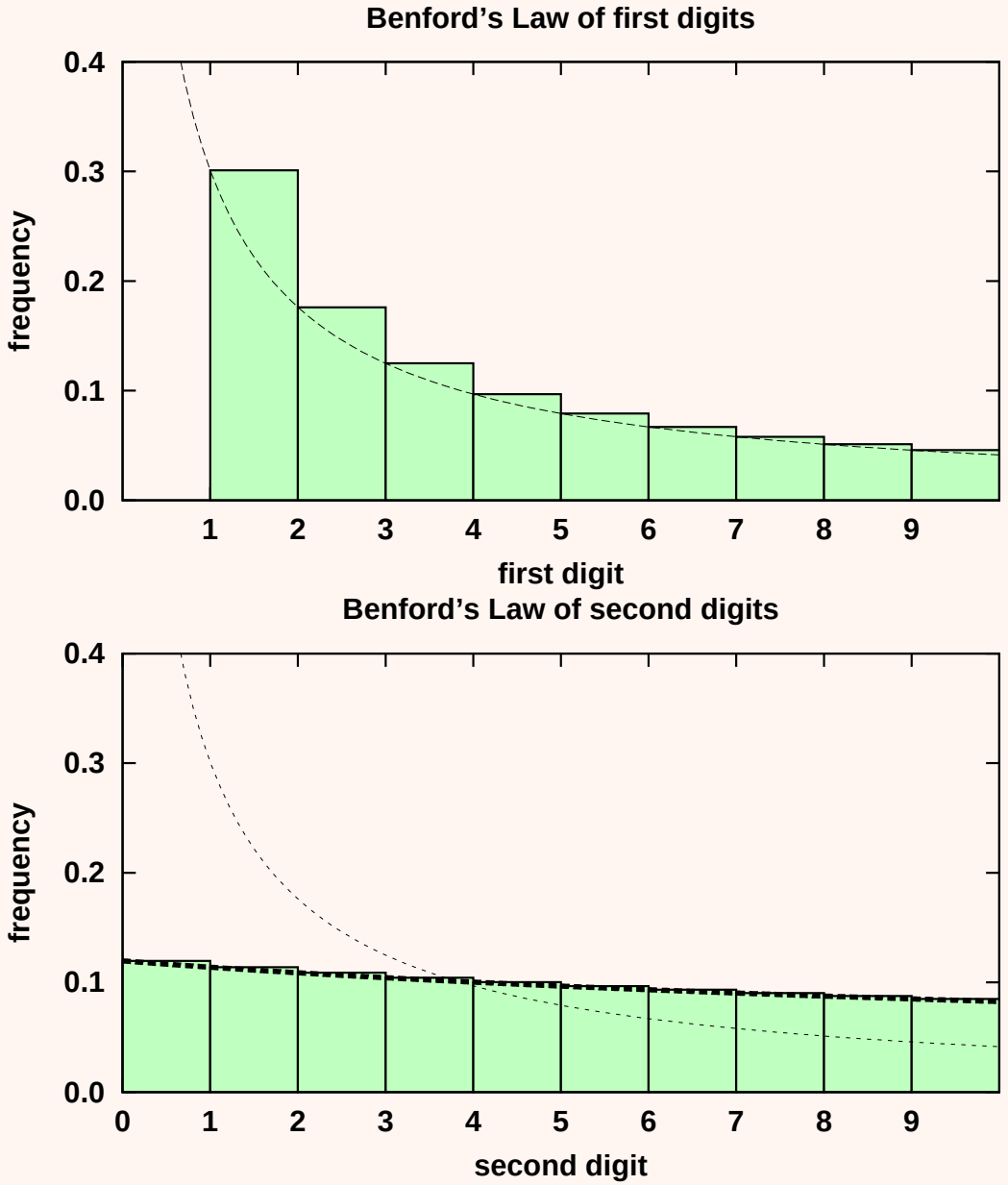
$$F_b = \frac{\log_{10}\left(\frac{ab+1}{ab}\right)}{\log_{10}\left(\frac{a+1}{a}\right)}$$

Here, b may be any of $0, 1, 2, \ldots, 9$.

Summed over all possible leading digits, the second-digit frequencies are **0.120, 0.114, 0.109, 0.104, 0.100, 0.097, 0.093, 0.090, 0.088**, and **0.085**.

In some cases, second-digit data have proved more useful than first-digit data, and have been used for examining election results for evidence of fraud (e.g., Argentina, Germany, Iran, Puerto Rico, USA, Venezuela).

Benford's Law pictorially



Benford's Law for arbitrary digits

For a number beginning with decimal digits $abc \cdots opq \cdots$,

$$\log(1+x) \approx (x - x^2/2 + x^3/3 - x^4/4 + \cdots), \qquad \textit{Taylor series,}$$

$$\log_{10}(1+x) \approx (x - x^2/2 + x^3/3 - x^4/4 + \cdots)/\log(10),$$

$$F_q = \frac{\log_{10}(\frac{abc\cdots opq+1}{abc\cdots opq})}{\log_{10}(\frac{abc\cdots op+1}{abc\cdots op})} = \frac{\log_{10}(1+\frac{1}{abc\cdots opq})}{\log_{10}(1+\frac{1}{abc\cdots op})} \approx \frac{abc\cdots op}{abc\cdots opq}$$

$$\to 1/10, \qquad \textit{for increasing } q.$$

For example, if $abcdefgh = 12345678$, then

$$F_9 = \frac{\log_{10}(\frac{123456789+1}{123456789})}{\log_{10}(\frac{12345678+1}{12345678})} \approx 0.099\,999\,996\,354\ldots$$

Thus, after the first few leading digits, **there is little difference in digit frequencies**.

Computational note: **use** `log1p(x)` **instead of** `log(1 + x)`.

Benford's Law and percentage growth

Consider a company with $1,000,000 revenues:

- Leading digit of 1: income increases by 100% to $2,000,000.
- Leading digit of 2: income increases by 50% to $3,000,000.
- Leading digit of 3: income increases by 33% to $4,000,000.
- Leading digit of 4: income increases by 25% to $5,000,000.
- ...
- Leading digit of 9: income increases by 11% to $10,000,000.

Suggestion: **If percentage growth is roughly constant, then smaller leading digits should be more common.**

Growth is more likely to be *geometric* than *arithmetic*.

Frequencies decrease [0.353, 0.177, 0.118, 0.088, 0.071, 0.059, 0.050, 0.044, and 0.039] but do not match Benford's Law.

Benford's Law: two observations

Benford's 'law of first digits' has a history over very many decades and has produced a literature which is remarkable in that it shows a lack of understanding that the **law is fundamental and general** *rather than specific to the properties of a particular data set.*

***B. K. Jones**, **Logarithmic distributions in reliability analysis**, Microelectronics Reliability **42**(4–5) 779–786 (2002).*

Wallace (2002) suggests that if the mean of a particular set of numbers is larger than the median and the skewness value is positive, the data set likely follows a Benford distribution. It follows that **the larger the ratio of the mean divided by the median, the more closely the set will follow Benford's Law**.

***C. Durtschi et al.**, **The Effective Use of Benford's Law to Assist in Detecting Fraud in Accounting Data**, Journal of Forensic Accounting **5**(1) 17–34 (2004).*

Benford's Law and mixed distributions

If distributions are selected at random (in any "unbiased way") and random samples are take from these distributions, then the significant-digit frequencies of the **combined sample** *will converge to Benford's distribution, even though the individual distributions selected may not closely follow the law.*

Theodore P. Hill, ***The First Digit Phenomenon***, *American Scientist,* ***86****(4) 358–363 July / August (1998).*

Benford's Law in other number bases

If Benford's Law holds for decimal numbers, then it also holds for other number bases, provided that those bases are not huge. Just change 10 to the base b in the logarithms in the digit-frequency formulas.
For example,

Digit	0	1	2	3	4	5	6	7
								Base $b = 2$
F_a		1.000						
F_b	0.585	0.415						
								Base $b = 4$
F_a		0.500	0.292	0.208				
F_b	0.304	0.261	0.230	0.206				
								Base $b = 8$
F_a		0.333	0.195	0.138	0.107	0.088	0.074	0.064
F_b	0.151	0.141	0.133	0.126	0.120	0.115	0.110	0.105

See **Theodore Hill**, ***Base-invariance implies Benford's Law***, *Proceedings of the American Mathematical Society* **123**(3) 887–895, March 1995.

Benford's Law observed in real data

Digit	0	1	2	3	4	5	6	7	8	9
							1990 US Census data (5148 values)			
F_a		0.298	0.215	0.113	0.082	0.098	0.056	0.055	0.034	0.049
F_b	0.166	0.090	0.096	0.081	0.100	0.122	0.076	0.073	0.066	0.130
								Atomic weights (110 values)		
F_a		0.391	0.309	0.045	0.036	0.055	0.036	0.036	0.036	0.055
F_b	0.173	0.045	0.109	0.100	0.145	0.145	0.055	0.055	0.091	0.082
								Country areas (1505 values)		
F_a		0.312	0.275	0.100	0.067	0.058	0.062	0.046	0.029	0.050
F_b	0.167	0.221	0.092	0.092	0.062	0.067	0.075	0.067	0.083	0.075
								Country population (163 values)		
F_a		0.301	0.202	0.092	0.135	0.055	0.067	0.055	0.037	0.055
F_b	0.147	0.153	0.110	0.098	0.098	0.123	0.086	0.043	0.049	0.092
								Infant mortality (208 values)		
F_a		0.361	0.293	0.043	0.072	0.062	0.087	0.038	0.014	0.029
F_b	0.303	0.139	0.077	0.058	0.077	0.077	0.106	0.067	0.043	0.053
								IBM 2010 annual financial report (6126 values)		
F_a		0.333	0.160	0.163	0.086	0.068	0.053	0.047	0.045	0.046
F_b	0.172	0.169	0.096	0.084	0.085	0.095	0.079	0.074	0.079	0.068
Fibonacci numbers: $f(n) = f(n-1) + f(n-2); f(2) = f(1) = 1$ (9994 values)										
F_a		0.301	0.176	0.125	0.097	0.079	0.067	0.058	0.051	0.046
F_b	0.120	0.114	0.109	0.105	0.100	0.097	0.094	0.090	0.088	0.085

When does Benford's Law apply?

Despite 140+ years since Newcomb's discovery, the mathematical conditions for, and derivation of, Benford's Law remain unsettled: see **Arno Berger and Theodore P. Hill**, ***Benford's law strikes back: no simple explanation in sight for mathematical gem***, *The Mathematical Intelligencer*, **33**(1) 85–91 (2011).

There is general agreement that the law applies to numbers whose distribution is **scale invariant**: if changing units of measure leaves the number distribution unchanged, then Benford's Law holds.

[**Roger S. Pinkham**, ***On the Distribution of First Significant Digits***, *Annals of Mathematical Statistics*, **32**(4) 1223–1230, December (1961)]

Thus, we can

- do accounting in dollars, euros, pesos, ruan, rubles, rupees, yen, ...;
- measure distances in metric or nonmetric units;
- measure areas in square furlongs, or square parsecs, or ...;
- count people, couples, families, arms, fingers, toes, ...

When does Benford's Law apply? [continued]

The numbers in many mathematical sequences and physical distributions obey Benford's Law **exactly**, or at least closely, including:

- geometric sequences, and asymptotically-geometric sequences, like the *Fibonacci numbers* (1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, 144, ...), and also the *Lucas numbers* (2, 1, 3, 4, 7, 11, 18, 29, 47, 76, ...) which obey $L(n) = L(n-1) + L(n-2)$, with $L(0) = 2$ and $L(1) = 1$;
- iterations like $x \leftarrow 3x + 1$, starting with $x = \textit{random number}$,
- powers of integers;
- logarithms of uniformly-distributed random numbers;
- prime numbers;
- reciprocals of all of the above;
- reciprocals of *Riemann zeta function* zeros;
- finite-state *Markov chains*;
- *Boltzmann–Gibbs* and *Fermi–Dirac* distributions (approximate), and *Bose–Einstein* distributions (exact).

When is Benford's Law inapplicable?

Sequences for which Benford's Law **does not hold** include:

- arithmetic sequences.
- random numbers from most common distributions;
- digit subsets of irrational and transcendental numbers;
- US telephone numbers (limited prefixes, leading digit never 1, last four digits all used);
- bounded sequences with restricted leading digits (hours of day; days of week, month, or year; house numbers; human ages (and heights and weights); . . .)

Where do Benford's Law publications appear?

About **1420** publications are listed in

http://www.math.utah.edu/pub/tex/bib/benfords-law.html

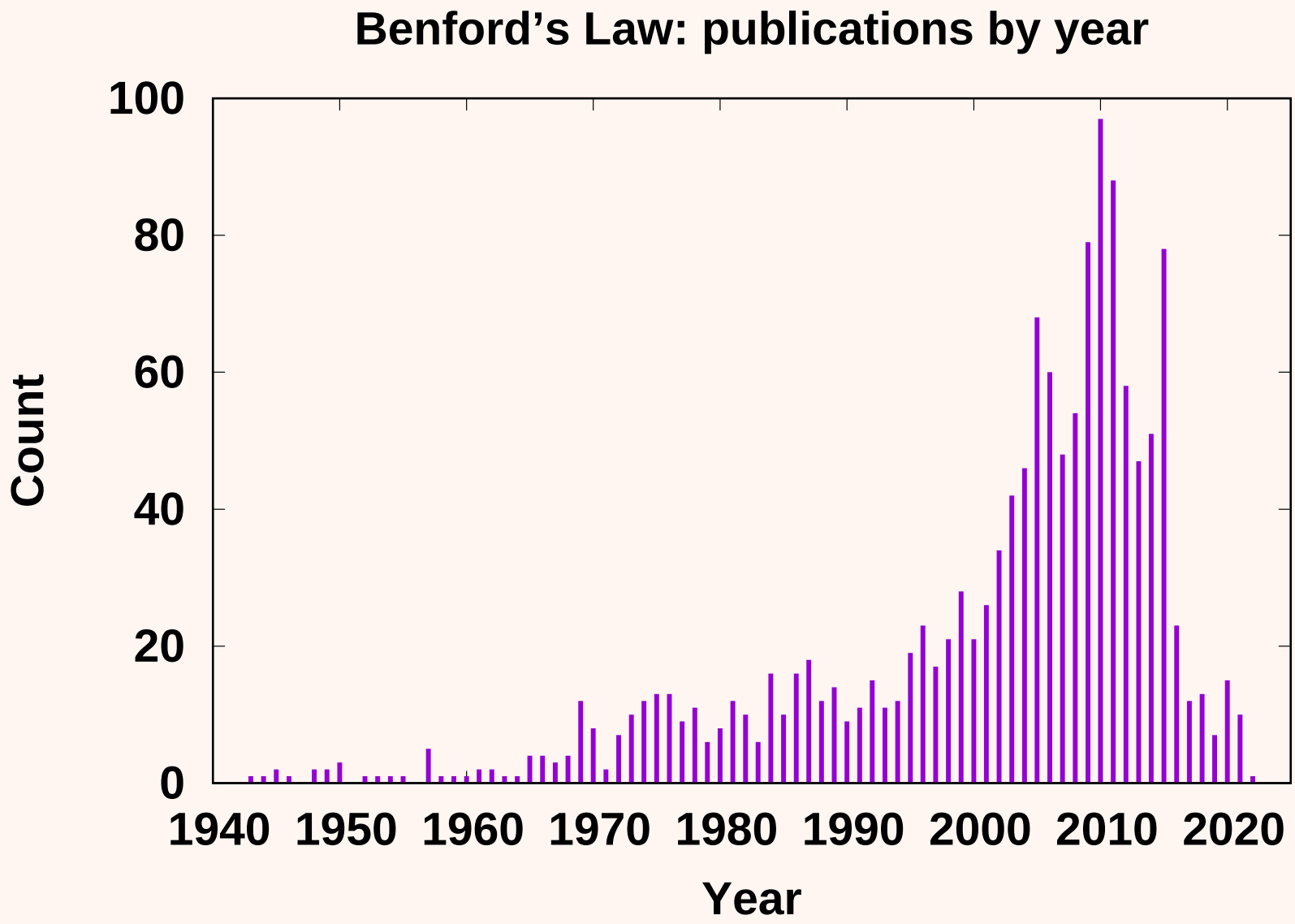
and about **1900** are at

http://www.benfordonline.net/

Benford's Law articles appear in at least **515** journals in these fields:

accounting
aerobiology
auditing
astronomy
bible studies
biology
botany
business
chaos theory
chemistry
computer science
conflict resolution
criminology
demographics
drug design
earthquake detection
economics
electoral studies
engineering
finance
forensics
gambling
geography
geophysics
human resources
imaging science
library science
marketing
mathematics
medicine
networking
neuroscience
nuclear engineering
nuclear science
operations research
physics
probability
psychology
signal processing
simulation
statistics
stock-market trading
volcanology

Benford's Law literature growth



Benford's Law in accounting

Fraud and deception are common when money or politics are involved. However, many who practice in that area are unaware of Benford's Law. Their cooked data may differ sufficiently from the distribution predicted by Benford's Law that their crimes can be detected.

Several tax authorities now use Benford's Law tests in their auditing software to find tax cheats.

Fraud in numerical research data is sometimes suspected, and Benford's Law may help detect it: see **John P. A. Ioannidis**, ***Why Most Published Research Findings Are False***, *PLoS Medicine*, **2**(8) 696–701, August (2005).

However, be sure first that Benford's Law is applicable, and that your statistics are good: see **Andreas Diekmann and Ben Jann**, ***Benford's Law and Fraud Detection: Facts and Legends***, *German Economic Review*, **11**(3) 397–401, August (2010).

A simple test of fraud

A 200-coin-flips experiment should produce six consecutive heads or tails with high probability, but few humans would generate such data.

```
hoc> for (k = 1; k <= 200; ++k) printf("%d", randint(0,1))
```

Three experiments produce (with zeros changed to dots):

```
...1.111.111..1.1.1.1111...1.1.11..1111...1.1.1.1.
.1.....1..11......111.111.1....11..1.1.11..11..1..
11.1..1.11..1....1...1111111.11....1...1....1.1.11
1.1..11111111...111.1....11.111...1....11..1..1111

1.1.1.1.11...1....1.11111......1..1.1...1.....1..1
11..1.1..1.1..1.11.11.1.11......1.1.1.1...1...1.11
..11111111..1.11....1111....1...1111.1.11..11.1...
11111.11111111....1..11111.1.11..11....1.1..1...1.

.11....1111.11.111.11.11.1...1.1111.111..11..1....
111....11.1..11..1..1.11...1...1111.111......11.11
.1.1111...1.111.111.111.11..1..1..111.11.1111...11
..11.1..1..111..11.1..1.11..11.11.1..111111..1.1..
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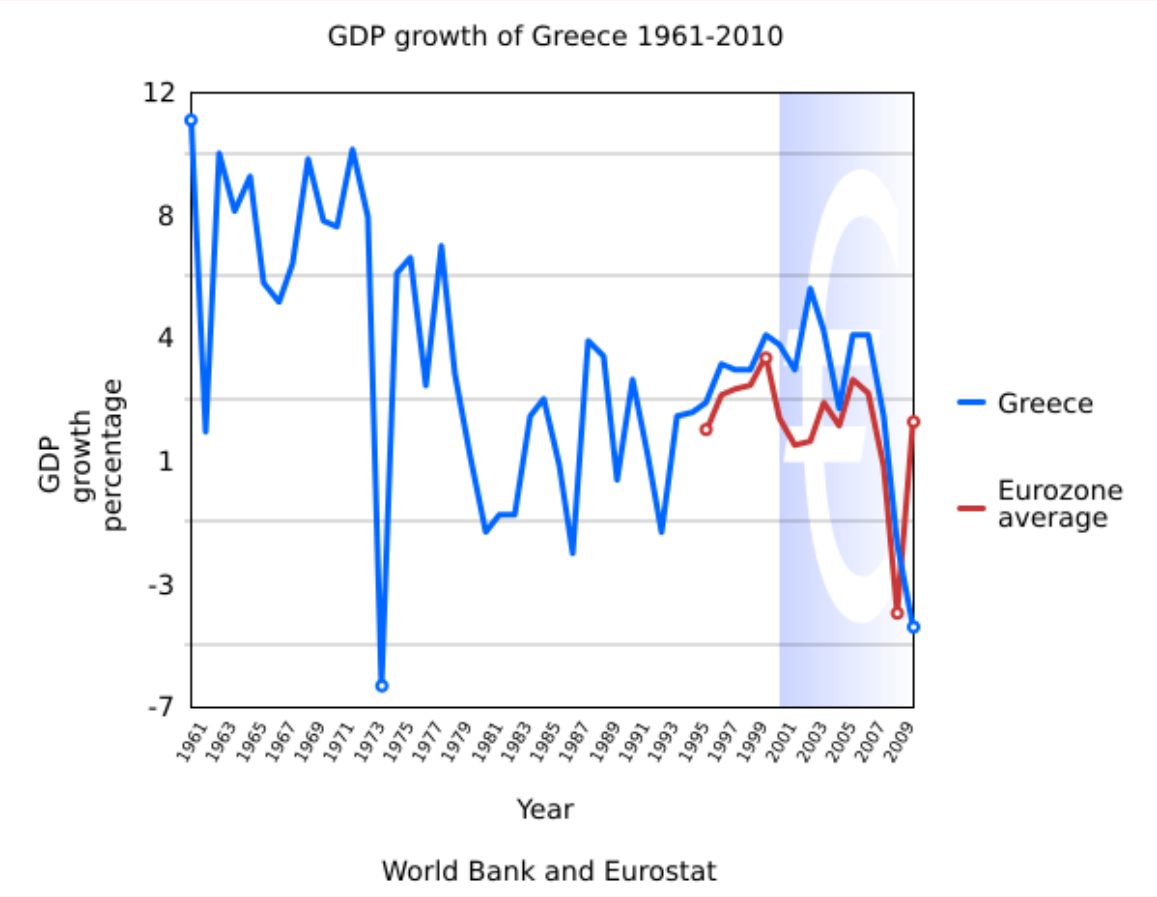
Benford's Law and the 2011 Greek debt crisis



See **Bernhard Rauch et al.**, ***Fact and Fiction in EU-Governmental Economic Data***, *German Economic Review* **12**(3) 243–255, August 2011, and **Hans Christian Müller**, ***How an arcane statistical law could have prevented the Greek disaster*** :

http://economicsintelligence.com/2011/07/28/

How to generate data in Benford's Law distribution?

If a simulation involves dimensioned data whose distribution should be **scale invariant**, then generate starting values from

$$\mathbf{10^{(random\ number\ uniform\ on\ [}a,b\mathbf{])}}$$

Other distributions

Benford's Law has received wide interest and applications, but not all data conform to it.
We look briefly at four other important distributions that model real-world data.

Stigler's Law

In unpublished notes of 1945, and first presented at a 1975 talk at the University of Chicago, **George J. Stigler** (*1982 Sveriges Riksbank Prize in Economic Sciences in Memory of Alfred Nobel*) proposed an alternative distribution of leading digits arising from a more complex formula:

$$F_d = \frac{1}{9}(d\ln(d) - (d+1)\ln(d+1) + (1 + \frac{10}{9}\ln(10)))$$

	1	2	3	4	5	6	7	8	9
Benford	0.3010	0.1761	0.1249	0.0969	0.0792	0.0669	0.0580	0.0512	0.0458
Stigler	0.2413	0.1832	0.1455	0.1174	0.0950	0.0764	0.0605	0.0465	0.0342

See **Joanne Lee, Wendy K. Tam Cho, and George G. Judge**, ***Stigler's approach to recovering the distribution of first significant digits in natural data sets***, *Statistics & Probability Letters*, **80**(2) 82–88, 15 January (2010).

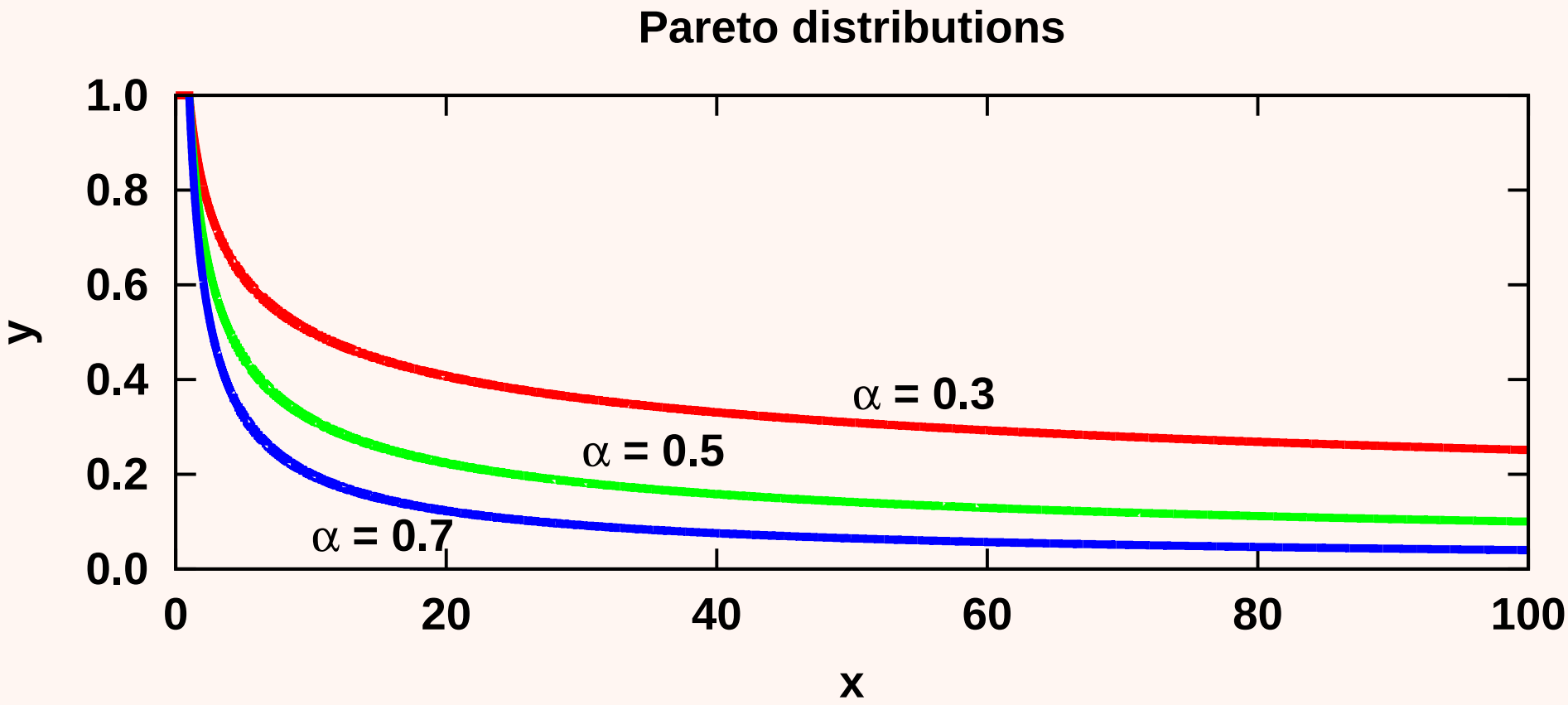
Pareto distribution

Italian economist and mathematician **Vilfredo Federico Pareto** (1848–1923) introduced the 80–20 rule in economics (80% of the wealth is owned by 20% of the people, which was true at the time in Italy, and found to be similar in other countries).
He developed the **Pareto distribution**, in which a random variable X has the property that the probability that it is greater than some number x is given by

$$\Pr(X > x) = \begin{cases} (x_m/x)^{\alpha}, & \text{for } x_m \leq x, \\ 1, & \text{otherwise.} \end{cases}$$

The positive value x_m is a cutoff, and as $\alpha \to \infty$, the Pareto distribution approaches a *Dirac delta function*, $\delta(x - x_m)$. When this models the distribution of wealth, the exponent α is called the **Pareto index**.
Teaser: See online biographies for the relation of Pareto's economic models to the rise of Fascism in Italy in the 1920s.

Pareto distributions pictorially



Zipf's law

In 1932, American linguist **George Kingsley Zipf** (1902–1950) developed a rule that has become known as **Zipf's Law**:

> *If S is some stochastic (random) variable, the probability that S exceeds s is proportional to $1/s$.*

The variable S might be, for example, the population of a city (small cities are more numerous than large ones). See the December 2011 ***National Geographic*** for a story on the dramatic growth of large cities around the world.

Zipf's Law is a special case of the **Pareto distribution**.

See

http://www.nslij-genetics.org/wli/zipf/

for an online bibliography.

Heaps' law

In a 1978 book, ***Information retrieval, computational and theoretical aspects***, **Harold Stanley Heaps** made an empirical observation from linguistics that the proportion of words from a vocabulary grows exponentially with the number of words in the text of documents:

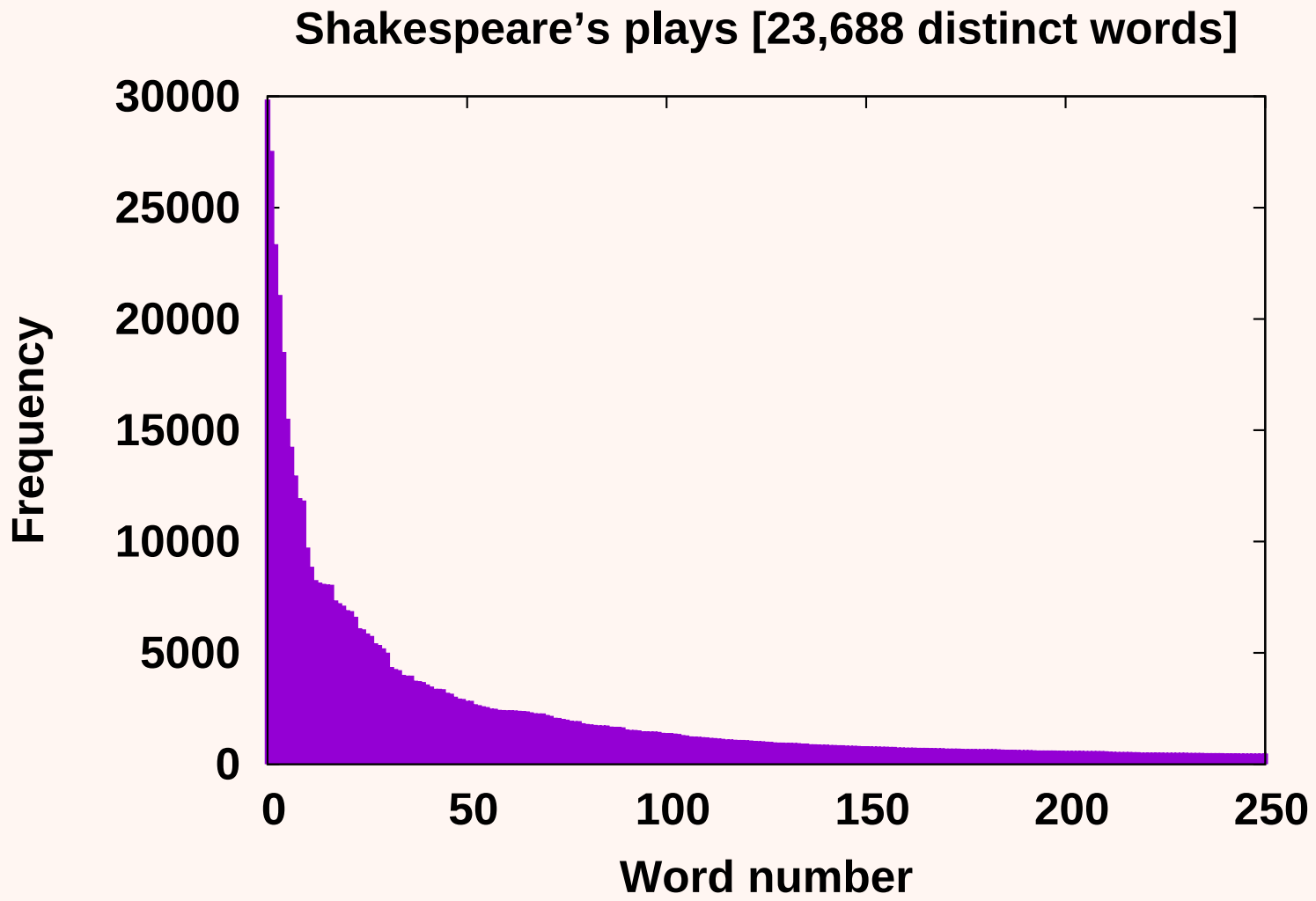
$$V_R(n) = Kn^{\beta}.$$

Here, n is the text size, and K and β are empirical parameters, and for human languages, $\beta \approx 0.4$ to 0.6.
Conclusion: if $\beta < 1$, then increasing n (taking larger and larger samples of text) results in *diminishing returns*. It is hard to find large enough text samples that include all, or even most, of the words in the vocabulary.

Consider what **Heaps' Law** means for Web searches, database retrievals, learning foreign languages, ...

Heaps' Law in Shakespeare

A small vocabulary captures most of the content, but large numbers of words are lost: 16 words (25%), 85 (50%), 250 (64%), 636 (75%), 991 (80%), 5466 (95%), 10455 (98%), 14565 (99%), ...



How to learn more

Many of the important papers on the distributions presented in this talk can be found in

http://www.math.utah.edu/pub/tex/bib/benfords-law.html

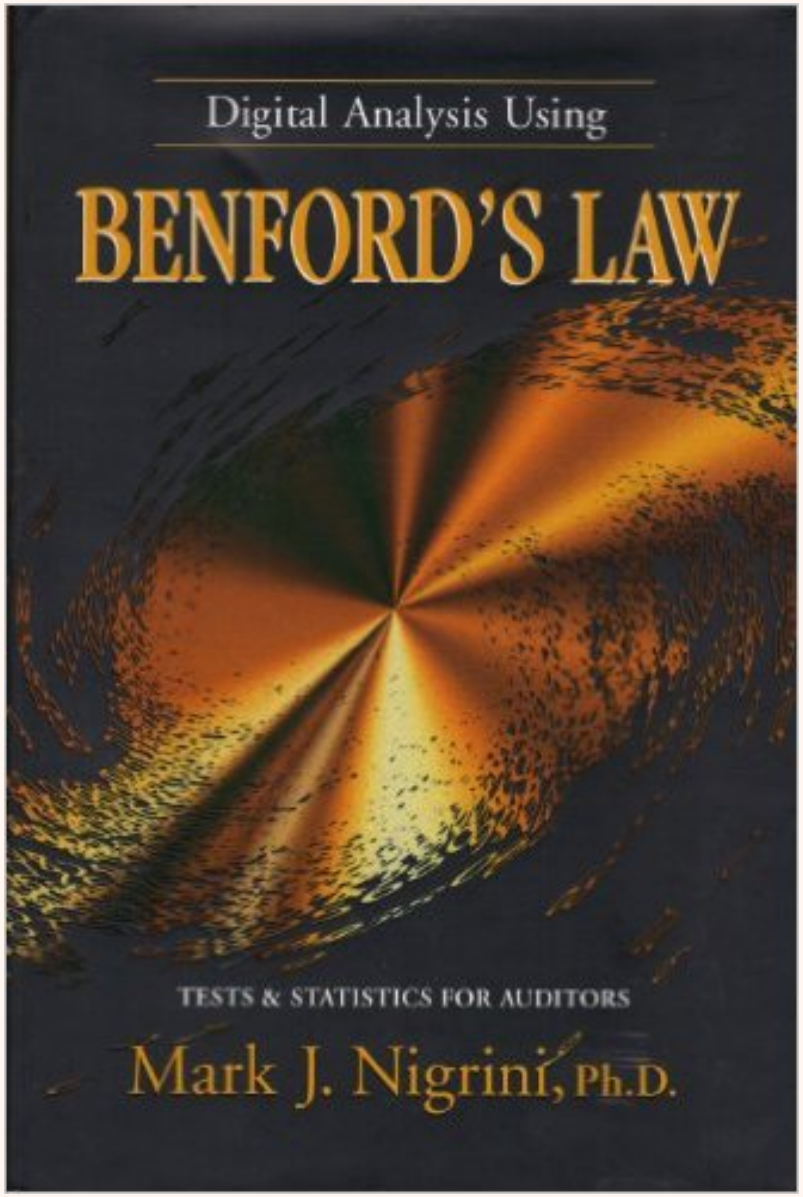
Of particular note is the survey by **Mark E. J. Newman**, ***Power laws, Pareto distributions and Zipf's law***, *Contemporary Physics*, **46**(5) 323–351, September (2005),

http://dx.doi.org/10.1080/00107510500052444

Mathematical details about the current state of Benford's Law research are given by **Arno Berger and Theodore P. Hill**, ***A basic theory of Benford's Law***, *Probabability Surveys* **8** 1–126 (2011),

http://dx.doi.org/10.1214/11-PS175

How to learn more [continued]



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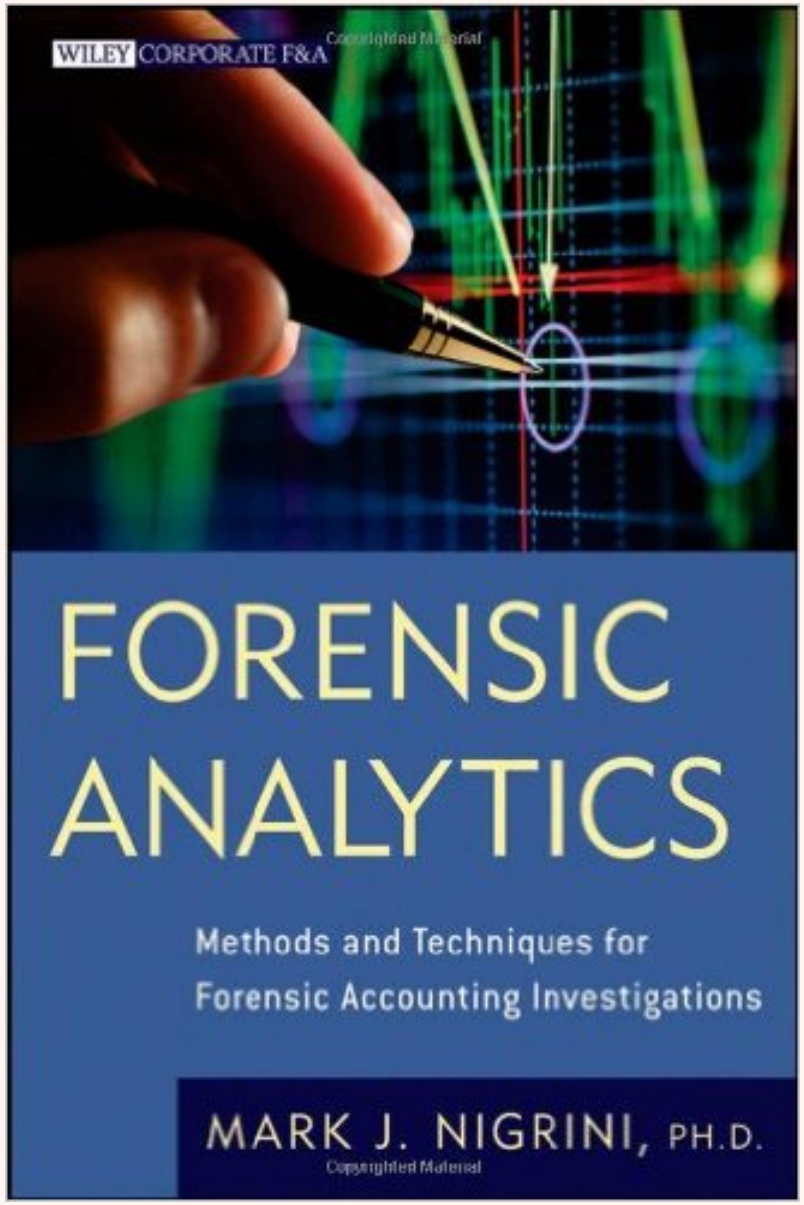

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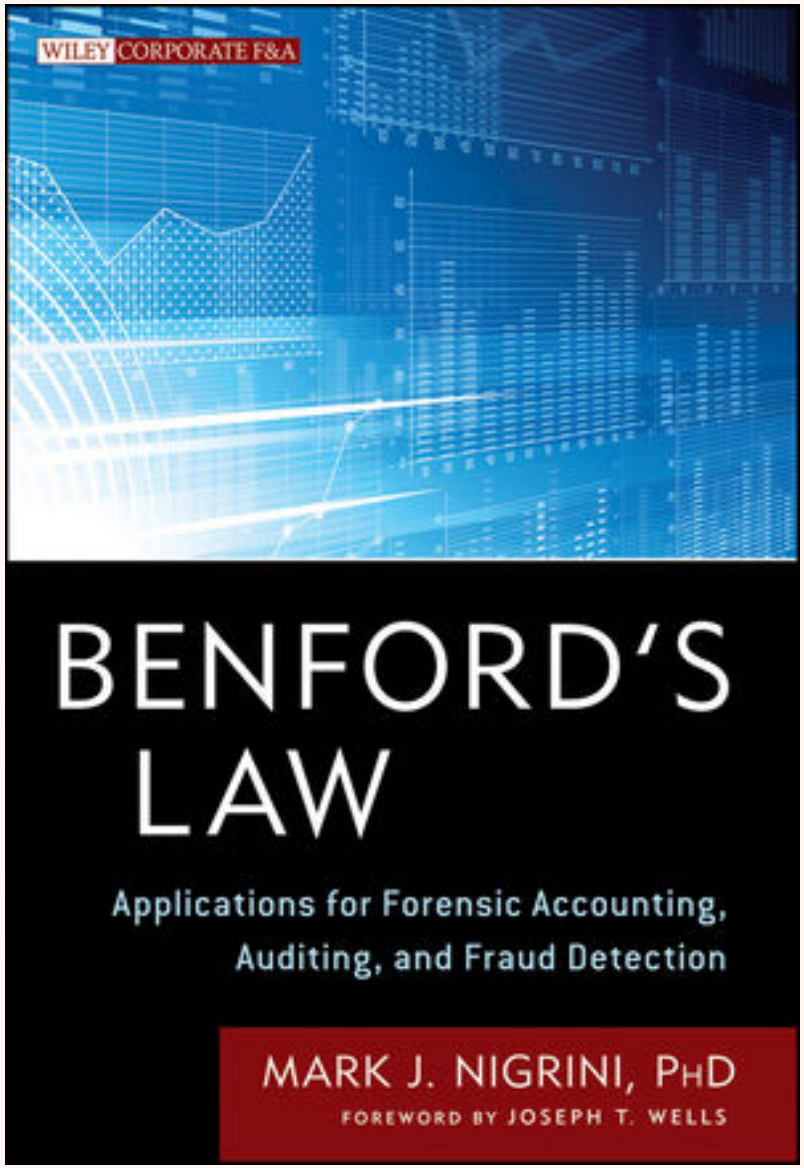
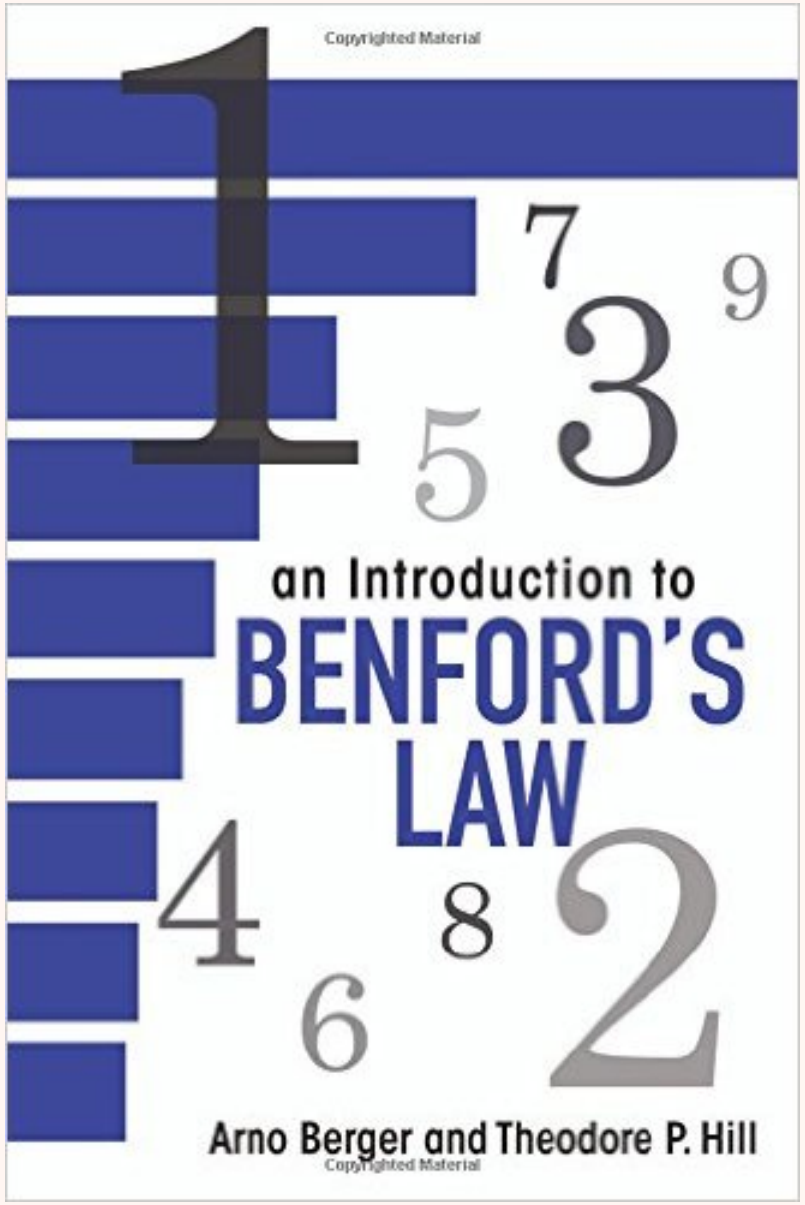
How to learn more [continued]

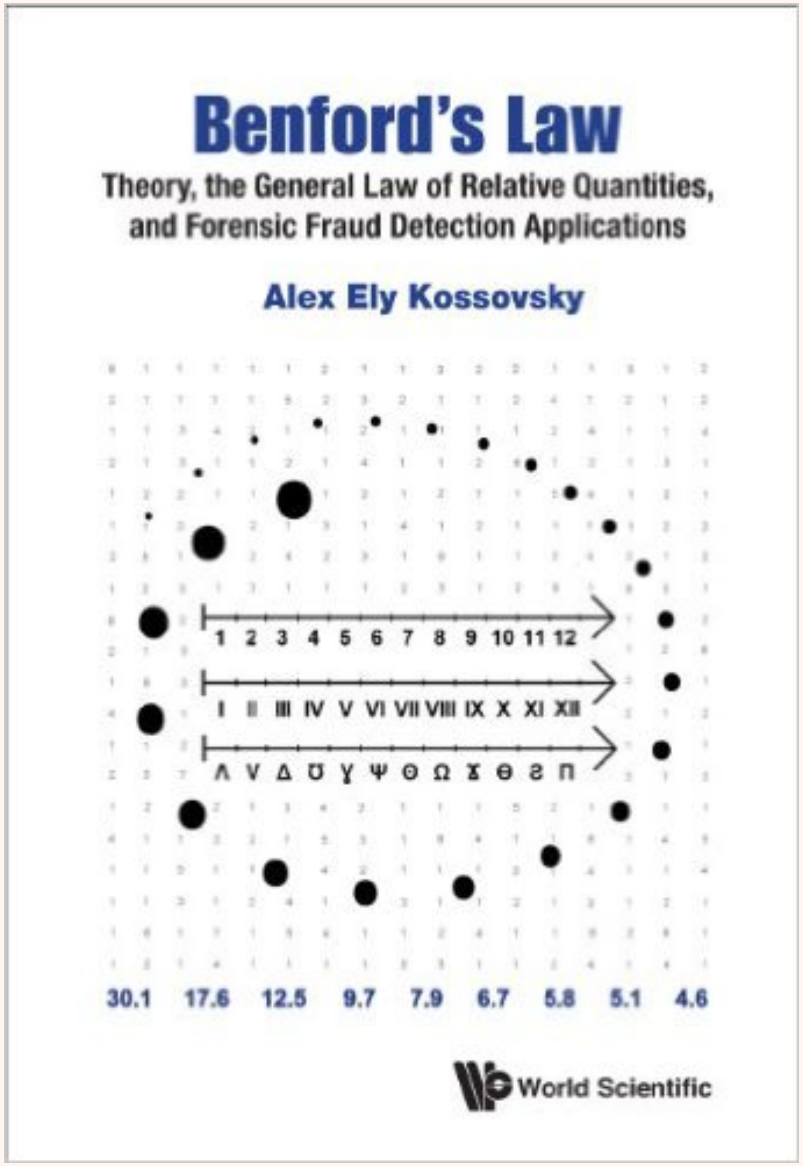


2012

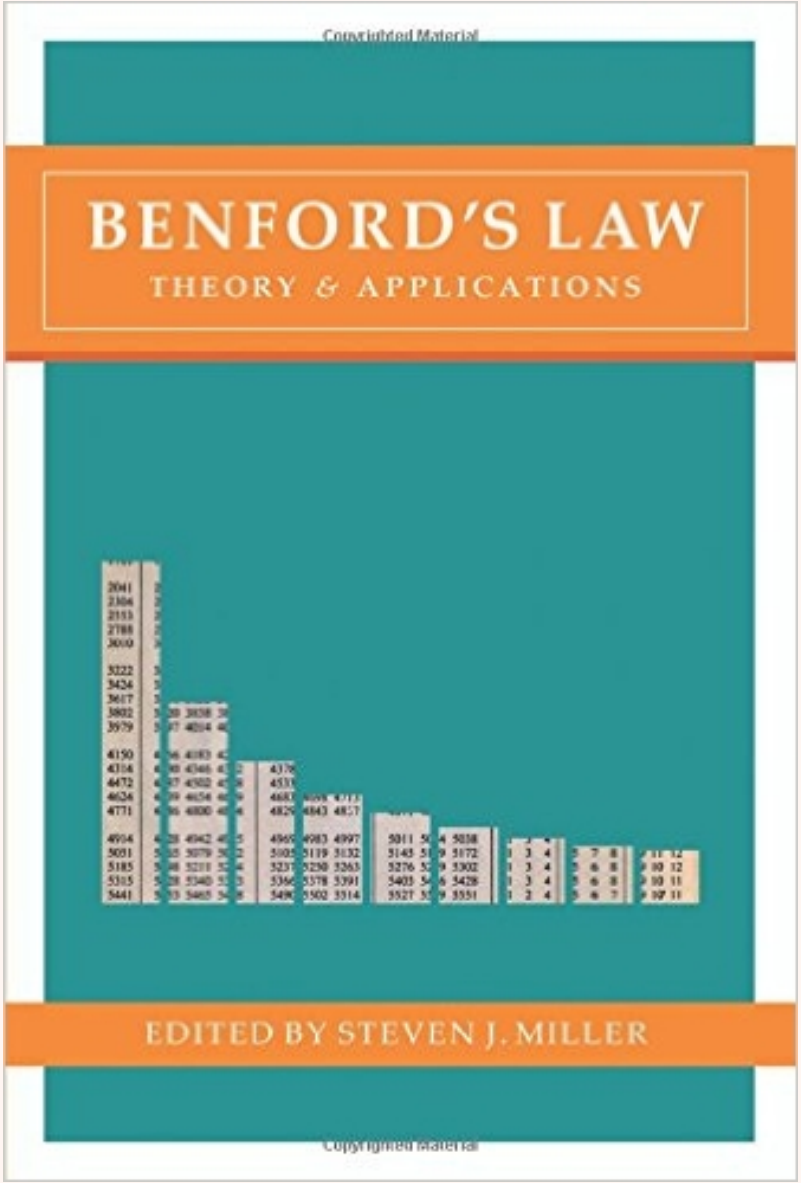


2015

How to learn more [continued]

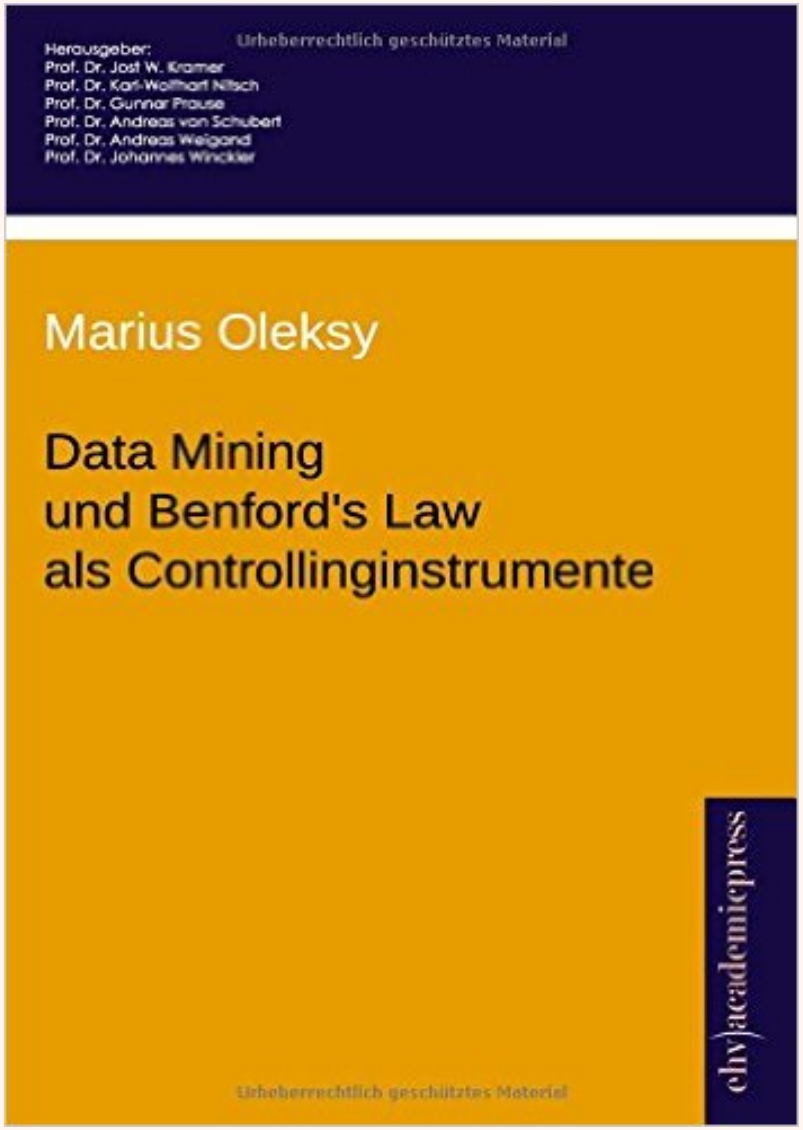


2015



2015

How to learn more [continued]



2014

2015