4800-11

Change of basis

vector spaces

F.

linear

$$\begin{tikzcd} & V \arrow[r, "f"] & V \arrow[dr, "g^{-1}"] & \\ F^n \arrow[ur, "g"] & & & F^n \end{tikzcd}$$

$$g^{-1}\circ f\circ g : F^n \to F^n$$

• Represent as a matrix. A.

Two different bases:

$$g_1 : F^n \to V \qquad \Big\| \qquad f : V \to V$$
$$g_2 : F^n \to V$$

Q: How are the matrices A_1 and A_2 related?

$$\underbrace{(g_2^{-1} g_1)}_{B^{-1}} \; g_1^{-1}\circ f \circ g_1 \; \underbrace{(g_1^{-1}\cdot g_2)}_{B} \qquad A_1$$

$$g_2^{-1}\circ f \circ g_2 \qquad A_2$$

$$F^n \xrightarrow{g_2} V \xrightarrow{g_1^{-1}} F^n$$

$$g_1^{-1} \circ g_2 = B$$

change of basis matrix

!

$$\boxed{A_2 = B^{-1} \circ A_1 \circ B}$$

(conjugation by B)

Note:

$$\det(A_2) = \det(B^{-1}A_1B)$$

$$= \det(B^{-1}) \cdot \det(A_1) \cdot \det(B)$$

$$= \det(A_1).$$

So: det(A) doesn't depend on the choice of basis!

- $f: V \longrightarrow V$

$\det(f) =: \det(A)$

for <u>any</u> <u>choice</u> <u>of</u> <u>basis</u>.

<u>Q:</u> What else is independent of basis?

$$\mathrm{tr}(A) = a_{11} + a_{22} + \dots + a_{nn}$$

$$\left(\mathrm{tr}(I_n) = n, \quad \mathrm{tr}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = 0 \right)$$

...

Proposition:

$$tr(AB) = tr(BA)$$

Proof: $A = (a_{ij})$

$B = (b_{ij})$

$$(AB)_{kk} = \sum_{j=1}^{n} a_{kj} b_{jk}$$

$$(BA)_{kk} = \sum_{j=1}^{n} b_{kj} a_{jk}$$

$$\sum_k (AB)_{kk} = \underline{\underline{\sum_k \sum_j}} a_{kj} b_{jk}$$

$$\sum_k (BA)_{kk} = \underline{\underline{\sum_k \sum_j}} a_{jk} \underline{\underline{b_{kj}}}$$

$$\mathrm{tr}(\underline{A}B) = \mathrm{tr}(B\underline{A})$$

$$\underline{\mathrm{tr}(\underbrace{B^{-1}A}\underbrace{B})} = \mathrm{tr}(\underbrace{B}\,\underbrace{B^{-1}A})$$

$$= \underline{\underline{\mathrm{tr}(A)}}$$

(multi-linear)

$$\det(A) = \sum_{\sigma} \operatorname{sgn}(\sigma) \prod a_{i\,\sigma(i)}$$

$$\operatorname{tr}(A) = \sum_{k} a_{kk} \quad \text{(linear)}$$

---

what other polys. in the (a_{ij})'s

are invariant when

$$A \longleftrightarrow B^{-1}AB \; ?$$

<u>Remember</u>: The <u>char.</u> <u>poly</u>:

$$ch(A) = \det(xI_n - A)$$

$$\parallel$$

$$\det(\underline{B}^{-1}(xI_n - A)\underline{B})$$

$$= \det(x \cdot B^{-1}B - B^{-1}AB)$$

$$= \det(xI_n - B^{-1}AB)$$

$$\underline{\underline{ch(A) \overset{\checkmark}{=} ch(B^{-1}AB)}}$$

$\underline{(2\times 2)}$:

$$\mathrm{ch}\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$$

$$= \det\begin{pmatrix} x-a_{11} & -a_{12} \\ -a_{21} & x-a_{22} \end{pmatrix}$$

$$= (x-a_{11})(x-a_{22}) - a_{12}a_{21}$$

$$= x^2 - x\underline{a_{11}} - x\underline{a_{22}} + a_{11}a_{22} - a_{12}a_{21}$$

$$= x^2 - x(\mathrm{tr}(A)) + \det(A)$$

(3x3)

$$\mathrm{cl}(A) = \det \begin{pmatrix} x-a_{11} & -a_{12} & -a_{13} \\ -a_{21} & x-a_{22} & -a_{23} \\ -a_{31} & -a_{32} & x-a_{33} \end{pmatrix}$$

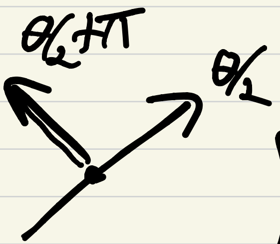
$$= (x-a_{11})(x-a_{22})(x-a_{33}) + \underbrace{\cdots}_{(\deg \le 1 \text{ in } x)}$$

$$= x^3 - \underline{x^2 (\mathrm{tr}(A))} + x(\underbrace{\quad}_{\text{(intermediate term)}}) - \underline{\det(A)}$$

Use ch(A) to find eigenvalues & eigenvectors.

Ex: Reflection of $\mathbb{R}^2$: $x^2 - 1$

$\Longrightarrow \lambda = \pm 1$ only eigenvalues



1

$$\begin{bmatrix} \cos(\theta) & \sin(\theta) \\ \sin(\theta) & -\cos(\theta) \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x \\ y \end{bmatrix}$$

$$\begin{bmatrix} & \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = -\begin{bmatrix} x \\ y \end{bmatrix}$$

Rotation:

$$A = \begin{bmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{bmatrix}$$

$$ch(A) = \det\begin{pmatrix} x-\cos(\theta) & \sin(\theta) \\ -\sin(\theta) & x-\cos(\theta) \end{pmatrix}$$

$$= x^2 - 2x\cos\theta + \cos^2\theta + \sin^2\theta$$

$$= x^2 - 2x\cos\theta + 1$$

Find roots: $$\frac{\cancel{2}\cos(\theta) \pm \overbrace{\sqrt{\cancel{4}\cos^2\theta - \cancel{4}1}}^{i\sin(\theta)}}{\cancel{2}}$$

"Eigenvalues" are:

$$\cos(\theta) \pm i\sin(\theta)$$

$$= \underline{e^{i\theta}} \quad \text{or} \quad \underline{\underline{e^{-i\theta}}}$$

"Eigenvectors" are:

$$\begin{bmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{bmatrix} \begin{bmatrix} 1 \\ i \end{bmatrix} = \begin{bmatrix} \cos(\theta) - i\sin(\theta) \\ \sin(\theta) + i\cos(\theta) \end{bmatrix}$$

$$= \begin{bmatrix} e^{-i\theta} \\ i \cdot e^{-i\theta} \end{bmatrix} = e^{-i\theta} \begin{bmatrix} 1 \\ i \end{bmatrix} \checkmark$$

When $F = \mathbb{R}$, then

replacing $\mathbb{R}^n$ by $\mathbb{C}^n$

and $F = \mathbb{R}$ by $\mathbb{C}$

(with the same matrix A)

is extension of scalars.

(Useful for finding eigenvalues)

Example:

$$A = \begin{bmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 1 \\ i & -i \end{bmatrix} \begin{bmatrix} e^{-i\theta} & 0 \\ 0 & e^{i\theta} \end{bmatrix} \begin{bmatrix} 1 & 1 \\ i & -i \end{bmatrix}^{-1}$$

$$A^n = \begin{bmatrix} 1 & 1 \\ i & -i \end{bmatrix} \begin{bmatrix} e^{-i\theta} & \\ & e^{i\theta} \end{bmatrix}^n \begin{bmatrix} 1 & 1 \\ i & -i \end{bmatrix}^{-1}$$

$$= \begin{bmatrix} & \end{bmatrix} \begin{bmatrix} e^{-in\theta} & 0 \\ & e^{in\theta} \end{bmatrix} \begin{bmatrix} & \end{bmatrix}$$

Def: $f: V \to V$ is semisimple if V has a basis of eigenvectors.

Thm: (a) If $ch(f)$ $\left(= ch(A)\right)$ has n distinct roots $\lambda_1, \dots, \lambda_n$, then $f: V \to V$ is semisimple.

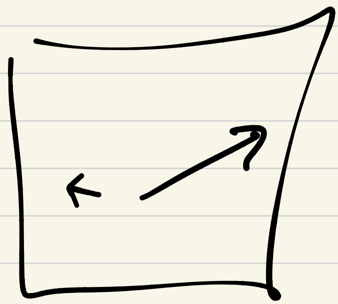
(b) Every orthogonal transformation is semisimple $/ \mathbb{C}$

<u>Pf</u>: (a) If $ch(A)$ has distinct roots $\lambda_1, \dots, \lambda_n$, then each root comes w/ an <u>eigenvector</u>

$$A \cdot \vec{v}_1 = \lambda_1 \vec{v}_1$$

$$A \cdot \vec{v}_2 = \lambda_2 \vec{v}_2$$

$$\vdots$$



<u>Need to show</u>: $\vec{v}_1, \dots, \vec{v}_n$ are <u>linearly independent</u>!

(**)

Suppose $c_1\vec{v}_1 + \cdots + c_n\vec{v}_n = 0$

Then

$$A(c_1\vec{v}_1 + \cdots + c_n\vec{v}_n) \overset{\text{"}}{=} 0$$

(*)

$$c_1\lambda_1\vec{v}_1 + \cdots + c_n\lambda_n\vec{v}_n = 0$$

$(*) - \lambda_1 (**)$:

$$c_2(\lambda_2-\lambda_1)\vec{v}_2 + \cdots + c_n(\lambda_n-\lambda_1)\vec{v}_n = 0$$

proceed by induction! $(\vec{v}_n = 0)$

($f: \mathbb{R}^n \to \mathbb{R}^n$)

(b) Suppose A is orthogonal:

$$|A\vec{v}| = |\vec{v}| \quad \forall \vec{v} \in \mathbb{R}^n$$

Maybe after extending scalars to $\mathbb{C}$:

Find an eigenvalue of $ch(A)$.

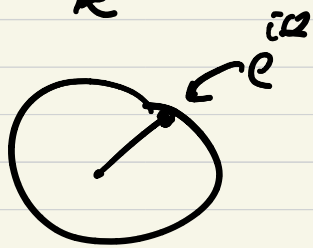
(in $\mathbb{C}$)

$x^n + \ldots \qquad \lambda$ root

Notice: $|\lambda| = 1$ because $|A\vec{v}| = |\lambda\vec{v}| = |\vec{v}|$

If $\lambda \in \mathbb{R}$, then $\lambda = \pm 1$

If $\lambda \in \mathbb{C}$, then $\lambda = e^{i\theta}$



<u>Suppose</u> $\lambda \in \mathbb{R}$. $\underline{\underline{\lambda = \pm 1}}$

<u>Then</u>

find an eigenvector $\vec{v}$.

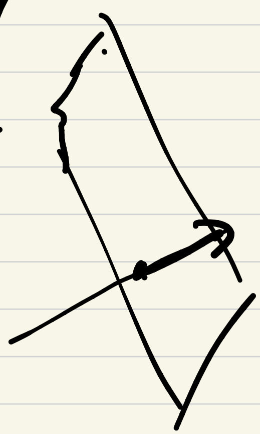
$$\underline{\underline{\left[\ A\vec{v} = \pm \vec{v}\ \right]}}$$

<u>Because</u> A is <u>orthogonal</u>,

$$A = \begin{bmatrix} \vec{v}_1 & \cdots & \vec{v}_n \end{bmatrix}$$

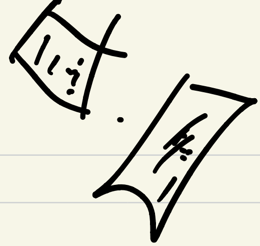
<u>orthonormal</u>

$\underline{W^{\perp}}$



$W = \mathbb{R} \cdot \vec{v}$

$$A\vec{v} = \pm \vec{v}$$



A orthogonal, and

$A : \underline{\underline{W}} \to \underline{W}$

for $W \subseteq V$, then

$\underline{A} : \underline{\underline{W}}^{\perp} \to \underline{\underline{W}}^{\perp}$ *

So we can do induction

<u>assume</u>: $A : W^{\perp} \to W^{\perp}$,

is <u>semisimple</u>.

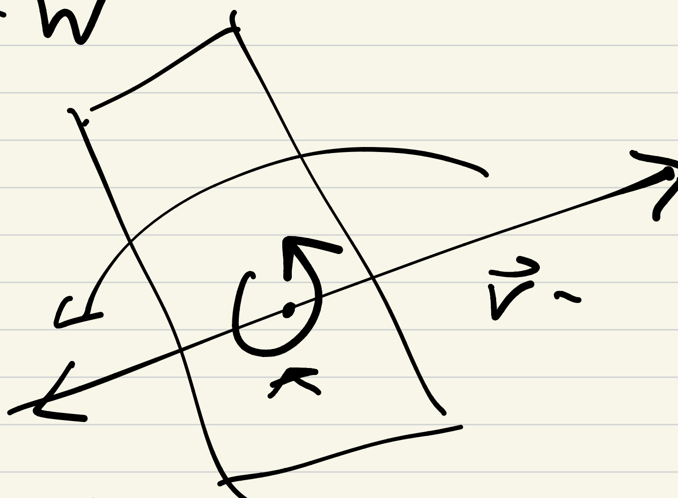
In $\mathbb{R}^3$:

$$\boxed{\begin{array}{l} A: W \to W \\ \Rightarrow A: W^\perp \to W^\perp \end{array}}$$

$ch(A) = x^3 + \_\_\_$

λ <u>real</u> <u>root</u> + eigen<u>vector</u>

$\mathbb{R}^2 = W^\perp$



$\to W = \mathbb{R}^1$

$\vec{v}$ ~ <u>axis</u>

$A: W^\perp \to W^\perp$ <u>orthogonal</u>: