4800-14

More groups:

Conjugation:

$$* \quad \rho_c : G \longrightarrow Aut(G) \quad *$$

$$\rho_c(g)(h) = g h g^{-1}$$

This breaks G up into conjugacy classes

$G = S_3$; $\{1\}$, $\{(1\,2),(1\,3),(2\,3)\}$

$\{(1\,2\,3),(1\,3\,2)\}$.

The conjugacy class containing h $\quad h^G$

is $\{ghg^{-1} \mid g \in G\}$

- Abelian groups $\quad \{h\}$

- Dihedral gps:

$\{1\}, \{x, x^{-1}\}, \{x^2, x^{-2}\}, \sim$

$\{y, x^2y, \dots\}$ $\{y, xy, \dots\}$

- $O(2, \mathbb{R})$: rotations $\quad S^1$

$\quad\quad\quad\quad\quad\quad \shortparallel \quad\quad\quad\quad\quad \shortparallel$

$SO(2,\mathbb{R})$ $\quad \boxed{O^-(2,\mathbb{R})}$

$\quad\quad \cup$

$\{1\}, \{r_\theta, r_{-\theta}\} \dots$ $\quad$ one class

Given: $f : G \to H$

a group homomorphism, then

- $I = f(G) \subset H$ is the image subgroup
- $K = f^{-1}(1) \subset G$ is a **normal** subgroup

i.e. $\underline{K}$ is a union of conjugacy classes

if $\underline{k} \in K$ and $\underline{g} \in G$ then $gkg^{-1} \in K$

if $f(k) = 1$, then

$$f(gkg^{-1}) = f(g)\,\cancel{f(k)}\,f(g^{-1})$$

$$= f(g) \cdot f(g^{-1})$$

$$= f(1) = 1$$

Examples: Normal

S_3: $\{1, \underbrace{(1\,2\,3), (1\,3\,2)}_{\text{conj. class}}\}$

conj. class

S_3: $\{1, (1\,2)\}$ not Normal

not a conj. class

S_4 : $K_4 = \{1, (12)(34), (13)(24), (14)(23)\}$

conj. class

Normal

$A_4 = \{1, (123), (132), (124), (142), (134), (143), (234), (243), (12)(34), (13)(24), (14)(23)\}$

$A_4 = \ker(\mathrm{sgn})$; $\mathrm{sgn}: S_4 \to \{\pm 1\}$

$C_n \subset D_{2n}$ is <u>normal</u>

$$(x^l y)\, x^k (x^l y)^{-1} = x^{-k}.$$

$\{1, y\}$ is not <u>normal</u> (for $n > 2$)

$$x y x^{-1} = x^2 y$$

<u>Def</u>: Given a <u>subgp.</u> $H \subseteq G$, there are right and left cosets of H:

Left cosets:

$$gH = \{g \cdot h \mid h \in H\}$$

Right cosets

$$Hg = \{h \cdot g \mid h \in H\}.$$

Observation: If G is finite, then

G is a disjoint union of left cosets gH, and they all have $|H|$ elements.

$$\underline{g_1 h_1} = \underline{g_2 h_2}$$

$$\Rightarrow \quad g_2 = g_1 h_1 h_2^{-1}.$$

so $\quad g_2 \in g_1 H$

and $\quad g_1 \in g_2 H$

so $\quad \underline{\overbrace{g_1 H} = g_2 H}$

$$g_1 h_1 = g_1 h_2 \Rightarrow \underline{h_1 = h_2} \quad X.$$

$$|G| = |H| \,(\#\text{ of left cosets})$$

$$\Longrightarrow \boxed{|H| \text{ divides } |G|}.$$

In particular, if $g \in G$, then the order of g divides $|G|$.

E.g. What are the subgroups of C_p?

p prime

$\{1\}, C_p$

$$\boxed{|H| \text{ divides } |G| \quad \underline{\text{Lagrange's}}\ \underline{\text{Thm}}}$$

Notation: $\rightarrow H \subset G$ subgp.

$K \trianglelefteq G$ normal subgp.

Proposition:

(a) If $K \triangleleft G$ then the left and right cosets of K are the same (!)

*(b) If $K \triangleleft G$, then

$\rightarrow (g_1 K)\cdot(g_2 K) = (g_1 g_2)K$

makes the left cosets into a group!

Pf (a)

$$gK = Kg \quad ?$$

$gkg^{-1} = k'$ because K is normal.

$$gK \subseteq Kg$$

So $\underline{gk = k'g}$, some k'.

$$\underline{\underline{gK \subseteq Kg \subseteq gK}}$$

(3) To define:

$$\text{"}(g_1 K)\cdot(g_2 K)\text{"} = (g_1 g_2)K,$$

we need to check:

$$g_1 K = g_1' K$$

$$g_2 K = g_2' K,$$

then $g_1 g_2 K = g_1' g_2' K$

$$(g_1 K)(g_2 K) = (g_1 K)(K g_2) \checkmark$$

$$= g_1 K g_2 = g_1 g_2 K.$$

Example:

$$K_4 \trianglelefteq S_4$$

$$\begin{array}{ccc} \uparrow & & \uparrow \\ 4 & & 24 \end{array}$$

$$S_4 / K_4 = \text{gp. of cosets}$$

(6 element)

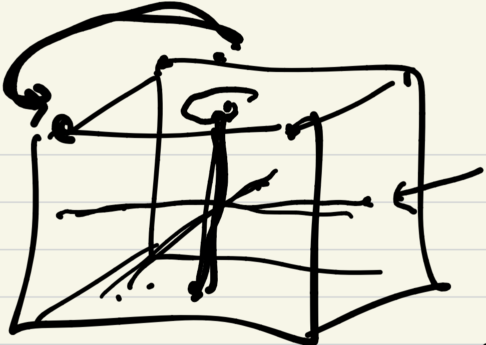
$$\| $$

$$S_3$$

✓

$$S_4 \xrightarrow{f} S_3$$

$$\boxed{\ker(f) = K_4}$$



$$G = \mathrm{Aut}(\mathrm{Cube}) \xrightarrow{f} \mathrm{Aut}([3]) = S_3$$

$\mathrm{Aut}(\mathrm{Cube}) \cong S_4$

$[3] \longleftrightarrow \{$axles of the cubes$\}$

$f^{-1}(1) = \{1$, 180° rotations about each face$\}$

$\|$

K_4!

Unitary Gps (analogous to orthogonal gps)

· Hermitian inner product on $\mathbb{C}^n$:

(generalizes $z \cdot \bar{z} = |z|^2$)

$$\underbrace{(z_1, \dots, z_n)}_{= \vec{z}}, \underbrace{(w_1, \dots, w_n)}_{= \vec{w}} \in \mathbb{C}^n$$

$$\langle \vec{z}, \vec{w} \rangle = \sum_{i=1}^{n} z_i \overline{w_i} \in \mathbb{C}$$

$$\langle \vec{z}, \vec{z} \rangle = \sum z_i \overline{z_i}$$

$$= \sum |z_i|^2 > 0$$

<u>unless</u> $\vec{z} = \vec{0}$

<u>Note:</u>

$$\langle \vec{z}, \vec{w} \rangle = \overline{\langle \vec{w}, \vec{z} \rangle}$$

$$\sum z_i \overline{w_i} \xrightarrow{\text{conj.}} \sum w_i \overline{z_i}$$

$$\langle c\vec{z}, \vec{w} \rangle = c \langle \vec{z}, \vec{w} \rangle$$

$$\langle \vec{z}, c\vec{w} \rangle = \overline{c} \langle \vec{z}, \vec{w} \rangle.$$

A $\mathbb{C}$-linear map:

$$f: \mathbb{C}^n \to \mathbb{C}^n$$

is <u>unitary</u> if

$$\langle f(\vec{z}), f(\vec{w}) \rangle = \langle \vec{z}, \vec{w} \rangle.$$

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f is a symmetry of the unit sphere in $\mathbb{C}^n$

$$S^{2n-1} = \{ \overset{\vec{z}}{\overset{\parallel}{(z_1, \ldots, z_n)}} \mid \langle \vec{z}, \vec{z} \rangle = 1 \}$$

$$(z_1, \dots, z_n) = (s_1 + it_1, \dots, s_n + it_n)$$

$$\langle \vec{z}, \vec{z} \rangle = \underline{s_1^2 + t_1^2 + \dots + s_n^2 + t_n^2}$$

$$=$$

$$1$$

$$S^1 \subset \mathbb{C}$$

$$S^3 \subset \mathbb{C}^2$$

$$S^5 \subset \mathbb{C}^3$$

f is a symmetry of S^{2n-1}

<u>and</u> if $\vec{z} \perp \vec{w}$ (i.e. $\langle \vec{z}, \vec{w} \rangle = 0$)

then $f(\vec{z}) \perp f(\vec{w})$.

To specify a unitary

$n \times n$ matrix A:

$$A = \left(\vec{u}_1, \vec{u}_2, \cdots, \vec{u}_n \right)$$

unit vectors

$|\vec{u}_i| = 1$, $\langle \vec{u}_i, \vec{u}_j \rangle = 0$

$$\begin{pmatrix} 1 \\ 0 \\ \vdots \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ \vdots \\ 0 \end{pmatrix}, \cdots$$

$$\underbrace{\begin{pmatrix} \vec{u}_1 & \cdots & \vec{u}_n \end{pmatrix}}_{A} \underbrace{\begin{pmatrix} \overline{\vec{u}_1} \\ \vdots \\ \overline{\vec{u}_n} \end{pmatrix}}_{\overline{A}^T} = \underbrace{\begin{pmatrix} \langle \vec{u}_i, \vec{u}_j \rangle \end{pmatrix}}_{I_n}$$

i.e. $$\boxed{A\overline{A}^T = I_n}$$

$$\Rightarrow \det(A) \cdot \underbrace{\det(\overline{A}^T)}_{\substack{= \det(\overline{A}) \\ = \overline{\det(A)}}} = 1$$

$$\det(A) = e^{i\theta}$$

$$\{e^{i\theta}\} = U(1)$$

$$\det : \{\text{unitary matrices}\} \to U(1)$$

Def: $U(n)$ unitary $n\times n$ matrices

$SU(n)$ unitary matrices of $\det = 1$

$$0 \to SU(n) \to U(n) \xrightarrow{\det} U(1) \to 0$$

$$SU(n) \trianglelefteq U(n)$$

Thm: $SU(2) = S^3$

More precisely,

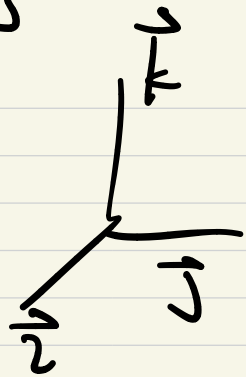
$SU(2) =$ gp. of unit quaternions (in $\mathbb{H}$)

$$\mathbb{H} = \mathbb{R} + \mathbb{R}\vec{i} + \mathbb{R}\vec{j} + \mathbb{R}\vec{k}$$

non-commutative field.
(division algebra)

$$\vec{i}^2 = \vec{j}^2 = \vec{k}^2 = -1$$
$$\vec{i}\vec{j} = \vec{k}, \ \vec{j}\vec{i} = -\vec{k}, \ \text{etc}.$$

$$\mathbb{H} \ni a + \overset{\in \mathbb{R}^3}{\vec{v}}$$



$$\left[\begin{aligned} &(a+\vec{v}) \cdot (b+\vec{w}) \\ &= \underline{ab} + \underline{a\vec{w}} + \underline{b\vec{v}} + \underline{\vec{v} \times \vec{w}} \end{aligned} \right]$$

$\underline{SU(2)}$ = unit quaternions.

$$SU(2) \longrightarrow \mathrm{Aut}(SU(2))$$

<u>conjugacy classes</u>!