4800-21

$Z[G]$ = vector space of class functions on G

$\alpha : G \to \mathbb{C}$ is in $Z[G]$

if α is constant on conjuclasses.

(ie. $\alpha(g) = \alpha(hgh^{-1}) \; \forall h$)

<u>E.g.</u> $G = S_3$

$C_1 = \{\underline{id}\}$, $C_2 = \{\underline{(12), (13), (23)}\}$

$C_3 = \{\underline{(123), (132)}\}$

$\alpha(id) \in \underline{\mathbb{C}}$

$\alpha(12) = \alpha(13) = \alpha(23) \in \underline{\mathbb{C}}$

$\alpha(123) = \alpha(132)$

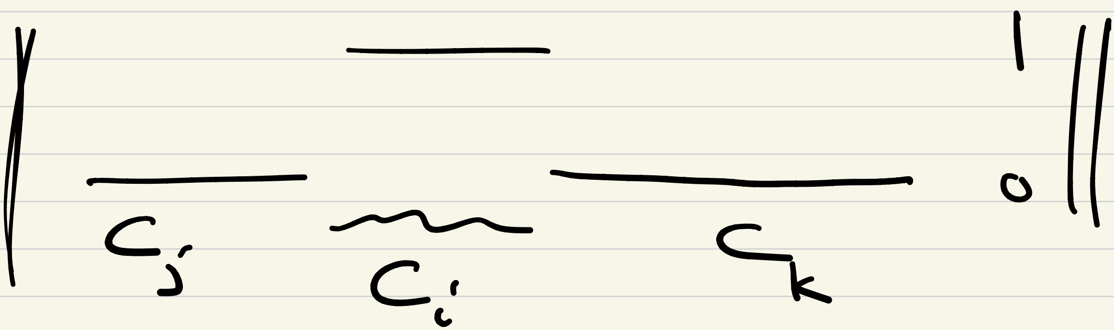
$\wedge$

$\underline{\mathbb{C}}$

dim $Z[G]$ = # of conj. classes.

Basis: Step functions

$$\delta_i(g) = \begin{cases} 1 & \text{if } g \in C_i \\ 0 & \text{if } g \notin C_i \end{cases}$$



$$\left[\ \alpha = \sum_i \lambda_i \delta_i \ \right]$$

(value of α on $g \in C_i$)

Character tables:

Expand χ_ρ (irred rep) in terms of δ_i

$$\underline{\chi_\rho(g)} = tr(\rho(g))$$

$$= tr(\rho(\underline{hgh^{-1}}))$$

so $\chi_\rho \in Z[G]$.

$$\left[\begin{array}{lccc} & \underset{1}{id} & \underset{3}{(1\,2)} & \underset{2}{(1\,2\,3)} \\ \chi_\rho = & 2\delta_1 + & \underline{0\delta_2} & \underline{-1\delta_3} \end{array}\right]$$

Inner product on

$Z[G]$:

$$\overline{(\alpha, \beta)} = \frac{1}{|G|} \sum_{g \in G} \alpha(g) \cdot \overline{\beta(g)}$$

class functions

E.g. $S_3 = G$

$$\underline{(\delta_i, \delta_j)} = \frac{1}{|G|} \sum_{g \in S_3} \underline{\delta_i(g)} \cdot \underline{\delta_j(g)}$$

$$= 0$$

$$(\delta_i, \delta_i) = \frac{1}{|G|} \cdot \sum_{g \in C_i} 1^2 = \frac{|C_i|}{|G|}$$

Thm: The characters of irreps of G form an orthonormal basis of $\underline{Z[G]}$.

In particular, $\#\,\underline{\text{irreps}} = \#\,\underline{\text{conj. classes}}$

Preliminaries:

Given (V_1, ρ_1) and (V_2, ρ_2) two reps. of G. Then:

$\underline{V_1} \oplus \underline{V_2}$ has basis

$\underline{e_i}, \underline{f_j}$ $\Rightarrow \chi_{V_1 \oplus V_2} = \chi_{V_1} \oplus \chi_{V_2}$

basis for V_1 basis for V_2

$$\chi_{V_1 \oplus V_2} = \chi_{V_1} + \chi_{V_2}$$

$$\chi_{V_1 \otimes V_2} = \chi_{V_1} \cdot \chi_{V_2}$$

$\hom(V_1, V_2)$ is a <u>G-rep</u>

$f: V_1 \to V_2$

in V_2

$$(g \cdot f)(v_1) = g \cdot f(g^{-1} v_1)$$

$$\underline{(gh)} \cdot f = (g \cdot h) \cdot f((gh)^{-1} v_1)$$

$$= g \cdot h \cdot f(h^{-1}(g^{-1} v_1))$$

$= g(h \cdot f)$?!

$$\boxed{\chi_{\hom(V_1, V_2)}(g) = \chi_{V_1}(g^{-1}) \cdot \chi_{V_2}(g)}$$

$$\left[\hom(V_1, V_2) = V_1^{\vee} \otimes V_2\right]$$

<u>Key idea:</u> Given $\rho: G \to \mathrm{Aut}(V)$

<u>Construct a linear map:</u>

$$p(v) = \frac{1}{|G|} \sum_{g \in G} g \cdot v$$ ← <u>average</u>

$$p(v) = \frac{1}{|G|} \sum g \cdot v$$

$$p(h \cdot v) = \frac{1}{|G|} \sum_{g \in G} (g \cdot h) \cdot v$$

$$= \frac{1}{|G|} \sum_{g \in G} g \cdot v$$

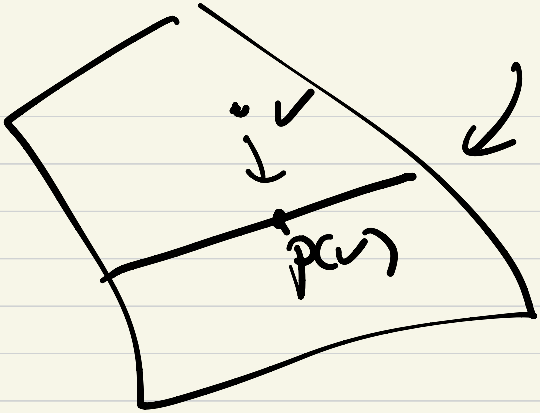
$$= p(v)$$

$$h \cdot p(v) = \frac{1}{|G|} \sum_{g \in G} h \cdot (g \cdot v)$$

$$= \frac{1}{|G|} \sum_{g \in G} (h \cdot g) v = p(v).$$

<u>so</u> $h \cdot p(v) = p(v)$ is <u>invariant</u>.

G acts trivially



$$p(p(v)) = p(v)$$

This is called an idempotent:

$$p : V \longrightarrow V$$

$$\cup$$

$$p(V)$$

$$p = \left[\begin{array}{ccc|c} 1 & & & \\ & \ddots & & * \\ & & 1 & \\ \hline & 0 & & 0 \end{array}\right]$$

$p(V)$: $p(e_1) = e_1, \ldots, p(e_k) = e_k$

$[\operatorname{tr}(p) = \dim(p(V)) = k]$

$$\boxed{p(v) = \frac{1}{|G|} \sum_{g} g \cdot v}$$

$$\boxed{\begin{aligned} tr(p) &= \dim(p(V)) \\ &= \frac{1}{|G|} \sum \underbrace{\chi_V(g)}_{} \end{aligned}}$$

$tr(\cdot g)$ $=$ $\chi_V(g)$

Let V_1, V_2 be irreps,

and consider

$$\hom(V_1, V_2)$$

$$\boxed{\dim \hom_G(V_1, V_2)} = \frac{1}{|G|} \sum \chi_{V_1}(g^{-1}) \cdot \chi_{V_2}(g)$$

Schur's Lemma $= 0$ or 1

Get:

$$0 = \frac{1}{|G|} \sum_g \chi_{V_1}(g^{-1}) \chi_{V_2}(g)$$

if $V_1 \not\simeq V_2$

$$1 = \frac{1}{|G|} \sum_g \chi_{V_1}(g^{-1}) \chi_{V_2}(g)$$

if $V_1 \simeq V_2$

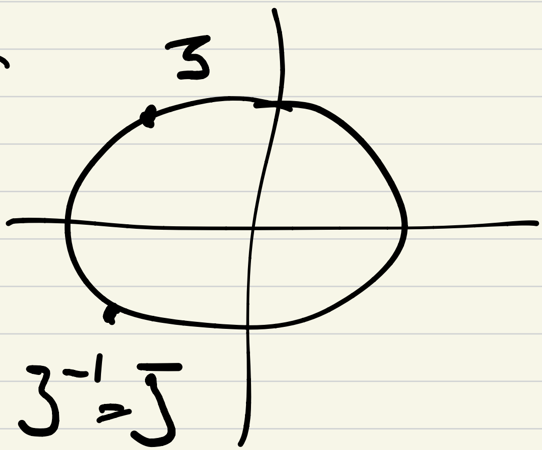
Rmk: $\chi_V(g^{-1}) = \overline{\chi_V(g)}$ (!)

$\mathrm{tr}(g^{-1}\cdot) = \overline{\mathrm{tr}(g\cdot)}$ why?

$$g = \begin{bmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{bmatrix}$$

since $g^{\cancel{\alpha}d} = 1$ so $\lambda_i^{d} = 1$

$$g^{-1} = \begin{bmatrix} \lambda_1^{-1} & & \\ & \ddots & \\ & & \lambda_n^{-1} \end{bmatrix} = \overline{\begin{bmatrix} \lambda_1 & & \\ & \ddots & \\ & & \overline{\lambda_n} \end{bmatrix}}$$



if $\zeta^d = 1$

From this, we get:

$$0 = \frac{1}{|G|} \sum \overline{\chi_{V_1}(g)} \cdot \chi_{V_2}(g)$$

$$= (\chi_{V_1}, \chi_{V_2}) \quad \text{if } V_1 \neq V_2$$

$$1 = \frac{1}{|G|} \sum \overline{\chi_{V_1}(g)} \chi_{V_1}(g)$$

$$= (\chi_{V_1}, \chi_{V_1})$$

They are orthonormal!

(1) Averaging map $P = \frac{1}{|G|} \sum g$.

(2) Character of $\mathrm{hom}(V_1, V_2)$

(3) Schur's Lemma, (4) $\overline{\chi(g^{-1})} = \chi(g)$.

Why are they a basis?

Leave for now!

Application: Hand me a rep.

$$(V,\rho) \rightsquigarrow \text{compute } (\chi_V,\chi_V).$$

If $(\chi_V,\chi_V)=1$, then V is irred!

$$\left(V=\bigoplus n_i U_i\right)$$

$$(\chi_V,\chi_V)=\left(\sum n_i \chi_{U_i}, \sum n_i \chi_{U_i}\right)$$

$$=\sum n_i^2 = 1 \Leftrightarrow V=U.$$

Application: Suppose U is simple

Then $(\chi_V, \chi_U) = n$

$\Rightarrow V = nU \oplus W$

$(\underline{nU} \oplus W, \underline{U}) = n + 0$

Concrete instance of this?

$\underline{\underline{S_4}}$: permutation rep.

$\underline{4 \text{ dim}}$

$$\sigma \cdot e_i = e_{\sigma(i)}$$

$$\begin{bmatrix} 0 & 1 & & \\ 1 & 0 & & \\ & & 1 & \\ & & & 1 \end{bmatrix}$$

	(id)	(12)	$(12)(34)$	(123)	(1234)
ρ_{perm}	4	2	0	1	0

$\langle e_1+e_2+e_3+e_4 \rangle$

<u>so</u> $\rho_{perm} = \underline{\underline{U_1}} \oplus \underline{\underline{U_2}}$

$$(\chi_V, \chi_\rho) = 4^2 + 2^2 \cdot 6 + 0^2 \cdot 3 + 1^2 \cdot 8 \cdot 0$$

$$= 16 + 24 + 8 = 48 / 24 = \boxed{2}$$

$$\rho_{perm} = \chi_{triv} \oplus \underline{\rho}_{\boxed{}\boxed{}\boxed{}}$$

$\Big[\ \rho = 3$ dim'l repn of $S_4\ \Big]$

Quest: Char tables for A_5, S_5