# 48co-6

# 4800-6

## Metric Spaces (Math 3210)

A <u>metric space</u> $M$

is a set w/ a

distance function (metric)

$$d : M \times M \to \mathbb{R}^{\geq 0} \text{ s.t.}$$

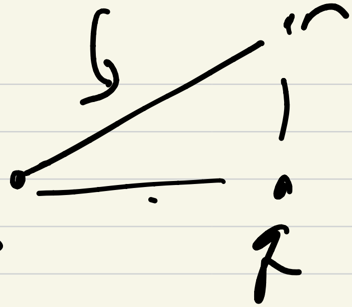
(0) $d(p,p) = 0$

(1) $d(p,q) > 0$ when $p \neq q$

(2) $d(p,q) = d(q,p)$

(3) <u>Triangle inequality</u>

$\triangle$ inequality




$$d(p,r) \leq d(p,q) + d(q,r).$$

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<u>Examples:</u> $\mathbb{R}^n$ has many <u>metrics</u>.

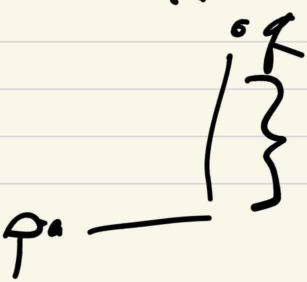
$\mathbb{R}^n$

$p = (p_1, \ldots, p_n)$

$q = (q_1, \ldots, q_n)$

(Max metric)

$$d_{max}(p,q) = \max_i |p_i - q_i|$$




$d_{max}(p,q)$.

$d((0,1),(3,8))$

$= \max\{|0-3|, |1-8|\}$

$= 7.$

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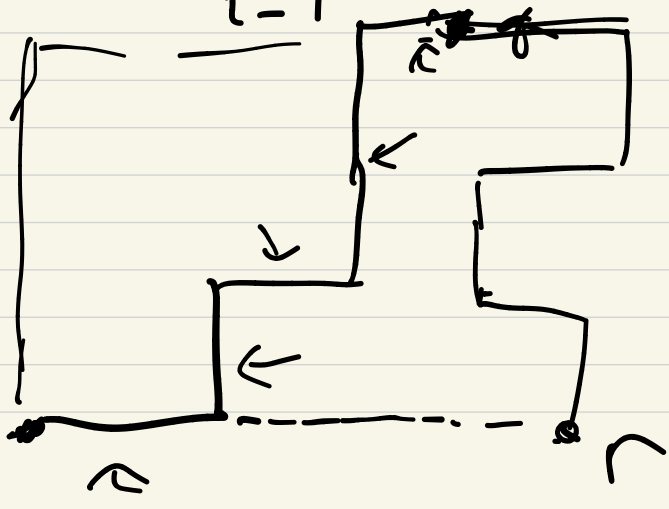
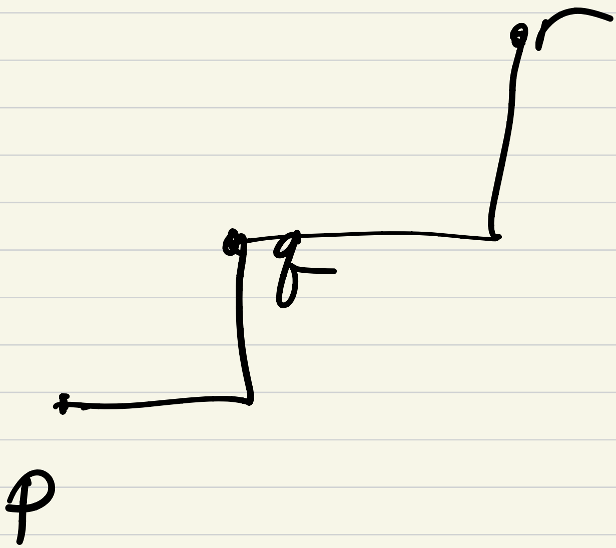
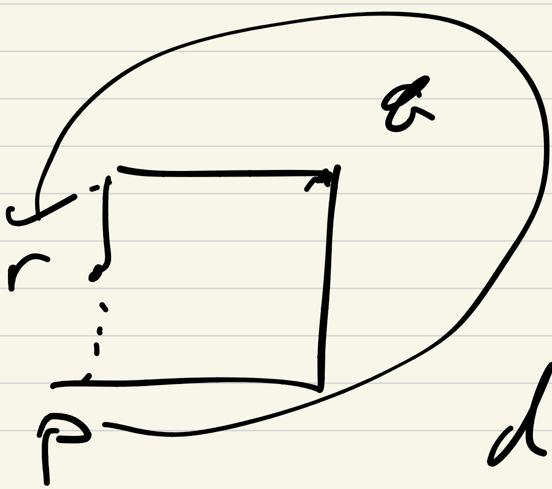
(Taxicab)
(Manhattan distance)

$$d_{Max}(p,q) = \sum_{i=1}^{n} |p_i - q_i|$$



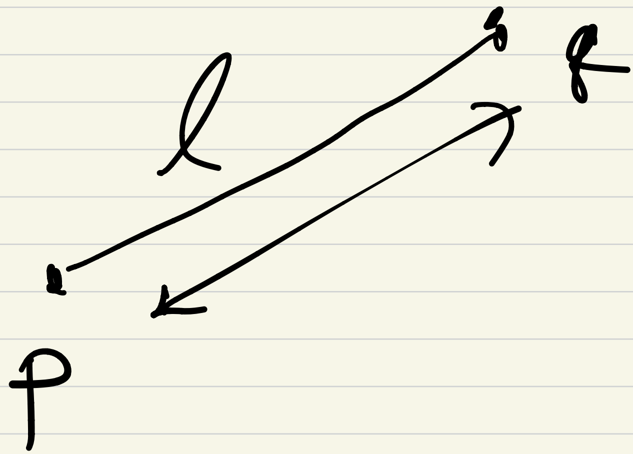



$$d(p,r) = d(p,q) + d(q,r)$$

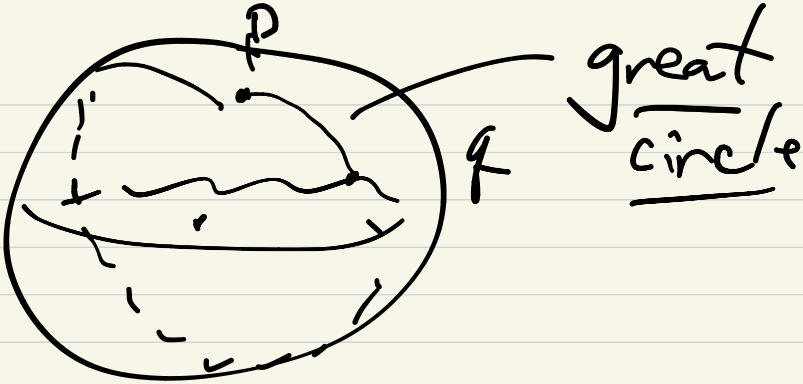



$$d(p,r) < d(p,q) + d(q,r).$$

(Euclidean metric)




$$\boxed{d(p,q) = \sqrt{\sum_{i=1}^{n} (p_i - q_i)^2}}$$




# Great circle metric on $S^2$

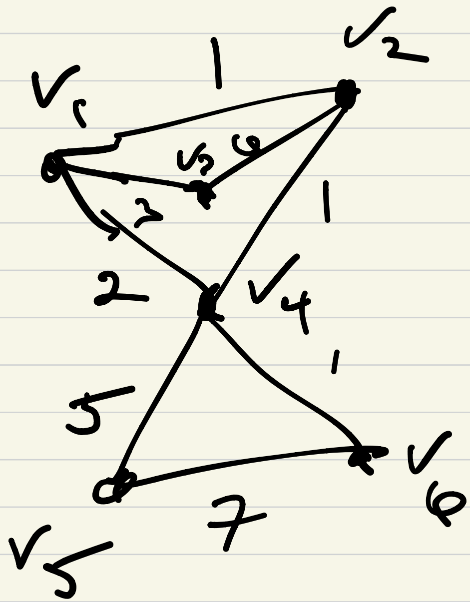

$d(p,q) =$

arclength ( $pq$ )

(great circle arc)

# Metric Graphs

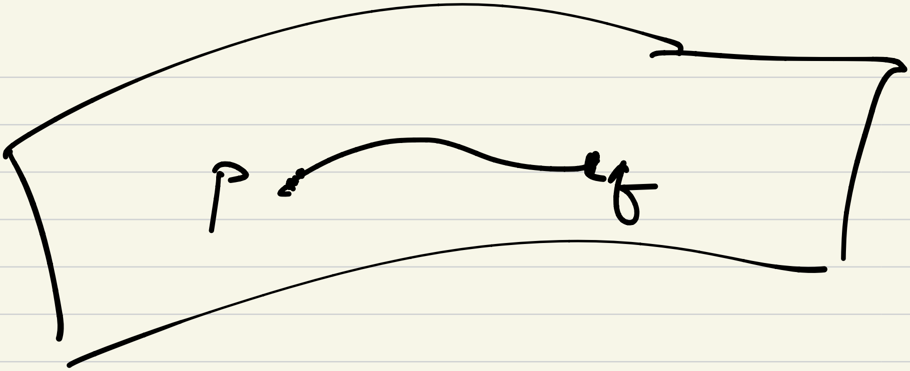
Connected Graph




Attach lengths to each edge.

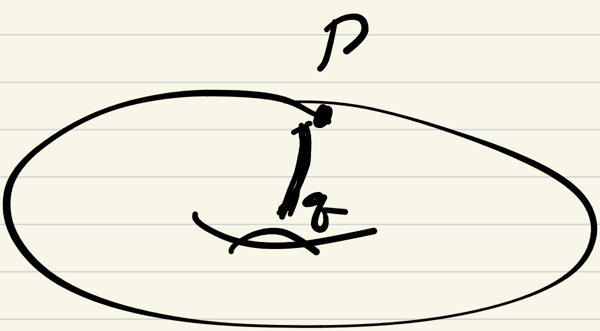
$d(v_1, v_5) = \underline{2+5}$

$d(v_i, v_j) =$ length of shortest path



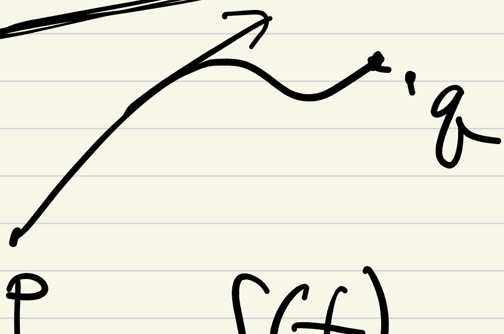

Metric on mfld

- Define lengths of paths.
- $d(p,q)$ = length of shortest path.



length of path in $\mathbb{R}^n$

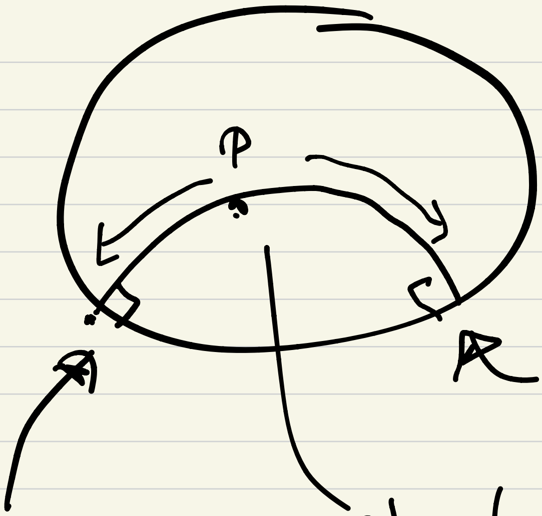
$$= \int_0^1 \overbrace{\|f'(t)\|}^{\text{speed}} \, dt$$




$f(t)$

$f(0)=p, \ f(1)=q$

(!)




**hyperbolic metric on $D$**

**shortest path**

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## Category of Metric Sp

**Objects:** Metric spaces $(M, d)$

**Morphisms:** Distance decreasing functions. $f : M \to N$

Example: $f: \mathbb{R}^n \longrightarrow \mathbb{R}^n$

$$f(p) = \tfrac{1}{2} p$$

$$d(f(p), f(q)) =$$

$$d\left(\tfrac{1}{2}p, \tfrac{1}{2}q\right) = \tfrac{1}{2} d(p,q)$$

$$f^{-1}(q) = 2q$$

not distance decreasing

$$\underline{\underline{Met}} = \left( \begin{array}{c} \text{Metric spaces} \\ + \\ \underline{\text{Distance}}\ \underline{\text{decreasing}} \\ \underline{\text{fcns}} \end{array} \right)$$

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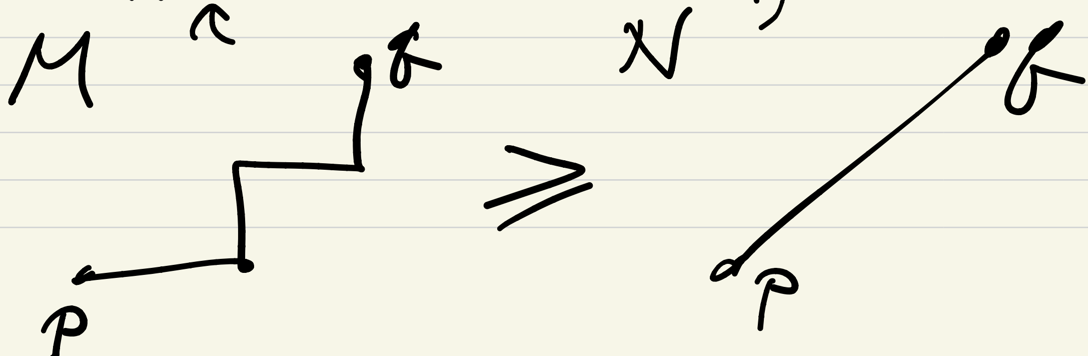
Ex:
$$(f(x) = x)$$
$$f : \mathbb{R}^n \to \mathbb{R}^n$$

as a function from

$$(\mathbb{R}^n, d^{1}) \xrightarrow{f} (\mathbb{R}^n, d)$$

Man Eucl



A symmetry in Met

is a distance-preserving (isometry)

bijection $f : (M, d) \to (M, d)$.

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Study: Symmetries of

$(\mathbb{R}^n, d)$

Euclidean.

**Start with** $(\mathbb{R}, |p-q|)$

$$d(p,q) = |p-q|$$

$$\sqrt{(p-q)^2} = |p-q|$$

**Examples of symetries of $\mathbb{R}$ :**

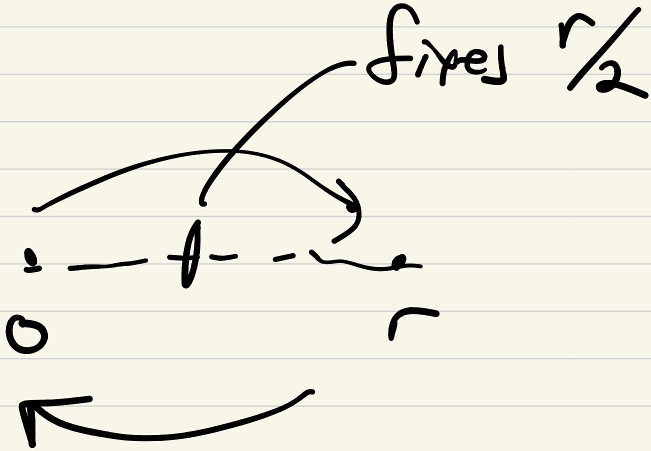
**Reflect across** $0$: $f(x) = -x$

**Translation**: $f(x) = x + r$

any real #

Try: $f(x) = -x + r$

Reflection: across $r/2$




$\tau_r(x) = x + r$

$\rho_s(x) = -x + 2s$

**Prop**: Every symmetry of $\mathbb{R}$ is either $\tau_r$ or $\rho_s$.

**Pf**: Suppose $f: \mathbb{R} \to \mathbb{R}$ is a symmetry, let $f(0) = r$.

**Then**

$$g(0) = (\tau_{-r} \circ f)(0) = \tau_{-r}(r)$$
$$= 0.$$

$$\begin{tikzcd} 0 \arrow[r, bend left, "f"] & r \arrow[l, "\tau_{-r}"] \end{tikzcd}$$

<u>Claim:</u> If $g : \mathbb{R} \to \mathbb{R}$

is a symmetry and

$g(0) = 0$, then

$$g(p) = p \quad \text{(identity)}$$

or

$$g(p) = -p \quad \text{(reflection)}$$

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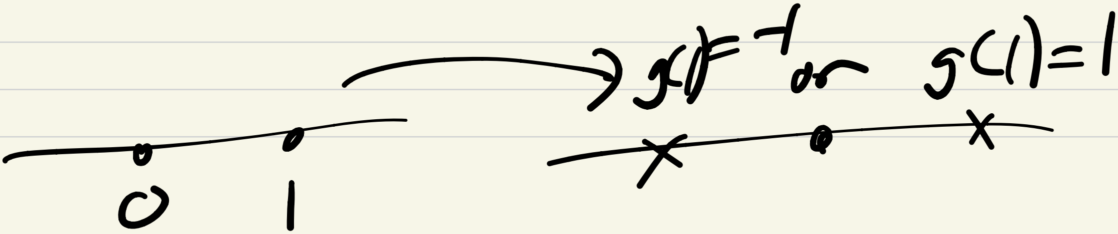
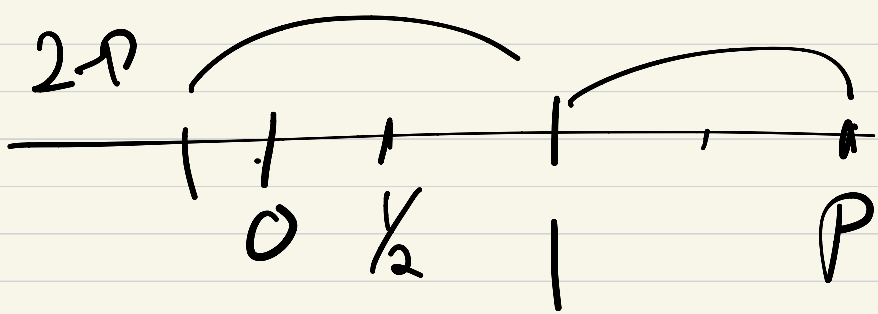
<u>Pf:</u> $g(0) = 0 \Rightarrow g(1) = \pm 1$



But if $g(0)=0$

and $g(1)=1$

then $g(p)=p$ for all $p$.



$g\left(\frac{p}{2}\right)=\pm\frac{p}{2}$ (preserving distance to 0)

$g(p)=p$ or $1-(p-1)=2-p$

Similarly if $g(0) = 0$

$g(1) = -1$,

then $(\rho_0 \circ g)(0) = 0$

$(1) = 1$

$\Rightarrow \rho_0 \circ g(p) = p$

$\Rightarrow g(p) = -p$. $\triangleleft$

Start with $f : \mathbb{R} \to \mathbb{R}$

$$\tau_{-r} \circ f(p) = p$$

or

$$\tau_{-r} \circ f(p) = -p$$

$$\Longrightarrow \quad f(p) = p + r = \tau_r$$

$$f(p) = -p + r = \rho_{r/2}$$

# Dot product:

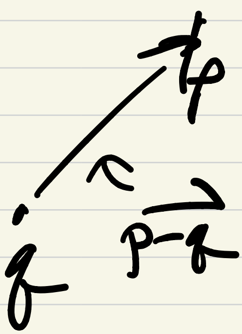
$$\vec{v} = (v_1, \dots, v_n)$$

$$\vec{w} = (w_1, \dots, w_n)$$

$$\vec{v} \cdot \vec{w} = v_1 w_1 + \dots + v_n w_n$$

$$\cdot : \mathbb{R}^n \times \mathbb{R}^n \longrightarrow \mathbb{R}$$

- Commutative
- $\vec{v} \cdot \vec{v} = |\vec{v}|^2$



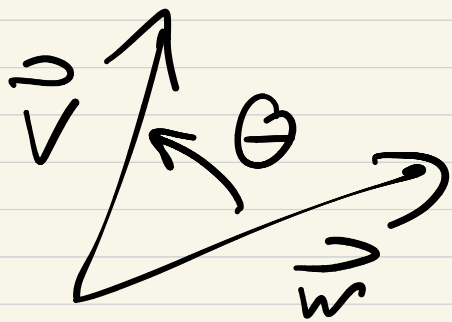

$$(d(p,q))^2 = (\overrightarrow{p-q}) \cdot (\overrightarrow{p-q})$$

- a <u>bilinear</u>:

$$(\vec{v}_1+\vec{v}_2)\cdot\vec{w} = \vec{v}_1\cdot\vec{w} + \vec{v}_2\cdot\vec{w}$$

- <u>computes</u> <u>angles</u> <u>between</u> <u>vectors</u>

$$\cos(\theta) = \frac{\vec{v}\cdot\vec{w}}{|\vec{v}|\cdot|\vec{w}|}$$




$$\vec{v}\cdot\vec{w} = 0$$

$$\Leftrightarrow \vec{v} \perp \vec{w}$$

**Prop:** [seeking symmetries of $\mathbb{R}^n$ that fix 0]

(a) If $\vec{v}_1, \dots, \vec{v}_n$ are mutually perpendicular unit vectors in $\mathbb{R}^n$, then

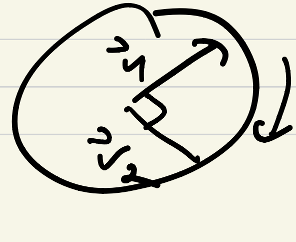
$$\phi(x_1, \dots, x_n) = x_1\vec{v}_1 + \cdots + x_n\vec{v}_n$$

is a symmetry of $\mathbb{R}^n$

(it is also linear!)

**Ex:** $\mathbb{R}^2$


or



With matrices:

$$\begin{bmatrix} \vec{v}_1 & \vec{v}_2 & \cdots & \vec{v}_n \end{bmatrix} \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} =$$

$$\phi(x_1, \ldots, x_n)$$

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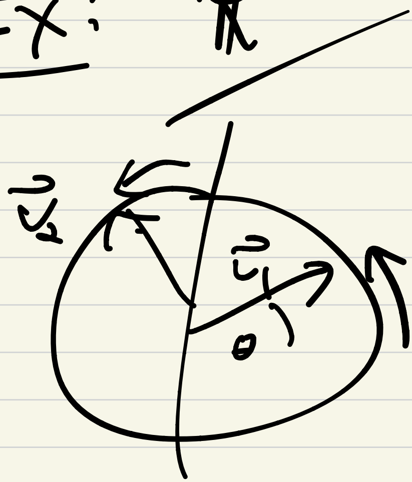
(b) These are the only symmetries of $\mathbb{R}^n$ that fix the origin!

Ex: $\mathbb{R}^2$




$$\vec{v}_1 = (\cos(\theta), \sin(\theta))$$

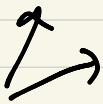
$$\vec{v}_2 = (\cos(\theta + \pi/2), \sin(\theta + \pi/2))$$

or

$$\vec{v}_2 = (\cos(\theta - \pi/2), \sin(\theta - \pi/2))$$

$$\begin{bmatrix}\cos(\theta) & \overset{-\sin(\theta)}{\overset{\shortparallel}{\cos(\theta+\pi/2)}} \\ \sin(\theta) & \underset{\cos(\theta)}{\underset{\shortparallel}{\sin(\theta+\pi/2)}}\end{bmatrix}$$

$\det = 1$



= matrix for rotation by $\theta$

$= \phi_\theta$

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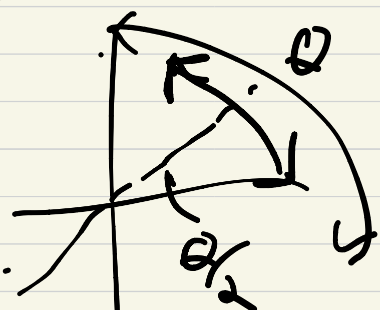
$$\begin{bmatrix}\cos(\theta) & \sin(\theta) \\ \sin(\theta) & -\cos(\theta)\end{bmatrix}$$




= matrix for reflecting across $y = \tan(\theta/2)\,x$

**Example:** In $\mathbb{R}^3$,

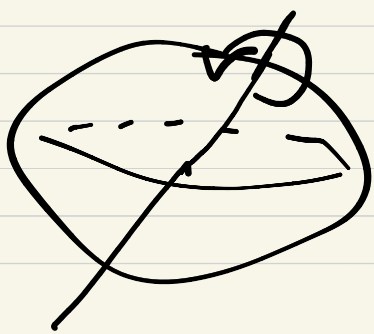
every symmetry

$$\phi(x_1, x_2, x_3) = x_1\vec{v}_1 + x_2\vec{v}_2 + x_3\vec{v}_3$$

were

$$\det\begin{bmatrix}\vec{v}_1 & \vec{v}_2 & \vec{v}_3\end{bmatrix} = 1$$

is a rotation around a fixed axis.



1, -1, -1