Categories, Manifolds and Representations

Math 4800, Fall 2020

1. Sets. In this section we look at two categories of sets, both of which implicitly appear in basic mathematics courses. One is the "small" category of subsets of a fixed universe set U, in which the identity map is the only symmetry, and the other is the category of all sets, in which the permutations of the elements of a (finite) set are its symmetries. We will also discuss how to transfer a symmetry, and the sign and cycle notation of a permutation.

Subsets. Let U be a fixed "universe" set. The category $\mathfrak{S}ub_U$ is defined by:

- The objects of $\mathfrak{S}ub_U$ are the subsets $S \subseteq U$.
- The morphisms of $\mathfrak{S}ub_U$ are set inclusions $S \subseteq T$.

This is a category with very few morphisms. The sets:

$$\hom(S,T)$$

are either empty (if S is not contained in T) or else consist of the single "inclusion" morphism $\subseteq$. Nevertheless, this gives a category since:

- The composition of inclusions is an inclusion, and
- The "reflexive" inclusions $S = S$ are the identity morphisms.

Every object S of $\mathfrak{S}ub_U$ has a single morphism (inclusion) to the universe U and there is a single morphism (inclusion) from the empty subset to each S. Equalities are the only isomorphisms since $S \subseteq T$ and $T \subseteq S$ together imply that $S = T$.

Symmetry. The "reflexive" inclusion $S = S$ is the only symmetry of an object S.

Remark. The objects of the category $\mathfrak{S}ub_U$ are the elements of the **power set** $\mathcal{P}(U)$ of U. A category in which the objects are elements of a set is called **small**.

Counting. If $|U| = n$, then the power set of U has 2^n elements, of which:

$$\binom{n}{m} \text{ are subsets with exactly } m \le n \text{ elements}$$

yielding the familiar sum formula for the rows of Pascal's triangle:

$$1 + \binom{n}{1} + \binom{n}{2} + \cdots + \binom{n}{n} = 2^n$$

Special Properties. The operations:

$$S \cap T \text{ (intersection)}, \ S \cup T \text{ (union) and } \overline{S} \text{ (complement)}$$

are defined on objects of $\mathfrak{S}ub_U$, and are often illustrated with Venn diagrams. Even though the category of subsets is rigid with no interesting symmetries, there is still interesting mathematics in these operations on subsets of U.

Without reference to a universe set U, we have the category of all sets.

Sets. The category $\mathfrak{S}ets$ (as previewed in §0) is defined by:

- The objects of $\mathfrak{S}ets$ are the sets S.
- The morphisms of $\mathfrak{S}ets$ are the set mappings $f : S \to T$.

Russell's Paradox. The collection of sets is **not** a set, i.e. $\mathfrak{S}ets$ is not small.

Proof. Suppose on the contrary that U is the set of all sets (including U itself!). Then
$$X = \{S \in U \mid S \notin S\} \subset U$$
would be a subset of U, hence also a set.

- Suppose $X \in X$. Then $X \notin X$ by the definition of X.
- Suppose $X \notin X$. Then $X \in X$ by the definition of X.

This is a paradox. It is an impasse which can only be resolved by revoking the "set" status of the collection of all sets. Calling it a collection instead seems like fudging, but it can be done rigorously. We will simply accept the fudging and move on. $\square$

On the other hand the functions from one set to another:
$$\hom(S, T) = \{f : S \to T\}$$
do constitute a set. (There is no self-referential issue to create a paradox.)

Special Properties. Two operations on sets, analogous to $\cap$ and $\cup$, are:
$$S \times T \text{ (Cartesian product) and } S \sqcup T \text{ (Disjoint union)}$$

The first is the set of ordered pairs of elements:
$$S \times T = \{(s, t) \mid s \in S,\ t \in T\}$$
and the second is a set that can be divided into complementary subsets, isomorphic to S and T respectively. One may make use of ordered pairs to construct this by "marking" the elements of S and T differently, so that we can distinguish elements they may have in common. E.g.
$$S \sqcup T = \{(s, -1) \mid s \in S\} \cup \{(t, 1) \mid t \in T\} \subset (S \cup T) \times \{-1, 1\}$$

Example. For sets R, S, T, the composition of functions is a function:
$$\circ : \hom(S, T) \times \hom(R, S) \to \hom(R, T);\ (f, g) \mapsto f \circ g$$
and the associative property asserts that the two functions:
$$\hom(T, U) \times \hom(S, T) \times \hom(R, S) \to \hom(R, U)$$
defined by:
$$(f, g, h) \mapsto f \circ (g \circ h) \text{ and } (f, g, h) \mapsto (f \circ g) \circ h$$
are the same.

Remark. The convention for composing functions is responsible for the unfortunate arrangement of the sets R, S, T in the Cartesian product above. Aesthetically, one might want to see the composition as a set mapping from:
$$\hom(R, S) \times \hom(S, T) \to \hom(R, T)$$
to "cancel" the inner S's, but this would require reversing the order of composition: "$(g \circ f)(r) = f(g(r))$" which would make us dizzy. So we'll sacrifice the aesthetics.

If S has m elements and T has n elements, then:
$$S \times T \text{ has } mn \text{ elements and } S \sqcup T \text{ has } m + n \text{ elements}$$
so these operations are intimately related to the arithmetic of natural numbers.

Warning. Keep in mind that the disjoint union of S and T is not their ordinary union since elements in the intersection of S and T are distinct in the disjoint union. For example, $S \sqcup S$ is not the set S, but rather two copies of the set S.

More Properties. The **image** via $f : S \to T$ of a subset $R \subset S$ is:

$$f(R) = \{f(s) \mid s \in R\} \subseteq T$$

and f is **surjective** if $f(S) = T$. The **inverse image** of $U \subseteq T$ via f is the subset:

$$f^{-1}(U) = \{s \in S \mid f(s) \in U\} \subseteq S$$

and f is **injective** if:

$$f^{-1}(\{t\}) = \begin{cases} \emptyset \ \text{ if } \ t \notin f(U) \\ \\ \text{a singleton set } \{s\} \text{ that we use to define } s := f^{-1}(t) \text{ otherwise} \end{cases}$$

As a function from S to $f(S)$, an injective function has a two-sided inverse function $f^{-1} : f(S) \to S$, and if f is surjective, then f^{-1} is the inverse of f itself.

Counting. If S has m elements and T has n elements, then:

(i) $\hom(S, T)$ has n^m elements,

(ii) the set of injective functions in (i) has $n(n-1)\cdots(n-m+1)$ elements,

(iii) if $n = m$, then the set of isomorphisms in (i) has $n!$ elements.

Remarks. The *surjective* functions from S to T (when $m > n$) are much harder to count than the injective functions (when $m < n$). In the category of sets, the empty set $\emptyset$ has an "empty" function $f : \emptyset \to T$ to each set T, including the empty set itself (the identity morphism $1_\emptyset$). There is an isomorphism between *finite* sets S and T if and only if S and T have the same number of elements, and in that case a function $f : S \to T$ is an isomorphism if and only if it is injective. Thus (iii) follows from (ii). A pair of infinite sets is said to have the same **cardinality** if and only if there is an isomorphism (bijective function) between them.

For each natural number $n \in \mathbb{N}$, let:

$$[n] = \{1, 2, ..., n\}$$

An isomorphism $f : [n] \to S$ is called an **ordering** of a set S with n elements.

Symmetry. We begin with some general remarks about symmetries in a category.

If $f : S \to T$ is an isomorphism and $\sigma : S \to S$ is a symmetry of S, then:

$$f \circ \sigma \circ f^{-1} : T \to T$$

is a symmetry of T with inverse $f \circ \sigma^{-1} \circ f^{-1}$.

Note. We may define this without parentheses since the composition is associative. Notice also that for all symmetries σ and τ of S,

$$f \circ (\sigma \circ \tau) \circ f^{-1} = (f \circ \sigma \circ f^{-1}) \circ (f \circ \tau \circ f^{-1})$$

and that $f \circ 1_S \circ f^{-1} = 1_T$ and $f \circ \sigma \circ f^{-1} = f \circ \tau \circ f^{-1}$ if and only if $\sigma = \tau$.

Via this **transfer**, the symmetries of S are in bijection with symmetries of T, and inverses and compositions carry over to inverses and compositions (in §5, we will see that this transfer of symmetries is an isomorphism in the category of groups).

In the category $\mathfrak{Sets}$ of sets, an ordering transfers the symmetries of $[n]$ to the symmetries of a set S with n elements, and so any property of the symmetries of finite sets is a consequence of the corresponding property of the symmetries of $[n]$.

The Sign of a Permutation. Given $f : [n] \to [n]$ in the category $\mathfrak{S}ets$, let:

$$\mathrm{sgn}(f) = \prod_{1 \le i < j \le n} \frac{f(j) - f(i)}{j - i}$$

Remark. The factors of the product are unchanged if i and j are reversed:

$$\frac{f(j) - f(i)}{j - i} = \frac{f(i) - f(j)}{i - j}$$

so the factors only depend on the subsets $\{i, j\}$, and not their ordering.

The following Proposition justifies referring to this as the **sign** of a permutation of the n numbers $1, 2, ..., n$.

Proposition 1.1. (a) $\mathrm{sgn}(f) = 0$ if and only if f is **not** a symmetry.

(b) If $\sigma : [n] \to [n]$ is a symmetry, then $\mathrm{sgn}(\sigma) = 1$ or -1.

(c) If $f, g : [n] \to [n]$ and $h = g \circ f$, then $\mathrm{sgn}(h) = \mathrm{sgn}(f) \cdot \mathrm{sgn}(g)$.

Proof. A function $f : [n] \to [n]$ fails to be a symmetry if and only if $f(i) = f(j)$ for some $i \neq j$, if and only if $\mathrm{sgn}(f) = 0$. This is (a).

For (b), since σ is a symmetry, it follows that the pairs $\{\sigma(i), \sigma(j)\}$ vary over all two-element subsets of $[n]$ as $\{i, j\}$ vary over all the two-element subsets of $[n]$. Thus $\prod_{i<j} |\sigma(j) - \sigma(i)| = \prod_{i<j} |j - i|$ and $\prod_{i<j}(\sigma(j) - \sigma(i)) = \pm \prod_{i<j}(j - i)$, which gives (b). Notice that it may be the case that $i < j$ but $\sigma(i) > \sigma(j)$. In fact, the number of such "order switchings" determines whether $\mathrm{sgn}(\sigma)$ is $+1$ or -1.

For (c), if f or g fails to be a symmetry then h fails and $\mathrm{sgn}(h) = 0 = \mathrm{sgn}(f) \cdot \mathrm{sgn}(g)$. Otherwise, g in particular is a symmetry, and:

$$\mathrm{sgn}(h) = \prod_{i<j} \frac{f(g(j)) - f(g(i))}{j - i} = \prod_{i<j} \frac{f(g(j)) - f(g(i))}{g(j) - g(i)} \cdot \frac{g(j) - g(i)}{j - i}$$

The product of the second factors gives $\mathrm{sgn}(g)$, and the product of the first factors (and the remarks above) gives $\mathrm{sgn}(f)$. $\square$

Transpositions. A symmetry σ of $[n]$ **transposes** i and $j \neq i$ if:

$$\sigma(i) = j,\ \sigma(j) = i \text{ and } \sigma(k) = k \text{ for all } k \neq i, j$$

in which case we write $\sigma = t_{i,j}$. (Notice that $t_{i,j} \circ t_{i,j} = 1_{[n]}$.)

Proposition 1.2. (a) The sign of each transposition $t_{i,j}$ is -1.

(b) Every symmetry of $[n]$ can be built by composing the transpositions:

$$t_{1,2}, t_{2,3}, ..., t_{n-1,n} \text{ (in some order, often with repetitions)}$$

Proof. To see (a), suppose $i < j$. Then the pair $\{i, j\}$ switches order under $t_{i,j}$ to $\{j = t_{i,j}(i), i = t_{i,j}(j)\}$ as do the $2(j - i - 1)$ pairs $\{i, k\}$ and $\{k, j\}$ for $i < k < j$ and no other pairs! Thus the sign of $t_{i,j}$ is the product of an odd number of -1's.

To see (b), suppose that σ is a symmetry of $[n]$ and $\sigma(n) = m$. Then

$$t_{n-1,n} \circ \cdots \circ t_{m,m+1} \circ \sigma$$

sends n to n, and is therefore a symmetry of $[n - 1]$. Call this τ. By induction on n, we may assume that τ is built from the transpositions $t_{1,2}, ..., t_{n-2,n-1}$ and then (b) follows, since $\sigma = t_{m+1,m} \circ \cdots \circ t_{n-1,n} \circ \tau$. $\square$

Example. Applying the proof of (b) to the transposition $t_{1,4}$ (a symmetry of [4]):

$$\tau = t_{3,4} \circ t_{2,3} \circ t_{1,2} \circ t_{1,4}, \text{ where } \tau(1) = 3, \tau(2) = 1, \tau(3) = 2$$

and applying the proof again, $t_{2,3} \circ \tau = t_{1,2}$ and so:

$$t_{2,3} \circ (t_{3,4} \circ t_{2,3} \circ t_{1,2}) \circ t_{1,4} = t_{1,2}$$

Undoing the compositions and using $t = t^{-1}$, we obtain the desired composition:

$$t_{1,4} = t_{1,2} \circ t_{2,3} \circ t_{3,4} \circ t_{2,3} \circ t_{1,2}$$

Corollary 1.3. If σ is a symmetry of $[n]$, and $\sigma = t_{i_1,j_1} \circ \cdots \circ t_{i_m,j_m}$ is an expression of σ as a composition of m transpositions (which exists by Proposition 1.2 (b)), then:

$$\text{sgn}(\sigma) = (-1)^m$$

Since the sign is well-defined, it follows that if $\sigma = t_{i_1,j_1} \circ \cdots \circ t_{i_l,j_l}$ is another such expression, then l and m are either both even or both odd.

Definition 1.4. A permutation σ is **even** if $\text{sgn}(\sigma) = +1$ and **odd** if $\text{sgn}(\sigma) = -1$.

Cycle Notation is an extremely useful way of recording the symmetries of $[n]$.

For a symmetry σ, a **cycle** (or orbit) is a sequence starting at some $m \in [n]$:

$$m, \sigma(m), (\sigma \circ \sigma)(m) = \sigma^2(m), (\sigma \circ \sigma \circ \sigma)(m) = \sigma^3(m), ...$$

Of course this list eventually repeats, but in fact the list **cycles.** That is:

Proposition 1.5. Given σ and m, there is an integer $d > 0$ such that:

$$m, \sigma(m), ..., \sigma^{d-1}(m) \text{ cycle through different numbers, and } \sigma^d(m) = m$$

Proof. Eventually there are repetitions in the sequence. Let:

$$\sigma^k(m) = \sigma^{k+d}(m)$$

be a repetition with the smallest "gap" between them. Then all the elements:

$$\sigma^k(m), \sigma^{k+1}(m), ..., \sigma^{k+d-1}(m)$$

are different, and by applying $(\sigma^{-1})^k$ to each of these, we get the Proposition. $\square$

The cycle notation for a symmetry σ of $[n]$ starts with the cycle:

$$1, \sigma(1), \cdots, \sigma^{d_1-1}(1)$$

and encloses it in parentheses. Then it takes the smallest number **not** in the cycle above and uses it to build a second cycle **that does not overlap with the first**:

$$m, \sigma(m), \cdots, \sigma^{d_2-1}(m)$$

and continues on until all numbers from 1 to n are in exactly one of the cycles.

Example. The symmetry of [5] given by:

$$\sigma(1) = 3, \sigma(2) = 4, \sigma(3) = 5, \sigma(4) = 2, \sigma(5) = 1$$

has as its first cycle: (1 3 5)

and then as its second cycle: (2 4)

so that in cycle notation, we would write $\sigma = (1\ 3\ 5)(2\ 4)$.

Notice that all the values of $\sigma(m)$, and therefore all the information about the symmetry σ is recovered from the cycle notation.

Example. There are six permutations of [3].

- The identity $1_{[3]} = (1)(2)(3)$ is an even permutation.
- The three transpositions are odd permutations.

$$t_{1,2} = (1\ 2)(3) \quad \text{and} \quad t_{1,3} = (1\ 3)(2) \quad \text{and} \quad t_{2,3} = (1)(2\ 3)$$

- Two additional permutations, both of which are even.

$$(1\ 2\ 3) = t_{1,3} \circ t_{1,2} \quad \text{and} \quad (1\ 3\ 2) = t_{1,2} \circ t_{1,3}$$

Assignment 1.

1. Sketch the directed graph picture for the category $\mathfrak{S}ub_{[3]}$.

2.* How many surjective functions are there from $[m]$ to [2]?
Devise a strategy for counting all the surjective functions from $[m]$ to $[n]$.

3. Create a 6×6 composition table for the symmetries of [3].
(Since composition is not commutative, let's agree that $\sigma \circ \tau$ goes in the box in the row corresponding to σ and the column corresponding to τ).

4. Find all the symmetries of [4], their cycle notation and their signs.
Devise a strategy for finding all the symmetries of $[n]$.
How many even permutations of [5] are there?

Extended Exercise. The Cartesian product $X \times Y$ of X and Y in the category $\mathfrak{S}ets$ has the following property:

- There are projection functions $p : X \times Y \to X$ and $q : X \times Y \to Y$.
- If S is any set with functions $f : S \to X$ and $g : S \to Y$, then

$$(f, g) : S \to X \times Y \text{ defined by } (f, g)(s) = (f(s), g(s))$$

has the property that $p \circ (f, g) = f$ and $q \circ (f, g) = g$.

Moreover, (f, g) is the **unique** function in $\hom(S, X \times Y)$ with this property!

Let X and Y be objects of a category $\mathcal{C}$.

Definition 1.A. The **product** of X and Y (if it exists) is a triple:

$$(Z, p : Z \to X, q : Z \to Y)$$

with the following *universal* property:

To each triple $(S, f : S \to X, g : S \to Y)$ there is a unique $h \in \hom(S, Z)$ with:

$$f = p \circ h \text{ and } g = q \circ h$$

5. Show that the intersection is the product in the cateogry $\mathfrak{S}ub_U$! That is, $(S \cap T, S \cap T \subseteq S, S \cap T \subseteq T)$ is the product of S and T in the category $\mathfrak{S}ub_U$.

Definition 1.B. The **coproduct** of X and Y (if it exists) is a triple:

$$(Z, i : X \to Z, j : Y \to Z)$$

with the universal property

To each triple $(S, f : X \to S, g : Y \to S)$ there is a unique $h \in \hom(Z, S)$ with:

$$f = h \circ i \text{ and } g = h \circ j$$

7. Find the coproduct of X and Y in the category of sets.

8. Find the coproduct of S and T in the category of subsets of U.