Categories, Symmetry and Manifolds

Math 4800, Fall 2020

Lecture 6. Continuous functions are defined in a Calculus class in terms of limits. Continuity is abstractly defined in the category of *topological spaces,* which includes Euclidean spaces and all metric spaces, as well as topological spaces with no metric. Continuity is the key to paring down the multitude of representations of the circle (an infinite abelian group) to the set of characters given by the complex exponentials. From this point of view a Fourier series is the expansion of a (class) function in terms of characters.

Definition 6.1. (a) A **topological space** is a set X equipped with a collection of **open subsets** $U \subset X$ with the following properties:

(i) $\emptyset$ and X are open subsets of X.

(ii) The intersection of two open sets is an open set.

(ii) The union of arbitrarily many open sets is an open set.

The complement of an open set is a **closed** set.

(b) If a collection $\{U_\lambda \mid \lambda \in \Lambda\}$ of subsets of X satisfies:

(i) The union $\cup_{\lambda \in \Lambda} U_\lambda = X$ and

(ii) For each $\mu, \nu \in \Lambda$, the intersection $U_\mu \cap U_\nu$ is a union of sets U_λ.

then the U_λ are a **basis** for a topology whose open sets are the unions:

$$U = \cup_{\lambda \in S} U_\lambda \text{ over all subsets of the indexing set } \Lambda$$

Remark. Every topology has basis, namely the collection of **all** open sets $U \subset X$. The point about a basis is to determine a topology using far fewer of the open sets. Two bases are equivalent of they generate the same topology on X.

Example. (i) If (X, d) is a metric space, then the open balls:

$$B_{p,r} = \{q \in X \mid d(p, q) < r\}$$

indexed by $p \in X$ and $r > 0$ are a basis for the **metric space topology** on X. Property (ii) is a consequence of the triangle inequality!

(ii) The Euclidean topology on $\mathbb{R}^n$ is the topology associated to the Euclidean metric. There is a countable basis of open sets for this topology given by the balls $B_{p,r}$ such that $p = (p_1, ..., p_n)$ has rational coordinates and $r \in \mathbb{Q}$.

Definition 6.2. A mapping $f : X \to Y$ of topological spaces is **continuous** if the inverse image $f^{-1}(U)$ of every open set $U \subset Y$ is open in X, or equivalently if the inverse image of every closed set is closed.

Notice that continuity can be checked on any basis for the topology on Y, since

$$f^{-1}(\cup U_\lambda) = \cup f^{-1}(U_\lambda)$$

Notice also that if $f : X \to Y$ and $g : Y \to Z$ are continuous, then:

$$(g \circ f)^{-1}(U) = f^{-1}(g^{-1}(U)) \text{ is open in } X \text{ for all open sets } U \subset Z$$

and therefore $g \circ f$ is continuous.

Example. (i) In the **discrete** topology on X, all subsets are open (and closed!).

(ii) In the **trivial** topology on X, only $\emptyset$ and X are open (and closed) sets.

These topologies are as far apart as it is possible to be. The identity map id_X is continuous as a map from the discrete topology to the trivial topology, but not the other way around (unless X is a single point).

The category $\mathfrak{T}op$ of topological spaces is defined by:

- The objects X of $\mathfrak{T}op$ are topological spaces.
- The morphisms $f : X \to Y$ in $\mathfrak{T}op$ are continuous maps.
- The isomorphisms are bijective continuous maps with continuous inverses. These are called **homeomorphisms**. Note that in the example above, the identity map from the discrete topology on X to the trivial topology on X is continuous, but the inverse map (which is again id_X) is not continuous, so in this case id_X is not a homeomorphism. This may bex confusing because notation is being abused. The identity maps in the category $\mathfrak{T}op$ are the identity from X to itself with **the same topology**. In that case, of course id_X is a homeomorphism.

Proposition 6.3. If $U_\lambda \subset X$ and $V_{\lambda'} \subset Y$ are bases for topologies, then

$$U_\lambda \times V_{\lambda'} \subset X \times Y$$

is a basis for the *product topology*, making $X \times Y$ (with Cartesian projections) the product of X and Y in the category of topological spaces. In the case of $\mathbb{R}^n$, the product topology (as an n-fold product $\mathbb{R}^n = \mathbb{R}^1 \times \cdots \mathbb{R}^1$) is the Euclidean topology.

Proof. If $U_\mu \cap U_\nu$ is a union of basis sets U_λ, and $V_{\mu'} \cap V_{\nu'}$ is a union of $V_{\lambda'}$, then $(U_\mu \times V_{\mu'}) \cap (U_\nu \times V_{\nu'})$ is a union of product sets $U_\lambda \times V_{\lambda'}$. The projection maps $\pi_X : X \times Y \to X$ and $\pi_Y : X \times Y \to Y$ are continuous since $\pi_X^{-1}(U_\lambda) = U_\lambda \times X = \cup_{\lambda'} U_\lambda \times V_{\lambda'}$ and similarly π_Y is continuous. Moreover, if Z is another topological space and $f : Z \to X$ and $g : Z \to Y$ are continuous, then

$$(f, g)^{-1}(U_\lambda \times V_{\lambda'}) = f^{-1}(U_\lambda) \cap g^{-1}(V_{\lambda'}) \text{ is open}$$

and it follows that the unique lifted map $(f, g) : Z \to X \times Y$ is continuous. So the product topology indeed defines a product in the category $\mathfrak{T}op$.

The product and Euclidean topologies on $\mathbb{R}^n$are the same because:

(i) Every Euclidean ball is a union of products of open intervals, and

(ii) Every product of open intervals is a union of Euclidean balls.

This is left as an exercise. □

Arithmetic Examples. The addition and multiplication maps:

$$+ : \mathbb{R}^1 \times \mathbb{R}^1 \to \mathbb{R}^1 \text{ and } * : \mathbb{R}^1 \times \mathbb{R}^1 \to \mathbb{R}^1$$

are continuous. Indeed, the inverse image of an interval (a, b) under the addition map is the region: $\{(x, y) \mid a < x+y < b\}$ which is an open set bounded between two parallel lines. Similarly, the inverse image of an interval (a, b) under multiplication is an open set bounded between closed plane curves (hyperbolas or the axes).

From this it follows that if $f, g : X \to \mathbb{R}$ are continuous real valued functions on a topological space, then $f + g$ and fg are continuous functions. Moreover, if $f, g : X \to \mathbb{R}^n$, then again $f + g$ is continuous, and if $f, g : X \to \mathbb{C}$, then fg is continuous as well.

Some important additional features of topological spaces include:

Definition 6.4. (a) A topological space X is **disconnected** if:

$$X = U_1 \cup U_2 \text{ for a pair of non-empty disjoint open subsets}$$

If there is no such pair of open sets, then X is **connected**.

(b) A topological space X is **Hausdorff** if for all $p, q \in X$, there are *neighborhoods*

$$p \in U_1 \text{ and } q \in U_2 \text{ with } U_1 \cap U_2 = \emptyset$$

i.e. p and q are separated by disjoint open sets.

(c) A topological space X is **compact** if it is Hausdorff and every *open cover*

$$X = \cup_{\lambda \in S} U_\lambda \text{ has a finite subcover } X = U_{\lambda_1} \cup \cdots \cup U_{\lambda_n}$$

Examples. (a) $\mathbb{R}$ is connected. Every open subset of $\mathbb{R}$ is a disjoint union of open invervals (a_λ, b_λ) and if $U \subset \mathbb{R}$ is open and not equal to $\mathbb{R}$, then there is a point a_λ or b_λ on the boundary of one of these intervals that is not in U. Every open set containing that point intersects the interval (a_λ, b_λ), and so cannot disconnect $\mathbb{R}$ together with U. Note. The closed subsets of $\mathbb{R}$ are not all disjoint unions of closed intervals, and are somehow more interesting than the open sets (e.g. Cantor sets).

(b) Metric space topologies are Hausdorff. The trivial topology is not Hausdorff. The discrete topology is Hausdorff with a vengeance!

(c) $\mathbb{R}$ is not compact. The union $\mathbb{R} = \cup_{n=1}^{\infty}(-n, n)$ has no finite subcover.

The following Proposition is left as an exercise.

Proposition 6.5. Let X and Y be topological spacexs.

(a) If X and Y are connected, then $X \times Y$ is connected.

(b) If X and Y are Hausdorff, then $X \times Y$ is Hausdorff.

(c) If X and Y are compact, then $X \times Y$ is compact.

Definition 6.6. If X is a topological space and $Y \subset X$, then the collection of subsets $U \cap Y \subset Y$ comprises the **induced topology** on X.

When we discuss whether a subset $Y \subset X$ of a topological space is connected or compact, it is with respect to the induced topology. Note that every subset of a Hausdorff topological space is Hausdorff, but a subset of a connected space need not be connected (e.g. $\mathbb{R} - \{0\} \subset \mathbb{R}$ is disconnected) and a subset of a compact space need not be compact. One way to see examples of this is via the:

Heine-Borel Theorem. $Y \subset \mathbb{R}^n$ is compact if and only if it is closed and bounded. Thus, for example, the unit circle:

$$S^1 = \{(x, y) \mid x^2 + y^2 = 1\}$$

is clearly bounded and closed since: $S^1 = h^{-1}(1)$ for the continuous map $h(x, y) = x^2 + y^2$, so by the Theorem, S^1 is compact. On the other hand if we remove the north pole $n = (0, 1)$, then $S^1 - n$ is bounded but not closed. In fact the two maps:

$$f : S^1 - n \to \mathbb{R} \text{ defined by } f(x, y) = \frac{x}{1 - y} \text{ and}$$

$$g : \mathbb{R} \to S^1 - n \text{ defined by } g(t) = \left(\frac{2t}{1 + t^2}, \frac{1 - t^2}{1 + t^2}\right)$$

are continuous inverse mappings, hence isomorphisms (homeomorphisms) in $\mathfrak{Top}$. So $(S^1 - n) \cong \mathbb{R}$ is not compact.

Here are some other examples:

Examples. The special linear group

$$\mathrm{SL}(2,\mathbb{R}) = \left\{ A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \mid \det(A) = 1 \right\} \subset \mathbb{R}^4$$

is closed, since $\mathrm{SL}(n,\mathbb{R}) = \det^{-1}(1)$ for the continuous map $\det(A) = ad - bc$, but it is not bounded. For example, the "one-parameter subgroup"

$$\left\{ \begin{bmatrix} t & 0 \\ 0 & t^{-1} \end{bmatrix} \mid t \in \mathbb{R}^* \right\} \subset \mathrm{SL}(2,\mathbb{R})$$

cannot be contained in any ball of finite radius. (This generalizes to $\mathrm{SL}(n,\mathbb{R})$)

The orthogonal group

$$\mathrm{O}(n,\mathbb{R}) = \{ A \mid AA^T = I_n \}$$

is both closed and bounded. It is closed because it is the inverse image:

$$f^{-1}(I_n) \text{ for the continuous map } f(A) = AA^T$$

It is bounded because the columns (or rows) of A consist of n vectors of length 1. Thus $O(n,\mathbb{R})$ is contained in the bounded subset:

$$S^{n-1} \times \cdots \times S^{n-1} \subset \mathbb{R}^n \times \cdots \times \mathbb{R}^n$$

Interestingly, $O(n,\mathbb{R})$ is not connected. It consists of two components:

$$\mathrm{SO}(n,\mathbb{R}) = O(n,\mathbb{R}) \cap \mathrm{SL}(n,\mathbb{R}) \text{ and } O(n,\mathbb{R})^- = O(n,\mathbb{R}) \cap \det^{-1}(-1)$$

that are homeomorphic via multiplication by any $A \in O(n,\mathbb{R})^-$.

Similarly, the unitary groups $U(n)$ are compact, but these are also connected.

Proposition 6.7. Suppose $f : X \to Y$ is a continuous map.

(a) If $Z \subset X$ is connected, then $f(Z) \subset Y$ is connected.

(b) If $Z \subset X$ is compact and Y is Hausdorff, then $f(Z)$ is compact.

The proof of this, too, is left as an exercise.

Definition 6.8. A **topological group** is a group G that is also a topological space, such that the group operations of multiplication and inverse:

$$m : G \times G \to G \text{ and } i : G \to G$$

are both continuous (and i is a homeomorphism since $i \circ i = \mathrm{id}_G$).

Example. (a) Since addition is continuous and $i(x) = -x$ is continuous, $(\mathbb{R}, +, 0)$ is an abelian topological group, as are $(\mathbb{R}^*, \cdot, 1)$ and $(\mathbb{R}^n, +, 0)$, $(\mathbb{C}^*, \cdot, 1)$. Note that $(\mathbb{C}^*, \cdot, 1)$ is connected, but $(\mathbb{R}^*, \cdot, 1)$ is disconnected.

(b) The circle $(S^1 = \mathbb{R}/2\pi\mathbb{R}, +, 0)$ is an abelian topological group.

(c) The groups $\mathrm{SL}(n,\mathbb{R})$, $\mathrm{SL}(n,\mathbb{C})$, $\mathrm{O}(n,\mathbb{R})$, $\mathrm{U}(n)$ are all topological groups. In particular, S^1 is isomorphic to $U(1)$ as topological groups via the map:

$$e^{it} : S^1 \to U(1)$$

This is, indeed, a complex representation of the group S^1.

Example (a) gives an interesting proof of the following "obvious" fact:

Observation. $\mathbb{R}^2$ and $\mathbb{R}$ are not homeomorphic topological spaces.

One nice way to see that $\mathbb{C}^*$ is connected is via the **exponential map**

$$exp : \mathbb{C} \to \mathbb{C}^*;\ exp(z) = e^{2\pi i z}$$

This is surjective, so $\mathbb{C}^*$ is connected because $\mathbb{C}$ is connected. Since removing one point disconnects $\mathbb{R}$ but doesn't disconnect $\mathbb{R}^2$, it follows that $\mathbb{R}$ and $\mathbb{R}^2$ cannot be homeomorphic. This generalizes via the use of *homology* to show that $\mathbb{R}^n$ and $\mathbb{R}^m$ are not homeomorphic when $n \neq m$, via a "topological invariant" that distinguishes the topological spaces $\mathbb{R}^m - 0$ and $\mathbb{R}^n - 0$.

Representation Theory of S^1. Since S^1 is abelian, all complex representations are characters:

$$\chi : S^1 \to \mathbb{C}^*$$

and if we forget the topology, then there are a lot of characters! For example,

$$f(t) = \begin{cases} e^{it} \text{ if } t \text{ is a rational multiple of } 2\pi \\ 1 \text{ if } t \text{ is an irrational multiple of } 2\pi \end{cases}$$

is a (totally discontinuous) character!

Since the conjugacy classes of an abelian group are singletons, a class function for the circle is just an *arbitrary* function:

$$f : S^1 \to \mathbb{C}$$

These are evidently unwieldy, so we use the topology of S^1 to streamline things:

Definition 6.8. A *continuous character* χ of a topological group G is the trace of a continuous complex representation:

$$\rho : G \to \mathrm{Aut}(V) \cong \mathrm{GL}(n, \mathbb{C}) \subset \mathbb{C}^{n^2}$$

Such a character χ defines a class function on the set of conjugacy classes of G that is a *continuous* function $\chi : G \to \mathbb{C}$ that is constant on conjugacy classes.

Proposition 6.9. The continuous representations of S^1 are the one-dimensional characters $\chi(t) = e^{int}$ for $n \in \mathbb{Z}$ (including the trivial character $(n = 0)$).

Proof. Under construction.