Abstract Algebra. Math 6310. Bertram/Utah 2022-23.

Homology and Homotopy and Functors (Oh My!)

Now we turn to chain complexes that are (usually) not exact.

Proposition 1. A morphism f of chain complexes:

$$\begin{array}{ccccccccc} \cdots & \to & C_{i+1} & \to & C_i & \to & C_{i-1} & \to & \cdots \\ & & f_{i+1}\downarrow & & f_i\downarrow & & f_{i-1}\downarrow & & \\ \cdots & \to & C'_{i+1} & \to & C'_i & \to & C'_{i-1} & \to & \cdots \end{array}$$

induces morphisms of homologies: $H_i(f) : H_i(C_\bullet) \to H_i(C'_\bullet)$.

Proof. Since the diagram commutes, we obtain:

$$f_i(\ker(d_i)) \subset \ker(d'_i) \text{ and } f_i(\mathrm{im}(d_{i+1})) \subset \mathrm{im}(d'_{i+1}) \text{ rendering}$$

$$H_i(f)(c_i + \mathrm{im}(d_{i+1})) := f_i(c_i) + \mathrm{im}(d'_{i+1})$$

a well-defined R-module homomorphism of homology modules. □

Different morphisms of chain complexes may induce the same maps on homology (e.g. any morphism to or from an exact complex induces the zero map). Also:

Definition. (a) A *chain homotopy* h between $f, g \in \mathrm{Hom}(C_\bullet, C'_\bullet)$ is a set of maps:

$$h_i : C_i \to C'_{i+1} \text{ raising the index by one!}$$

with the property that $(f_i - g_i)(c_i) = (d'_{i+1} \circ h_i - h_{i-1} \circ d_i)(c_i)$.

(b) We say $f \sim g$ if there is a chain homotopy between f and g.

Example. Let M and N be R-modules. Given a self-map of complexes:

$$\begin{array}{ccccccc} 0 & \to & M & \stackrel{i}{\to} & M \oplus N & \stackrel{q}{\to} & N \to & 0 \\ & & \downarrow 0 & & \downarrow g & & \downarrow 0 & \\ 0 & \to & M & \stackrel{i}{\to} & M \oplus N & \stackrel{q}{\to} & N \to & 0 \end{array}$$

then the morphism g is **not** required to be the zero map. Instead, we have:

$$g(m, n) = (f(n), 0) \text{ for some morphism } f : N \to M$$

On the other hand, each of these maps of complexes is *homotopic* to zero via:

$$h(m, n) = f(n) : M \oplus N \to M \text{ and } 0 : N \to M \oplus N$$

Singular Homology (a pop-up ad). Let X be a topological space.

$$\Delta_n = \{(x_0, ..., x_n) \mid x_0 + \cdots + x_n = 1\} \subset \mathbb{R}^{n+1}$$

is the n-simplex (a manifold with boundary), and

$$C_n(X) = \sum k_i \{f_i : \Delta_n \to X\}$$

is the free abelian group on the continuous maps $f : \Delta_n \to X$. with boundary:

$$\partial_n(f) = f|_{\text{boundary of } \Delta_n}$$

(with appropriate signs so that $\partial_{n-1} \circ \partial_n = 0$). The homology of the complex $(C_\bullet(X), \partial_\bullet)$ is the *singular homology* $H_i(X, \mathbb{Z})$, and a continuous map $f : X \to Y$ induces a chain map $f : C_\bullet(X) \to C_\bullet(Y)$ by composition, and therefore a map on singular homology groups. *Homotopic* continuous maps $f \sim g : X \to Y$ induce the same map on singular homology because they are used to create a chain homotopy between the chain maps! (Hence the name).

Remarks. (i) Chain homotopy is an equivalence relation on chain maps.

(ii) The chain maps that are homotopic to the zero map are a sub R-module:

$$\mathrm{Zero}(C_\bullet, C'_\bullet) \subset \mathrm{Hom}(C_\bullet, C'_\bullet)$$

Proposition 2. If $f \sim 0$, then f induces the zero map on all homology R-modules.

Proof. Let h be the chain homotopy satisfying the definition. Then

$$f_i(c_i) = d'_{i+1}(h_i(c_i)) - h_{i-1}(d_i(c_i)) = d'_{i+1}(h_i(c_i)) \in \mathrm{im}(d'_{i+1})$$

for all elements $c_i \in \ker(d_i)$. □

Proposition 3. If $f : C_\bullet \to C'_\bullet$ or $g : C'_\bullet \to C''_\bullet$ is homotopic to zero, then $g \circ f \sim 0$.

Proof. If h is the homotopy for f, then $g \circ h$ is the homotopy for $g \circ f$, etc. □

Remark. It is tempting to declare homotopic maps to be equal and replace:

$$\mathrm{Hom}(C_\bullet, C'_\bullet) \text{ with } \mathrm{Hom}(C_\bullet, C'_\bullet)/\mathrm{Zero}(C_\bullet, C'_\bullet)$$

This defines a "homotopy category" of chain complexes (by Proposition 3), in which the Hom spaces are the quotient modules. Zero complexes are still the zero objects and direct sums are still the products and coproducts. Isomorphisms are a bit different, though, since f and g are two-sided inverses (with $C''_\bullet = C_\bullet$) in the homotopy category if and only if $g \circ f \sim 1_{C_\bullet}$ and $f \circ g \sim 1_{C'_\bullet}$. However, in the homotopy category there are no kernels and cokernels with universal properties (in general), so the homotopy category is not an abelian category. It is instead a *triangulated* category, with a "mapping cone" playing the role of an intertwined kernel and cokernel.

The next lemma is a close relative of the snake lemma. It connects the homology objects of complexes in a short exact sequence (as objects of $\mathcal{C}h_{\mathcal{A}}$) via a long exact sequence (of objects of $\mathcal{A}$).

The Zigzag Lemma. Let $0 \to C'_\bullet \stackrel{f}{\to} C_\bullet \stackrel{g}{\to} C''_\bullet \to 0$ be a short exact sequence of complexes of R-modules. Then there are connecting "snake" morphisms:

$$\delta_i : H_i(C''_\bullet) \to H_{i-1}(C_\bullet)$$

in a long exact sequence of homology modules:

$$(*) \ \stackrel{\delta_{i+1}}{\to} H_i(C_\bullet) \stackrel{H_i(f)}{\to} H_i(C'_\bullet) \stackrel{H_i(g)}{\to} H_i(C''_\bullet) \stackrel{\delta_i}{\to}$$

Proof. Consider the diagram (with the complexes running vertically):

$$\begin{array}{ccccccccc}
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \to & C_{i+1} & \to & C'_{i+1} & \to & C''_{i+1} & \to & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \to & C_i & \stackrel{f_i}{\to} & C'_i & \stackrel{g_i}{\to} & C''_i & \to & 0 \\
 & & \downarrow d_i & & \downarrow d'_i & & \downarrow d''_i & & \\
0 & \to & C_{i-1} & \to & C'_{i-1} & \to & C''_{i-1} & \to & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & &
\end{array}$$

Via the snake lemma, we get snake maps: $\delta : \ker(d''_i) \to \mathrm{coker}(d_i) = C_{i-1}/\mathrm{im}(d_i)$ which we want to convert into maps on homology modules:

$$\delta_i : \ \ker(d''_i)/\mathrm{im}(d''_{i+1}) \to \ker(d_{i-1})/\mathrm{im}(d_i)$$

To do this, we need to show that δ_i is well-defined, i.e. that:

$$\delta(\ker(d_i'')) \subset \ker(d_{i-1}) \text{ and } \delta(\mathrm{im}(d_{i+1}'')) \subset \mathrm{im}(d_i)$$

and then we need to show that the sequence $(*)$ is exact at each of:

$$H_i(C_\bullet), H_i(C_\bullet') \text{ and } H_i(C_\bullet'')$$

Happy diagram chasing :-)

Remark. There is much more one can do along these lines. For example, one can ask, given a long exact sequence of complexes, whether there is a relationship among the homologies (there is). Or, in a more symmetric vein, one could consider a chain complex of chain complexes (a *double complex*) and ask for relations between the batch of homologies in one direction and those in the other direction. This is what *spectral sequences* are designed to do.

Functors. On the one hand, functors are the morphisms in a category whose objects are categories! On the other hand, one can ask whether a functor between abelian categories preserves exact sequences (in which case it is an *exact* functor). Of particular interest are functors that are nearly exact, converting short exact sequences to short left (or right) exact sequences.

Definition. (a) A *functor* $F : \mathcal{F} \to \mathcal{G}$ between categories is:

- A mapping of objects $F(X) = Y$, and
- A mapping of morphisms $F(f : X \to Y) = F(f) : F(X) \to F(Y)$

with the property that $F(1_X) = 1_{F(X)}$ and

$$F(f \circ g) = F(f) \circ F(g) \text{ for all composable morphisms}$$

(b) A *contravariant functor* $F : \mathcal{F} \to \mathcal{G}$ is a functor from $\mathcal{F}^{\mathrm{opp}}$ to $\mathcal{G}$, i.e. it is a functor with the property that the mapping on morphisms "reverses arrows."

Some Fundamental Examples.

(a) Forgetful functors (from more structure to less). E.g.

$$F_{Ab,Set} : \mathcal{A}b \to \mathcal{S}et$$

from the category of abelian groups to sets, taking:

$$F(A, +) = A \text{ as a set, and } F(f : A \to B) = f : F(A) \to F(B)$$

since every abelian group is a set and every homomorphism is a set mapping. Note that there are more set mappings between two abelian groups than there are group homomorphisms, but if two group homomorphisms are the same, then the set mappings are the same. On the other hand, the "forgetful" functor:

$$F_{Ab,Gr} : \mathcal{A}b \to \mathcal{G}r$$

from the category of abelian groups to arbitrary groups does not map onto all objects (since there are non-commutative groups), but:

$$(*)\ F_{Ab,Gr} :\ \mathrm{Hom}_{\mathcal{A}b}(A, B) \to \mathrm{Hom}_{\mathcal{G}r}(A, B)$$

is a bijection for all abelian groups A and B. That is, the homorphisms between abelian groups are the same whether they are regarded as abelian groups or just as groups. A functor that is an *injection* on all Homs (like $F_{Ab,Set}$) is *faithful* and a functor that is a *bijection* on all Homs (like $F_{Ab,Gr}$) is *fully faithful.*

(b) Each object A of an abelian category $\mathcal{A}$ determines two functors:

$$F_A : \mathcal{A} \to \mathcal{A}b \text{ and } F^A : \mathcal{A}^{\text{opp}} \to \mathcal{A}b$$

defined as follows.

$$F_A(B) = \text{Hom}(A, B) \text{ and } F_A(f : B \to C) = f_* : \text{Hom}(A, B) \to \text{Hom}(A, C) \text{ and}$$
$$F^A(B) = \text{Hom}(B, A) \text{ and } F^A(f : C \to B) = f^* : \text{Hom}(B, A) \to \text{Hom}(C, A)$$
$$\text{where } f_*(\phi) = f \circ \phi \text{ and } f^*(\phi) = \phi \circ f$$

To check the first: $F_A(1_B)(\phi) = (1_B)_*(\phi) = 1_B \circ \phi = \phi$ is the identity, and $(g \circ f)_*(\phi) = (g \circ f) \circ \phi = g \circ (f \circ \phi) = (g_* \circ f_*)(\phi)$. Now you check the second.

(c) Another Topology Popup Ad. Let $\mathcal{T}op_0$ be the category of pointed topological spaces (X, p), with morphisms $f : (X, p) \to (Y, q)$ consisting of continuous maps:

$$f : X \to Y \text{ with } f(p) = q$$

Then the *fundamental group* is the functor to the category of groups:

$$\pi_1 : \mathcal{T}op_0 \to \mathcal{G}r$$

where $\pi_1(X, p)$ is the group of (homotopy equivalence classes of) continuous loops based at p and loop concatenation is the (non-commutative) group operation. The continuous maps $f : (X, p) \to (Y, q)$ go to homomorphisms $f_* : \pi_1(X, p) \to \pi_1(Y, q)$ and if $f \sim g$ are homotopic maps, then $f_* = g_*$, so this is not a faithful functor!

Proposition 4. Let

$$0 \to B \stackrel{f}{\to} B' \stackrel{g}{\to} B'' \to 0$$

be a short exact sequence in an abelian category $\mathcal{A}$, i.e. f is a monomorphism and is the kernel of g, and g is an epimorphism and is the cokernel of f.

Then the following sequences of abelian groups are both (left!) exact.

(i) $0 \to F_A(B) \stackrel{f_*}{\to} F_A(B') \stackrel{g_*}{\to} F_A(B'')$ and (ii) $F^A(B) \stackrel{f^*}{\leftarrow} F^A(B') \stackrel{g^*}{\leftarrow} F^A(B'') \leftarrow 0$

Proof (of (i), with (ii) done analogously) Consider the diagram:

$$\begin{array}{ccccccc} & & A & & & & \\ & & \downarrow \phi & \searrow \psi & & & \\ 0 & \to & B & \stackrel{f}{\longrightarrow} C & \stackrel{g}{\longrightarrow} & D & \to 0 \end{array}$$

By definition, f is a monomorphism if and only if $f_*(\phi_1) = f_*(\phi_2)$ implies $\phi_1 = \phi_2$ i.e. f_* is an injective map of abelian groups, and if $g_*(\psi) = 0$, then ψ factors as $\psi = f \circ \phi = f_*(\phi)$ for some $\phi : A \to B$ since f is the kernel of g. □

Definition. A short exact sequence of objects of an abelian category $\mathcal{A}$:

$$0 \to B \stackrel{f}{\to} B' \stackrel{g}{\to} B'' \to 0$$

is *split* by any morphism $h : B'' \to B'$ such that $g \circ h = 1_{B''}$ and one then obtains:

$$(f, h) : B \oplus B'' \to B', \text{ an isomorphism with the (co)product}$$

Example. In the category of R-modules, every short exact sequence:

$$0 \to M \stackrel{f}{\to} N \stackrel{g}{\to} R^n \to 0$$

culminating in a free R-module splits by sending each generator $e_i \in R^n$ to any n_i such that $g(n_i) = e_i$.

Proposition 5. An exact sequence $0 \to B \xrightarrow{f} B' \xrightarrow{g} B'' \to 0$ splits if and only if:

$$F_A(g) = g_* : \mathrm{Hom}(A, B') \to \mathrm{Hom}(A, B'')$$

is surjective for all A.

Proof. Suppose g_* is surjective for $A = B''$. Choose $h \in \mathrm{Hom}(B'', B')$ so that:

$$g_*(h) = 1_{B''} \in \mathrm{Hom}(B'', B'')$$

Then by definition, h splits the sequence! On the other hand, if $h : B'' \to B'$ splits the sequence, then for any object A, we have $g_* h_*(\psi) = \psi$ for all $\psi \in \mathrm{Hom}(A, B'')$ and so g_* is surjective. □

Instead of focusing on the sequence, we can instead focus on the object A.

Definition. An object P of $\mathcal{A}$ is **projective** if $g_* : \mathrm{Hom}(P, B') \to \mathrm{Hom}(P, B'')$ is surjective whenever $g : B' \to B''$ is surjective.

Proposition 6. The direct summands of free modules are the projectives in $\mathcal{M}od_R$.

Proof. If P is a projective R-module, let $g : F \to P$ be a surjective morphism from a free module (which always exists) with kernel module K and exact sequence:

$$0 \to K \to F \to P \to 0$$

Then as in the proof of Proposition 5, we may consider the surjection:

$$g_* : \mathrm{Hom}(P, F) \to \mathrm{Hom}(P, P)$$

and choose $h \in \mathrm{Hom}(P, F)$ with the property that $g_*(h) = 1_P$. Then h splits the sequence, and one obtains the isomorphism $(f, h) : K \oplus P \cong F$.

Conversely, let F be a free R-module and $g : B' \to B''$ a surjection. Then any $\psi : F \to B''$ lifts to $\phi : F \to B'$ by letting $\{e_\lambda\}$ freely generate F and choosing(!) $b'_\lambda = \phi(e_\lambda)$ so that $g(b'_\lambda) = \psi(e_\lambda)$. This shows that free modules are projective. Now if P is a summand of F, let:

$$h : P \to F \text{ and } q : F \to P \text{ be the inclusion and projection, with } q \circ h = 1_P$$

Given $\psi : P \to B''$, we let $\phi : F \to B'$ lift the map $\psi \circ q : F \to B''$, then $((g \circ \phi) \circ h)(p) = ((\psi \circ q) \circ h)(p) = \psi(p)$ and so $\phi \circ h$ lifts ψ. □

Corollary. An R-module with torsion is never projective.

Meanwhile, in Opposite Land....

Definition. A short exact sequence $0 \to B \xrightarrow{f} B' \xrightarrow{g} B'' \to 0$ is *left split* by any morphism $e : B' \to B$ with the property that $f \circ e = 1_B$, from which one obtains an isomorphism $(e, g) : B' \to B \oplus B''$ with the product.

The proofs of the following two Propositions are left as exercises.

Proposition 7. A short exact sequence is left split if and only if it is split.

Proposition 8. A sequence

$$0 \to B \to B' \to B'' \to 0$$

(left) splits if and only if each $F^A(f) = f^*$ is surjective.

Definition. An object I of $\mathcal{A}$ is *injective* if $F^A(f)$ is surjective when f is injective.

Given the similarities between F^A and F_A, the following may be a surprise.

Proposition 9. If R is a domain but not a field, then R is **not** injective in $\mathcal{M}od_R$.

Proof. Let $a \in R$ and let $a : R \to R$ be multiplication by a. Then recall that multiplciation by a is injective if and only if a is not a zero-divisor. For the map:

$$F^R(a) = a^* : \mathrm{Hom}(R,R) \to \mathrm{Hom}(R,R)$$

we see that $1_R = a^*(b)$ if and only if $ab = 1$, i.e. if and only if a is a unit.

Thus, if a is not a zero-divisor or a unit, then R is not an injective R-module. □

Remark. Of course R, being free, is a projective in $\mathcal{M}od_R$.

Proposition 10. The injectives in the category $\mathcal{A}b$ are the *divisible* groups, i.e. those abelian groups A with the property that $nx = a$ has a solution in A whenever $a \in A$ and $n \in \mathbb{Z}$.

Proof. Suppose first that $nx = a$ has no solution. Then $n \neq 0$, and:

$$n : \mathbb{Z} \to \mathbb{Z}$$

exhibits the non-injectivity of A, with $a \notin n^*(\mathrm{Hom}(\mathbb{Z}, A))$.

On the other hand, suppose A is divisible and consider $B \subset B'$ and $\phi : B \to A$. Then consider the chains (C, ϕ_C) of intermediary subgroups $B \subseteq C \subseteq B'$ equipped with morphisms $\phi_C : C \to A$ such that $\phi_C|_B = \phi$. Note that a chain is defined by the condition $(C, \phi_C) \subset (C', \phi_{C'})$ if and only if $C \subset C'$ and $\phi_{C'}|_C = \phi_C$.

Each such chain clearly has an upper bound, so by Zorn's lemma, there is a maximal element (C, ϕ_C). Now suppose that $b' \in B' - C$. Then either $nb' = x \in C$ for some n, in which case sending $b' + c$ to $\frac{1}{n}\phi_C(x) + \phi_C(c) \in A$ extends the map ϕ_C to the larger subgroup generated by C and b', or else $nb' \notin C$ for all n, in which case ϕ_C may be extended by sending b' to an arbitrary $a \in A$. In either case, (C, ϕ_C) is not a maximal element unless $C = B'$, proving the result with Zorn's lemma. □

Examples. The rationals $\mathbb{Q}$ (with $+$) and the groups:

$$\mathbb{Z}/p^\infty\mathbb{Z} = \{\zeta \in \mathbb{C} \mid \zeta^{p^n} = 1 \text{ for some } n\}$$

are divisible groups (for all primes p).

Corollary. Every (finitely generated) abelian group is a subgroup of an injective.

Note, however, that an injective (non-zero) abelian group is never finitely generated!